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GENERALIZATION OF WEIGHTED OSTROWSKI TYPE INEQUALITIES BY ABEL–GONTSCHAROFF POLYNOMIAL

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Abstract. We present a weighted generalization of Ostrowski type inequality for continous functions presented by Abel-Gontscharoff interpolating polynomial

1. Introduction

The well known *Ostrowski inequality* states:

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \leqslant \left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\left\|f'\right\|_\infty. \tag{1.1}$$

It holds for every $x\in[a,b]$ whenever $f:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$ and differentiable on $\langle a,b\rangle$ with bounded derivative. Ostrowski proved it in 1938. in [8] and since then it has been generalized in a number of ways. Over the last decades some new inequalities of this type have been intensively considered and applied in Numerical analysis and Probability (see [6], [7]).

The aim of this paper is to give a weighted generalization of Ostrowski type inequality for functions presented by Abel-Gontscharoff interpolating polynomial. For this purpose we will first introduce Abel-Gontcharoff interpolation.

Let $-\infty<a<b<\infty$, and $a\leqslant a_1\leqslant\ldots\leqslant a_n\leqslant b$ be given knots. We denote $\mathbf{a}=(a_1,\ldots,a_n)$. It is well known, that for $f\in C^n[a,b]$ a unique polynomial $P_A(t)$ of degree $(n-1)$ exists (see [1]), fulfilling one of the following Abel-Gontcharoff conditions:

$$P_A^{(i)}(a_{i+1})=f^{(i)}(a_{i+1});\ 0\leqslant i\leqslant n-1. \tag{1.2}$$

The associated error $e_A(t)$ can be represented in terms of the Green's function $G_{\mathbf{a},n}(t,s)$ of the boundary value problem

$$z^{(n)}(t)=0$$

$$z^{(i)}(a_{i+1})=0,\ 0\leqslant i\leqslant n-1$$

$$e_A(t)=\int_a^b G_{\mathbf{a},n}(t,s)f^{(n)}(s)\,ds,\ t\in[a,b]$$

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and it is given by

$$G_{\mathbf{a},n}(t,s)=\begin{cases}\sum_{i=0}^{k-1}\frac{T_{\mathbf{a},i}(t)}{(n-i-1)!}(a_{i+1}-s)^{n-i-1}, & a_k\leqslant s\leqslant t\\ -\sum_{i=k}^{n-1}\frac{T_{\mathbf{a},i}(t)}{(n-i-1)!}(a_{i+1}-s)^{n-i-1}, & t\leqslant s\leqslant a_{k+1}\\ & k=0,\dots,n\end{cases} \tag{1.3}$$

where $a_0=a$, $a_{n+1}=b$ and

$$T_{\mathbf{a},i}(t)=\frac{1}{1!2!\dots i!}\begin{vmatrix}1 & a_1 & a_1^2 & \dots & a_1^{i-1} & a_1^i\\ 0 & 1 & 2a_2 & \dots & (i-1)a_2^{i-2} & ia_2^{i-1}\\ \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & \dots & (i-1)! & i!a_i\\ 1 & t & t^2 & \dots & t^{i-1} & t^i\end{vmatrix} \tag{1.4}$$

$$=\int_{a_1}^{t}\int_{a_2}^{t_1}\dots\int_{a_i}^{t_{i-1}}dt_i dt_{i-1}\dots dt_1\,(t_0=t).$$

The first few are

$$T_{\mathbf{a},0}(t)=1,$$
$$T_{\mathbf{a},1}(t)=t-a_1,$$
$$T_{\mathbf{a},2}(t)=1/2\left[t^2-2a_2t+a_1(2a_2-a_1)\right],$$
$$T_{\mathbf{a},3}(t)=1/2\left[(t-a_3)^3/3-(a_2-a_3)^2(t-a_1)-(a_1-a_3)^3\right].$$

The following result holds (see [1]).

THEOREM 1. *Let $f\in C^n[a,b]$ and let P_A be its Abel-Gontscharoff interpolating polynomial. Then for $a=a_0\leqslant a_1\leqslant\dots\leqslant a_n\leqslant a_{n+1}=b$ it holds*

$$f(t)=P_A(t)+e_A(t) \tag{1.5}$$
$$=\sum_{i=0}^{n-1}T_{\mathbf{a},i}(t)f^{(i)}(a_{i+1})+\int_a^b G_{\mathbf{a},n}(t,s)f^{(n)}(s)\,ds$$

where G_n is the Green's functions, defined by (1.3).

