Mathematical
Inequalities
& Applications
Volume 22, Number 4 (2019), 1271–1282 doi:10.7153/mia-2019-22-87

WEIGHTED OSTROWSKI TYPE INEQUALITIES BY LIDSTONE POLYNOMIALS

ANDREA AGLIĆ ALJINOVIĆ, LJILJANKA KVESIĆ AND JOSIP PEČARIĆ

(*Communicated by M. Praljak*)

Abstract. We present a weighted generalization of Ostrowski type inequality for differentiable functions of class C^n presented by Lidstone interpolating polynomial.

1. Introduction

The aim of this paper is to give a weighted generalization of Ostrowski type inequality for differentiable functions of class C^n presented by Lidstone interpolating polynomial. For this purpose we will first present basic facts about Lidstone interpolation.

Lidstone series was introduced in 1929 by Lidstone [4]. It approximates a given function in the neighborhood of two symmetric points on $[0,1]$.

For $f \in C^{(2n)}([0,1])$ Widder [8] gave the following fundamental interpolation result

$$f(t)=\sum_{k=0}^{n-1}\left[f^{(2k)}(0)\Lambda_k(1-t)+f^{(2k)}(1)\Lambda_k(t)\right]+\int_0^1 G_n(t,s)f^{(2n)}(s)ds. \tag{1.1}$$

Here Λ_n is the Lidstone polynomial of degree $2n+1$ (uniquely) defined by the relations

$$\begin{aligned}&\Lambda_0(t)=t,\\ &\Lambda_n''(t)=\Lambda_{n-1}(t),\\ &\Lambda_n(0)=\Lambda_n(1)=0, \quad n\geqslant 1,\end{aligned} \tag{1.2}$$

G_1 is the homogeneous Green's function

$$G_1(t,s)=\begin{cases}(t-1)s, & \text{if } s\leqslant t,\\ (s-1)t, & \text{if } t\leqslant s,\end{cases} \tag{1.3}$$

and with the successive iterates of G_1 we obtain G_n

$$G_n(t,s)=\int_0^1 G_1(t,u)G_{n-1}(u,s)\,du, \quad n\geqslant 2. \tag{1.4}$$

Mathematics subject classification (2010): 26D15, 26D20.
Keywords and phrases: Ostrowski type inequality, Lidstone polynomials.

Paper MIA-22-87

Let's denote the Lidstone interpolating polynomial

$$P_L^n(t) = \sum_{k=0}^{n-1} \left[f^{(2k)}(0)\Lambda_k(1-t) + f^{(2k)}(1)\Lambda_k(t) \right],$$

and the associated error

$$e_L(t) = \int_0^1 G_n(t,s) f^{(2n)}(s) ds.$$

The Lidstone polynomial $\Lambda_n(t)$ can be expressed as [1], [3]

$$\Lambda_n(t) = \int_0^1 G_n(t,s) s\, ds. \tag{1.5}$$

Identity (1.1) can be adapted for $f \in C^{(2n)}([a,b])$ by using the linear transform $t \mapsto \frac{t-a}{b-a}$, so we have

$$\begin{aligned} f(t) = & \sum_{k=0}^{n-1} (b-a)^{2k} \left[f^{(2k)}(a)\Lambda_k\left(\frac{b-t}{b-a}\right) + f^{(2k)}(b)\Lambda_k\left(\frac{t-a}{b-a}\right) \right] \\ & + (b-a)^{2n-1} \int_a^b G_n\left(\frac{t-a}{b-a}, \frac{s-a}{b-a}\right) f^{(2n)}(s) ds. \end{aligned} \tag{1.6}$$

We will also need the *weighted Montgomery identity*, obtained by J. Pečarić in [7]

$$f(x) = \int_a^b w(t) f(t)\, dt + \int_a^b P_w(x,t) f'(t)\, dt, \tag{1.7}$$

where $w : [a,b] \to [0,\infty\rangle$ is some normalized weight function i.e. integrable function satisfying $\int_a^b w(t)\, dt = 1, W : \mathbb{R} \to [0,1]$

$$W(t) = \begin{cases} 0, & t < a, \\ \int_a^t w(x) dx, & t \in [a,b], \\ 1, & t > b, \end{cases}$$

and $P_w(x,t)$ is the weighted Peano kernel

$$P_w(x,t) = \begin{cases} W(t), & a \leqslant t \leqslant x, \\ W(t) - 1, & x < t \leqslant b. \end{cases} \tag{1.8}$$