We will also need the *weighted Montgomery identity*, obtained by J. Pečarić in [9]

$$f(x)=\int_a^b w(t)f(t)\,dt+\int_a^b P_w(x,t)f'(t)\,dt \tag{1.6}$$

where $w:[a,b]\to[0,\infty\rangle$ is some normalized weight function i.e. integrable function satisfying $\int\limits_a^b w(t)\,dt = 1$,

$$W(t) = \begin{cases} 0, & t < a, \\ \int\limits_a^t w(x)\,dx, & t \in [a,b], \\ 1, & t > b. \end{cases}$$

and $P_w(x,t)$ is the weighted Peano kernel

$$P_w(x,t) = \begin{cases} W(t), & a \leqslant t \leqslant x, \\ W(t) - 1, & x < t \leqslant b. \end{cases} \tag{1.7}$$

In Section 2 we present weighted generalization of the Montgomery identity by using Abel-Gontscharoff interpolating polynomial. In Section 3 we derive Ostrowski type inequality for differentiable functions of class C^n. Finally, we obtain special cases $n = 2$ for uniform weighted function, as well as for normalized weight functions $w(t) = \frac{1}{\pi\sqrt{1-t^2}}$, $t \in \langle -1, 1\rangle$; $w(t) = \frac{2}{\pi}\sqrt{1-t^2}$, $t \in [-1,1]$; $w(t) = \frac{3}{2}\sqrt{t}$, $t \in [0,1]$ and $w(t) = \frac{1}{2\sqrt{t}}$, $t \in \langle 0, 1]$. For some other applications of Montgomery type identities for integral Ostrowski type inequalities we refer interested reader to [2], [3], [4], [5], [6].

2. Generalization of weighted Montgomery identity

THEOREM 2. *Suppose $n \geqslant 2$, $f \in C^n[a,b]$ and $w : [a,b] \to [0,\infty\rangle$ is some normalized weight function. Then for $a = b_0 \leqslant b_1 \leqslant \ldots \leqslant b_{n-1} \leqslant b_n = b$, $\mathbf{b} = (b_1, \ldots, b_{n-1})$, the following identity holds*

$$f(x) = \int_a^b w(t)f(t)\,dt + \sum_{i=0}^{n-2} f^{(i+1)}(b_{i+1}) \int_a^b P_w(x,t)T_{\mathbf{b},i}(t)\,dt + \int_a^b \left(\int_a^b G_{\mathbf{b},n-1}(t,s)P_w(x,t)\,dt \right) f^{(n)}(s)\,ds. \tag{2.1}$$

Proof. If we take $n-1$ knots $b_1 \leqslant \ldots \leqslant b_{n-1}$ instead of n and apply (1.5) to function $f'(t)$, we get the following identity

$$f'(t) = \sum_{i=0}^{n-2} T_{\mathbf{b},i}(t) f^{(i+1)}(b_{i+1}) + \int_a^b G_{\mathbf{b},n-1}(t,s) f^{(n)}(s)\,ds \tag{2.2}$$

By putting (2.2) in (1.6) we get identity (2.1). □

THEOREM 3. *Suppose $f \in C^n[a,b]$ and $w : [a,b] \to [0,\infty\rangle$ is some normalized weight function. Then for $a = a_0 \leqslant a_1 \leqslant \ldots \leqslant a_n \leqslant a_{n+1} = b$ following identity holds*

$$f(x) - \int_a^b w(t)f(t)\,dt = \sum_{i=0}^{n-1} f^{(i)}(a_{i+1})\left(T_{\mathbf{a},i}(x) - \int_a^b w(t)T_{\mathbf{a},i}(t)\,dt\right) + \int_a^b \left(G_{\mathbf{a},n}(x,s) - \int_a^b w(t)G_{\mathbf{a},n}(t,s)\,dt\right) f^{(n)}(s)\,ds. \tag{2.3}$$

Proof. If we multiply (1.5) with $w(t)$ and integrate from a to b, we get

$$\int_a^b w(t)f(t)\,dt = \sum_{i=0}^{n-1} f^{(i)}(a_{i+1}) \int_a^b w(t)T_{\mathbf{a},i}(t)\,dt + \int_a^b \int_a^b w(t)G_{\mathbf{a},n}(t,s)f^{(n)}(s)\,dsdt. \tag{2.4}$$

Also, for the $\mathbf{a} = (a_1,\ldots,a_n)$ we have

$$f(x) = \sum_{i=0}^{n-1} T_{\mathbf{a},i}(x)f^{(i)}(a_{i+1}) + \int_a^b G_{\mathbf{a},n}(x,s)f^{(n)}(s)\,ds. \tag{2.5}$$

From (2.4) and (2.5) we obtain (2.3). □

COROLLARY 1. *Suppose $f \in C^n[a,b]$ and $w : [a,b] \to [0,\infty\rangle$ is some normalized weight function. Then following identity holds*

$$f(x) - \int_a^b w(t)f(t)\,dt = \sum_{i=0}^{n-1} f^{(i)}(a_{i+1}) \int_a^b P_w(x,t)T'_{\mathbf{a},i}(t)\,dt + \int_a^b \left(\int_a^b P_w(x,t)\frac{\partial}{\partial x}G_{\mathbf{a},n}(t,s)\,dt\right) f^{(n)}(s)\,ds. \tag{2.6}$$

Proof. By applying the weighted Montgomery identity for the $T_{\mathbf{a},i}(t)$ and $G_{\mathbf{a},n}(x,s)$ we obtain next two identities

$$T_{\mathbf{a},i}(x) = \int_a^b w(t)T_{\mathbf{a},i}(t)\,dt + \int_a^b P_w(x,t)T'_{\mathbf{a},i}(t)\,dt, \tag{2.7}$$

$$G_{\mathbf{a},n}(x,s) = \int_a^b w(t)G_{\mathbf{a},n}(t,s)\,dt + \int_a^b P_w(x,t)\frac{\partial}{\partial t}G_{\mathbf{a},n}(t,s)\,dt. \tag{2.8}$$

By putting (2.7) and (2.8) into (2.3) we obtain (2.6). □

REMARK 1. Identities (2.1) and (2.6) coincide for $n \geqslant 2$, if we choose knots $b_i = a_{i+1}$, $(i = 1,\ldots,n-1)$. Namely, from the (1.4) for $b_i = a_{i+1}$ we can conclude that the following holds

$$T'_{\mathbf{a},i}(t) = T_{\mathbf{b},i-1}(t), \quad i \geqslant 1 \tag{2.9}$$

$$\frac{\partial}{\partial t}G_{\mathbf{a},n}(t,s) = G_{\mathbf{b},n-1}(t,s), \; i \geqslant 1$$

Since that for $i=0$ holds $T'_{\mathbf{a},0}(t)=0$, the first term in the sum in identity (2.6) is equal to zero. Also, for the same choice of knots we have

$$f^{(i)}(b_i)=f^{(i)}(a_{i+1}),\ i\geqslant 1$$

which proves assertion. Further, (2.3) and (2.6) for $n=1$ coincide with weighted Montgomery identity. Thus, for further generalizations we will use (2.1).

3. Ostrowski type inequalities

Here and hereafter for $p\geqslant 1$ we denote

$$\|f\|_p=\left(\int_a^b |f(t)|^p\,dt\right)^{\frac{1}{p}}$$

and

$$\|f\|_\infty = ess \sup_{t\in[a,b]} |f(t)|.$$

DEFINITION 1. A pair of two real numbers (p,q) are called *conjugate exponents* if $1<p,q<\infty$ and $\frac{1}{p}+\frac{1}{q}=1$. Formally, we will also define $p=1$ as conjugate to $q=\infty$ and vice versa.

THEOREM 4. *Suppose that all the assumptions of the Theorem 2 hold. Additionally assume (p,q) is pair of conjugate exponents $1\leqslant p,q\leqslant\infty$. Then following inequality holds*

$$\left|f(x)-\int_a^b w(t)f(t)\,dt-\sum_{i=0}^{n-2} f^{(i+1)}(b_{i+1})\int_a^b P_w(x,t)T_{\mathbf{b},i}(t)\,dt\right| \leqslant \|K(\cdot,x)\|_q \left\|f^{(n)}\right\|_p \tag{3.1}$$

where

$$K(s,x)=\int_a^b G_{\mathbf{b},n-1}(t,s)P_w(x,t)\,dt.$$

Proof. Applying Hölder inequality to the (2.1) we get (3.1). □

3.1. Case $n=2$ for uniform weight function $w(t)=\frac{1}{b-a}$, $t\in[a,b]$

COROLLARY 2. *Assume (p,q) is pair of conjugate exponents and $1\leqslant q<\infty$, $1<p\leqslant\infty$. If $f\in C^2[a,b]$ then for every $x\in\langle a,b\rangle$ following inequality holds*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt-f'(x)\left(x-\frac{a+b}{2}\right)\right|\leqslant\left(\frac{(x-a)^{2q+1}+(b-x)^{2q+1}}{(2q+1)\,2^q\,(b-a)^q}\right)^{1/q}\|f''\|_p.$$

Proof. We apply Theorem 4 with uniform weight function $w(t) = \frac{1}{b-a}$, $t \in [a,b]$ and knots $b_0 = a$, $b_1 = x$, $b_2 = b$. Thus we have

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\,dt - f'(x) \int_a^b P(x,t) T_{\mathbf{b},0}(t)\,dt \right| \leqslant \|K(\cdot,x)\|_q \left\| f'' \right\|_p$$

where

$$K(s,x) = \int_a^b G_{\mathbf{b},1}(t,s) P(x,t)\,dt$$

and

$$G_{\mathbf{b},1}(t,s) = \begin{cases} 0, & a \leqslant s \leqslant t \leqslant x, \\ -1, & a \leqslant t \leqslant s \leqslant x, \\ 1, & x \leqslant s \leqslant t \leqslant b, \\ 0, & x \leqslant t \leqslant s \leqslant b. \end{cases}$$

Since $T_{\mathbf{b},0}(t) = 1$ we have

$$\int_a^b P(x,t) T_{\mathbf{b},0}(t)\,dt = \int_a^b P(x,t)\,dt = \int_a^x \frac{t-a}{b-a}\,dt + \int_x^b \frac{t-b}{b-a}\,dt = x - \frac{a+b}{2}$$

and

$$K(s,x) = \int_a^x G_{\mathbf{b},1}(t,s) \frac{t-a}{b-a}\,dt + \int_x^b G_{\mathbf{b},1}(t,s) \frac{t-b}{b-a}\,dt.$$

If $s < x$

$$K(s,x) = -\int_a^s \frac{t-a}{b-a}\,dt = -\frac{(s-a)^2}{2(b-a)}$$

and if $s > x$

$$K(s,x) = \int_s^b \frac{t-b}{b-a}\,dt = -\frac{(s-b)^2}{2(b-a)}.$$

So the q-norm of $K(s,x)$ with respect to variable s is

$$\begin{aligned} \|K(\cdot,x)\|_q &= \left(\int_a^x \left| -\frac{(s-a)^2}{2(b-a)} \right|^q ds + \int_x^b \left| -\frac{(s-b)^2}{2(b-a)} \right|^q ds \right)^{1/q} \\ &= \left(\frac{(x-a)^{2q+1} + (b-x)^{2q+1}}{(2q+1)2^q (b-a)^q} \right)^{1/q} \end{aligned}$$

and the proof is done. □

COROLLARY 3. *If $f \in C^2[a,b]$ then for every $x \in \langle a,b \rangle$ following inequality holds*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\,dt - f'(x) \left(x - \frac{a+b}{2} \right) \right| \leqslant \frac{1}{2(b-a)} \max\left\{ (x-a)^2, (b-x)^2 \right\} \left\| f'' \right\|_1.$$

Proof. We apply Theorem 4 with uniform weight function $w(t) = \frac{1}{b-a}$, $t \in [a,b]$ and knots $b_0 = a$, $b_1 = x$, $b_2 = b$ and $p = 1$ $(q = \infty)$. Thus we have

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\,dt - f'(x) \int_a^b P(x,t) T_{\mathbf{b},0}(t)\,dt \right| \leqslant \|K(\cdot,x)\|_\infty \left\| f'' \right\|_1$$

where

$$K(s,x) = \int_a^b G_{\mathbf{b},1}(t,s) P(x,t)\,dt$$

and

$$G_{\mathbf{b},1}(t,s) = \begin{cases} 0, & a \leqslant s \leqslant t \leqslant x, \\ -1, & x \leqslant t \leqslant s \leqslant x, \\ 1, & x \leqslant s \leqslant t \leqslant b, \\ 0, & x \leqslant t \leqslant s \leqslant b. \end{cases}$$

As in the proof of the previous corollary we have

$$\int_a^b P(x,t) T_{\mathbf{b},0}(t)\,dt = x - \frac{a+b}{2}$$

and

$$K(s,x) = \int_a^x G_{\mathbf{b},1}(t,s) \frac{t-a}{b-a}\,dt + \int_x^b G_{\mathbf{b},1}(t,s) \frac{t-b}{b-a}\,dt$$

that is

$$K(s,x) = \begin{cases} -\frac{(s-a)^2}{2(b-a)}, & s < x, \\ \\ -\frac{(s-b)^2}{2(b-a)}, & x < s. \end{cases}$$

So the ∞-norm of $K(s,x)$ with respect to variable s is

$$\begin{aligned} \|K(\cdot,x)\|_\infty &= \sup_{s \in [a,b]} |K(s,x)| = \max \left\{ \sup_{s \in [a,x]} \left| -\frac{(s-a)^2}{2(b-a)} \right|, \sup_{s \in [x,b]} \left| -\frac{(s-b)^2}{2(b-a)} \right| \right\} \\ &= \frac{1}{2(b-a)} \max \left\{ (x-a)^2, (b-x)^2 \right\} \end{aligned}$$

and the proof is done. □

COROLLARY 4. *Assume (p,q) is pair of conjugate exponents and $1 \leqslant q < \infty$, $1 < p \leqslant \infty$. If $f \in C^2[a,b]$ then following inequality holds*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \leqslant \frac{1}{8} \left(\frac{(b-a)^{q+1}}{(2q+1)2} \right)^{1/q} \left\| f'' \right\|_p .$$

Proof. We take $x = \frac{a+b}{2}$ in the Corollary 2. □

COROLLARY 5. *$f \in C^2[a,b]$ then following inequality holds*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leqslant \frac{(b-a)}{8} \left\| f'' \right\|_1 .$$

Proof. We take $x = \frac{a+b}{2}$ in the Corollary 3. □

3.2. Case $n = 2$ for weight function $w(t) = \frac{1}{\pi\sqrt{1-t^2}}$, $t \in \langle -1, 1 \rangle$

COROLLARY 6. *If $f \in C^2[-1,1]$ then for every $x \in \langle -1,1 \rangle$ following inequality holds*

$$\left| f(x) - \frac{1}{\pi} \int_{-1}^1 \frac{f(t)}{\sqrt{1-t^2}}\, dt - x f'(x) \right| \leqslant \left(\frac{x}{\pi} \arcsin x + \frac{1}{\pi}\sqrt{1-x^2} + \frac{|x|}{2} \right) \left\| f'' \right\|_1 .$$

Proof. We apply Theorem 4 with weight function $w(t) = \frac{1}{\pi\sqrt{1-t^2}}$, $t \in \langle -1, 1 \rangle$ and knots $b_0 = -1$, $b_1 = x$, $b_2 = 1$ and $p = 1$ $(q = \infty)$. Thus we have

$$\left| f(x) - \frac{1}{\pi} \int_{-1}^1 \frac{1}{\sqrt{1-t^2}} f(t)\, dt - f'(x) \int_{-1}^1 P_w(x,t) T_{\mathbf{b},0}(t)\, dt \right| \leqslant \left\| K(\cdot, x) \right\|_\infty \left\| f'' \right\|_1$$

where

$$K(s,x) = \int_{-1}^1 G_{\mathbf{b},1}(t,s) P_w(x,t)\, dt$$

and

$$G_{\mathbf{b},1}(t,s) = \begin{cases} 0, & -1 \leqslant s \leqslant t \leqslant x, \\ -1, & x \leqslant t \leqslant s \leqslant x, \\ 1, & x \leqslant s \leqslant t \leqslant 1, \\ 0, & x \leqslant t \leqslant s \leqslant 1. \end{cases}$$

Since $T_{\mathbf{b},0}(t) = 1$ we have

$$\int_{-1}^1 P_w(x,t) T_{\mathbf{b},0}(t)\, dt = \frac{1}{\pi} \int_{-1}^x \left(\arcsin t + \frac{\pi}{2} \right) dt + \frac{1}{\pi} \int_x^1 \left(\arcsin t - \frac{\pi}{2} \right) dt = x$$

and

$$K(s,x) = \frac{1}{\pi} \int_{-1}^x G_{\mathbf{b},1}(t,s) \left(\arcsin t + \frac{\pi}{2} \right) dt + \frac{1}{\pi} \int_x^1 G_{\mathbf{b},1} \left(\arcsin t - \frac{\pi}{2} \right) dt$$

that is

$$K(s,x) = \begin{cases} -\frac{1}{\pi} \left(s \arcsin s + \sqrt{1-s^2} + \frac{s\pi}{2} \right), & s < x, \\ \frac{1}{\pi} \left(-s \arcsin s - \sqrt{1-s^2} + \frac{s\pi}{2} \right), & x < s. \end{cases}$$

Let's denote

$$\alpha(s) = -\frac{1}{\pi}\left(s\arcsin s + \sqrt{1-s^2} + \frac{s\pi}{2}\right),\ s \in [-1,1]$$

and

$$\beta(s) = \frac{1}{\pi}\left(-s\arcsin s - \sqrt{1-s^2} + \frac{s\pi}{2}\right),\ s \in [-1,1].$$

Since α is negative and decreasing on $[-1,1]$ and β is negative and increasing on $[-1,1]$ the ∞-norm of $K(s,x)$ with respect to variable s is

$$\begin{aligned}\|K(\cdot,x)\|_\infty &= \max\left\{\sup_{s\in[-1,x]}|\alpha(s)|,\ \sup_{s\in[x,1]}|\beta(s)|\right\} = \max\{-\alpha(x),-\beta(x)\}\\ &= \frac{1}{\pi}\max\left\{\left(x\arcsin x + \sqrt{1-x^2} + \frac{x\pi}{2}\right), x\arcsin x + \sqrt{1-x^2} - \frac{x\pi}{2}\right\}\\ &= \frac{1}{\pi}\left(x\arcsin x + \sqrt{1-x^2} + \frac{|x|\pi}{2}\right). \quad \square\end{aligned}$$

COROLLARY 7. *If $f \in C^2[-1,1]$ then for every $x \in \langle -1,1\rangle$ following inequality holds*

$$\left|f(x) - \frac{1}{\pi}\int_{-1}^{1}\frac{f(t)}{\sqrt{1-t^2}}dt - x f'(x)\right| \leqslant \left(\frac{1}{4} + \frac{1}{2}x^2\right)\|f''\|_\infty.$$

Proof. We apply Theorem 4 with weight function $w(t) = \frac{1}{\pi\sqrt{1-t^2}}$, $t \in \langle -1,1\rangle$ and knots $b_0 = -1$, $b_1 = x$, $b_2 = 1$ and $p = \infty$ $(q = 1)$. Similar as in previous corollary we have

$$\left|f(x) - \frac{1}{\pi}\int_{-1}^{1}\frac{1}{\sqrt{1-t^2}}f(t)\,dt - xf'(x)\right| \leqslant \|K(\cdot,x)\|_1 \left\|f''\right\|_\infty$$

and

$$K(s,x) = \begin{cases} -\frac{1}{\pi}\left(s\arcsin s + \sqrt{1-s^2} + \frac{s\pi}{2}\right), & s < x,\\ \frac{1}{\pi}\left(-s\arcsin s - \sqrt{1-s^2} + \frac{s\pi}{2}\right), & x < s.\end{cases}$$

The 1-norm of $K(s,x)$ with respect to variable s is

$$\begin{aligned}\|K(\cdot,x)\|_1 &= \frac{1}{\pi}\int_{-1}^{x}\left|s\arcsin s + \sqrt{1-s^2} + \frac{s\pi}{2}\right|ds + \frac{1}{\pi}\int_{x}^{1}\left|s\arcsin s + \sqrt{1-s^2} - \frac{s\pi}{2}\right|ds\\ &= \frac{1}{4} + \frac{1}{2}x^2. \quad \square\end{aligned}$$

3.3. Case $n = 2$ for weight function $w(t) = \frac{2}{\pi}\sqrt{1-t^2}$, $t \in [-1,1]$

COROLLARY 8. *If $f \in C^2[-1,1]$ then for every $x \in \langle -1,1\rangle$ following inequality holds*

$$\left|f(x) - \frac{2}{\pi}\int_{-1}^{1}\sqrt{1-t^2}f(t)\,dt - x f'(x)\right| \leqslant \left(\frac{x}{\pi}\arcsin x + \frac{2+x^2}{3\pi}\sqrt{1-x^2} + \frac{|x|}{2}\right)\|f''\|_1.$$

Proof. We apply Theorem 4 with weight function $w(t)=\frac{2}{\pi}\sqrt{1-t^2}$, $t\in[-1,1]$ and knots $b_0=-1$, $b_1=x$, $b_2=1$ and $p=1$ $(q=\infty)$. Thus we have

$$\left|f(x)-\frac{2}{\pi}\int_{-1}^{1}\sqrt{1-t^2}f(t)\,dt-f'(x)\int_{-1}^{1}P_w(x,t)T_{\mathbf{b},0}(t)\,dt\right|\leqslant\|K(\cdot,x)\|_\infty\left\|f''\right\|_1$$

where

$$K(s,x)=\int_{-1}^{1}G_{\mathbf{b},1}(t,s)P_w(x,t)\,dt.$$

Since $T_{\mathbf{b},0}(t)=1$ we have

$$\int_{-1}^{1}P_w(x,t)T_{\mathbf{b},0}(t)\,dt=\int_{-1}^{x}\left(\frac{1}{\pi}\arcsin t+\frac{1}{\pi}t\sqrt{1-t^2}+\frac{1}{2}\right)dt$$
$$+\int_{x}^{1}\left(\frac{1}{\pi}\arcsin t+\frac{1}{\pi}t\sqrt{1-t^2}-\frac{1}{2}\right)dt=x$$

and

$$K(s,x)=\int_{-1}^{x}G_{\mathbf{b},1}(t,s)\left(\frac{1}{\pi}\arcsin t+\frac{1}{\pi}t\sqrt{1-t^2}+\frac{1}{2}\right)dt$$
$$+\int_{x}^{1}G_{\mathbf{b},1}(t,s)\left(\frac{1}{\pi}\arcsin t+\frac{1}{\pi}t\sqrt{1-t^2}-\frac{1}{2}\right)dt$$

that is

$$K(s,x)=\begin{cases}\left(-\frac{1}{\pi}s\arcsin s-\frac{2}{3\pi}\sqrt{1-s^2}-\frac{1}{3\pi}s^2\sqrt{1-s^2}-\frac{1}{2}s\right), & s<x,\\[2ex] \left(-\frac{1}{\pi}s\arcsin s-\frac{2}{3\pi}\sqrt{1-s^2}-\frac{1}{3\pi}s^2\sqrt{1-s^2}+\frac{1}{2}s\right), & x<s.\end{cases}$$

Let's denote

$$\alpha(s)=-\frac{1}{\pi}s\arcsin s-\frac{2}{3\pi}\sqrt{1-s^2}-\frac{1}{3\pi}s^2\sqrt{1-s^2}-\frac{1}{2}s,\ s\in[-1,1]$$

and

$$\beta(s)=-\frac{1}{\pi}s\arcsin s-\frac{2}{3\pi}\sqrt{1-s^2}-\frac{1}{3\pi}s^2\sqrt{1-s^2}+\frac{1}{2}s,\ ,s\in[-1,1].$$

Since α is negative and decreasing on $[-1,1]$ and β is negative and increasing on $[-1,1]$ the ∞-norm of $K(s,x)$ with respect to variable s is

$$\|K(\cdot,x)\|_\infty=\max\left\{\sup_{s\in[-1,x]}|\alpha(s)|,\ \sup_{s\in[x,1]}|\beta(s)|\right\}=\max\{-\alpha(x),-\beta(x)\}$$
$$=\max\left\{\frac{x}{\pi}\arcsin x+\frac{2+x^2}{3\pi}\sqrt{1-x^2}+\frac{1}{2}x,\ \frac{x}{\pi}\arcsin x+\frac{2+x^2}{3\pi}\sqrt{1-x^2}-\frac{1}{2}x\right\}$$
$$=\frac{x}{\pi}\arcsin x+\frac{2+x^2}{3\pi}\sqrt{1-x^2}+\frac{|x|}{2}.\quad\square$$

COROLLARY 9. *If $f \in C^2[-1,1]$ then for every $x \in \langle -1,1\rangle$ following inequality holds*

$$\left| f(x) - \frac{2}{\pi}\int_{-1}^{1}\sqrt{1-t^2}dt - x f'(x)\right| \leqslant \left(\frac{1}{8}+\frac{1}{2}x^2\right)\left\|f''\right\|_\infty .$$

Proof. We apply Theorem 4 with weight function $w(t) = \frac{2}{\pi}\sqrt{1-t^2}$, $t \in [-1,1]$ and knots $b_0 = -1$, $b_1 = x$, $b_2 = 1$ and $p = \infty$ $(q = 1)$. Similar as in previous corollary we have

$$\left| f(x) - \frac{2}{\pi}\int_{-1}^{1}\sqrt{1-t^2}dt - x f'(x)\right| \leqslant \left\|K(\cdot,x)\right\|_1 \left\|f''\right\|_\infty$$

and

$$K(s,x) = \begin{cases} \left(-\frac{1}{\pi}s\arcsin s - \frac{2}{3\pi}\sqrt{1-s^2} - \frac{1}{3\pi}s^2\sqrt{1-s^2} - \frac{1}{2}s\right), & s < x, \\ \left(-\frac{1}{\pi}s\arcsin s - \frac{2}{3\pi}\sqrt{1-s^2} - \frac{1}{3\pi}s^2\sqrt{1-s^2} + \frac{1}{2}s\right), & x < s. \end{cases}$$

The 1-norm of $K(s,x)$ with respect to variable s is

$$\begin{aligned}\left\|K(\cdot,x)\right\|_1 &= \int_{-1}^{x}\left|-\frac{1}{\pi}s\arcsin s - \frac{2}{3\pi}\sqrt{1-s^2} - \frac{1}{3\pi}s^2\sqrt{1-s^2} - \frac{1}{2}s\right| ds \\ &\quad + \int_{x}^{1}\left|-\frac{1}{\pi}s\arcsin s - \frac{2}{3\pi}\sqrt{1-s^2} - \frac{1}{3\pi}s^2\sqrt{1-s^2} + \frac{1}{2}s\right| ds \\ &= \frac{1}{8}+\frac{1}{2}x^2. \quad \square\end{aligned}$$

3.4. Case $n = 2$ for weight function $w(t) = \frac{3}{2}\sqrt{t}$, $t \in [0,1]$

COROLLARY 10. *If $f \in C^2[0,1]$ then for every $x \in \langle 0,1\rangle$ following inequality holds*

$$\left| f(x) - \frac{3}{2}\int_{0}^{1}\sqrt{t}f(t)\,dt - f'(x)\left(x-\frac{3}{5}\right)\right| \leqslant \left(\frac{2}{5}\sqrt{x^5} + \max\left\{0,\frac{3}{5}-x\right\}\right)\left\|f''\right\|_1 .$$

Proof. We apply Theorem 4 with weight function $w(t) = \frac{3}{2}\sqrt{t}$, $t \in [0,1]$ and knots $b_0 = 0$, $b_1 = x$, $b_2 = 1$ and $p = 1$ $(q = \infty)$. Thus we have

$$\left| f(x) - \frac{3}{2}\int_{0}^{1}\sqrt{t}f(t)\,dt - f'(x)\int_0^1 P_w(x,t)T_{\mathbf{b},0}(t)\,dt\right| \leqslant \left\|K(\cdot,x)\right\|_\infty \left\|f''\right\|_1$$

where

$$K(s,x) = \int_0^1 G_{\mathbf{b},1}(t,s)P_w(x,t)\,dt.$$

Since $T_{\mathbf{b},0}(t) = 1$ we have

$$\int_0^1 P_w(x,t)T_{\mathbf{b},0}(t)\,dt = \int_0^x \sqrt{t^3}dt + \int_x^1\left(\sqrt{t^3}-1\right)dt = x - \frac{3}{5}$$

and

$$K(s,x) = \int_0^x G_{\mathbf{b},1}(t,s)\sqrt{t^3}dt + \int_x^1 G_{\mathbf{b},1}\left(\sqrt{t^3}-1\right)dt$$

that is

$$K(s,x) = \begin{cases} -\frac{2}{5}\sqrt{s^5}, & s < x, \\ -\frac{2}{5}\sqrt{s^5} + s - \frac{3}{5}, & x < s. \end{cases}$$

So the ∞-norm of $K(s,x)$ with respect to variable s is

$$\begin{aligned} \|K(\cdot,x)\|_\infty &= \sup_{s\in[0,1]} |K(s,x)| = \max\left\{ \sup_{s\in[0,x]} \left|-\frac{2}{5}\sqrt{s^5}\right|, \sup_{s\in[x,1]} \left|-\frac{2}{5}\sqrt{s^5}+s-\frac{3}{5}\right| \right\} \\ &= \max\left\{ \frac{2}{5}\sqrt{x^5}, \frac{2}{5}\sqrt{x^5} - x + \frac{3}{5} \right\}. \quad \square \end{aligned}$$

COROLLARY 11. *If $f \in C^2[0,1]$ then for every $x \in \langle 0,1\rangle$ following inequality holds*

$$\left| f(x) - \frac{3}{2}\int_0^1 \sqrt{t}f(t)\,dt - f'(x)\left(x-\frac{3}{5}\right) \right| \leqslant \left(\frac{1}{2}x^2 - \frac{3}{5}x + \frac{3}{14}\right)\|f''\|_\infty.$$

Proof. We apply Theorem 4 with weight function $w(t) = \frac{3}{2}\sqrt{t}$, $t \in [0,1]$ and knots $b_0 = 0$, $b_1 = x$, $b_2 = 1$ and $p = \infty$ $(q = 1)$. Similar as in previous corollary we have

$$\left| f(x) - \frac{2}{\pi}\int_{-1}^1 \sqrt{1-t^2}dt - f'(x)\left(x-\frac{3}{5}\right) \right| \leqslant \|K(\cdot,x)\|_1 \|f''\|_\infty$$

and

$$K(s,x) = \begin{cases} -\frac{2}{5}\sqrt{s^5}, & s < x, \\ -\frac{2}{5}\sqrt{s^5} + s - \frac{3}{5}, & x < s. \end{cases}$$

The 1-norm of $K(s,x)$ with respect to variable s is

$$\|K(\cdot,x)\|_1 = \int_0^x \left|-\frac{2}{5}\sqrt{s^5}\right| ds + \int_x^1 \left|-\frac{2}{5}\sqrt{s^5}+s-\frac{3}{5}\right| ds = \frac{1}{2}x^2 - \frac{3}{5}x + \frac{3}{14}. \quad \square$$

3.5. Case $n = 2$ for weight function $w(t) = \frac{1}{2\sqrt{t}}$, $t \in \langle 0,1]$

COROLLARY 12. *If $f \in C^2[0,1]$ then for every $x \in \langle 0,1\rangle$ following inequality holds*

$$\left| f(x) - \frac{1}{2}\int_0^1 \frac{1}{\sqrt{t}}f(t)\,dt - f'(x)\left(x-\frac{1}{3}\right) \right| \leqslant \left(\frac{2}{3}\sqrt{x^3} + \max\left\{0, \frac{1}{3}-x\right\}\right)\|f''\|_1.$$

Proof. We apply Theorem 4 with uniform weight function $w(t) = \frac{1}{2\sqrt{t}}$, $t \in \langle 0,1]$ and knots $b_0 = 0$, $b_1 = x$, $b_2 = 1$ and $p = 1$ $(q = \infty)$. Thus we have

$$\left| f(x) - \frac{1}{2}\int_0^1 \frac{1}{\sqrt{t}} f(t)\,dt - f'(x)\int_0^1 P_w(x,t)T_{\mathbf{b},0}(t)\,dt \right| \leqslant \|K(\cdot,x)\|_\infty \left\|f''\right\|_1$$

where

$$K(s,x) = \int_0^1 G_{\mathbf{b},1}(t,s)P_w(x,t)\,dt.$$

Since $T_{\mathbf{b},0}(t) = 1$ we have

$$\int_0^1 P_w(x,t)T_{\mathbf{b},0}(t)\,dt = \int_0^x \sqrt{t}dt + \int_x^1 \left(\sqrt{t} - 1\right)dt = x - \frac{1}{3}$$

and

$$K(s,x) = \int_0^x G_{\mathbf{b},1}(t,s)\sqrt{t}dt + \int_x^1 G_{\mathbf{b},1}(t,s)\left(\sqrt{t} - 1\right)dt$$

that is

$$K(s,x) = \begin{cases} -\frac{2}{3}\sqrt{s^3}, & s < x, \\ -\frac{2}{3}\sqrt{s^3} + s - \frac{1}{3}, & x < s. \end{cases}$$

So the ∞-norm of $K(s,x)$ with respect to variable s is

$$\begin{aligned} \|K(\cdot,x)\|_\infty &= \sup_{s\in[0,1]} |K(s,x)| = \max\left\{ \sup_{s\in\langle 0,x]} \left| -\frac{2}{3}\sqrt{s^3} \right|, \sup_{s\in[x,1]} \left| -\frac{2}{3}\sqrt{s^3} + s - \frac{1}{3} \right| \right\} \\ &= \max\left\{ \frac{2}{3}\sqrt{x^3}, \frac{2}{3}\sqrt{x^3} - x + \frac{1}{3} \right\}. \quad \square \end{aligned}$$

COROLLARY 13. *If $f \in C^2[0,1]$ then for every $x \in \langle 0,1\rangle$ following inequality holds*

$$\left| f(x) - \frac{1}{2}\int_0^1 \frac{1}{\sqrt{t}} f(t)\,dt - f'(x)\left(x - \frac{1}{3}\right) \right| \leqslant \left(\frac{1}{2}x^2 - \frac{1}{3}x + \frac{1}{10} \right) \left\|f''\right\|_\infty.$$

Proof. We apply Theorem 4 with weight function $w(t) = \frac{1}{2\sqrt{t}}$, $t \in \langle 0,1]$ and knots $b_0 = 0$, $b_1 = x$, $b_2 = 1$ and $p = \infty$ $(q = 1)$. Similar as in previous corollary we have

$$\left| f(x) - \frac{1}{2}\int_0^1 \frac{1}{\sqrt{t}} f(t)\,dt - f'(x)\left(x - \frac{1}{3}\right) \right| \leqslant \|K(\cdot,x)\|_1 \left\|f''\right\|_\infty$$

and

$$K(s,x) = \begin{cases} -\frac{2}{3}\sqrt{s^3}, & s < x, \\ -\frac{2}{3}\sqrt{s^3} + s - \frac{1}{3}, & x < s. \end{cases}$$

The 1-norm of $K(s,x)$ with respect to variable s is

$$\|K(\cdot,x)\|_1 = \int_0^x \left|-\frac{2}{3}\sqrt{s^3}\right| ds + \int_x^1 \left|-\frac{2}{3}\sqrt{s^3}+s-\frac{1}{3}\right| ds = \frac{1}{2}x^2 - \frac{1}{3}x + \frac{1}{10}. \quad \square$$

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