Here and hereafter for $p \geqslant 1$ we denote

$$\|f\|_p = \left(\int_a^b |f(t)|^p\, dt \right)^{\frac{1}{p}}$$

and

$$\|f\|_\infty = ess \sup_{t \in [a,b]} |f(t)|.$$

REMARK 1. From (1.7) we have

$$\left| f(x) - \int_a^b w(t) f(t)\, dt \right| \leqslant \|f'\|_\infty \int_a^b |P_w(x,t)|\, dt$$

and then by applying uniform weight function $w(t) = \frac{1}{b-a}$, $t \in [a,b]$

$$\int_a^b |P_w(x,t)|\, dt = \frac{1}{b-a}\left(\int_a^x (t-a)\, dt + \int_x^b (b-t)\, dt \right)$$
$$= \frac{(x-a)^2 + (x-b)^2}{2(b-a)} = \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right](b-a)$$

we obtain the well known *Ostrowski inequality* proved by Ostrowski [6] in 1938:

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t) dt \right| \leqslant \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right](b-a)\|f'\|_\infty . \tag{1.9}$$

It holds for every $x \in [a,b]$ whenever $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on $\langle a,b \rangle$ with bounded derivative.

In next sections we present weighted generalization of the Montgomery identity by using Lidstone interpolating polynomial. Then, in a similar way as in the previous remark, we derive Ostrowski type inequality for differentiable functions of class C^n. Finally, we obtain special cases $n = 2$ and $n = 3$ for uniform weighted function, as well as for $w(t) = \frac{3}{2}\sqrt{t}$, $t \in [0,1]$ and $w(t) = \frac{1}{2\sqrt{t}}$, $t \in \langle 0,1]$. For some other applications of Montgomery type identities and Ostrowski type inequalities we refer interested reader to [2], [5].

2. Generalization of weighted Montgomery identity via Lidstone polynomials

THEOREM 1. *Let $w : [a,b] \to [0,\infty\rangle$ be a normalized weight function and $f \in C^{(2n+1)}([a,b])$. Then the following identity holds*

$$f(x) = \int_a^b w(t) f(t)\, dt + \sum_{k=0}^{n-1} (b-a)^{2k} \left[f^{(2k+1)}(a) \int_a^b P_w(x,t) \Lambda_k\left(\frac{b-t}{b-a}\right) dt \right. \tag{2.1}$$
$$\left. + f^{(2k+1)}(b) \int_a^b P_w(x,t) \Lambda_k\left(\frac{t-a}{b-a}\right) dt \right]$$
$$+ (b-a)^{2n-1} \int_a^b \left(\int_a^b P_w(x,t) G_n\left(\frac{t-a}{b-a}, \frac{s-a}{b-a}\right) dt \right) f^{(2n+1)}(s) ds.$$

Proof. By applying (1.6) with $f' \in C^{(2n)}([a,b])$ instead of f, we get

$$f'(t) = \sum_{k=0}^{n-1} (b-a)^{2k} \left[f^{(2k+1)}(a) \Lambda_k\left(\frac{b-t}{b-a}\right) + f^{(2k+1)}(b) \Lambda_k\left(\frac{t-a}{b-a}\right) \right]$$
$$+ (b-a)^{2n-1} \int_a^b G_n\left(\frac{t-a}{b-a}, \frac{s-a}{b-a}\right) f^{(2n+1)}(s) ds. \tag{2.2}$$

Taking (2.2) in the weighted Montgomery identity (1.7), we obtain (2.1). □

THEOREM 2. *Let* $w:[a,b]\to[0,\infty\rangle$ *be a normalized weight function and* $f\in C^{(2n)}([a,b])$. *Then the following identity holds*

$$f(x)=\int_a^b w(t)f(t)\,dt+\sum_{k=0}^{n-1}(b-a)^{2k-1}\left[-f^{(2k)}(a)\int_a^b P_w(x,t)\Lambda_k'\left(\frac{b-t}{b-a}\right)dt\right.$$
$$\left.+f^{(2k)}(b)\int_a^b P_w(x,t)\Lambda_k'\left(\frac{t-a}{b-a}\right)dt\right]$$
$$+(b-a)^{2n-2}\int_a^b\left(\int_a^b P_w(x,t)\frac{\partial}{\partial t}G_n\left(\frac{t-a}{b-a},\frac{s-a}{b-a}\right)dt\right)f^{(2n)}(s)ds. \tag{2.3}$$

Proof. By multiplying (1.6) with $w(t)$ and integrating with respect to t from a to b we obtain

$$\int_a^b w(t)f(t)dt=\sum_{k=0}^{n-1}(b-a)^{2k}\left[f^{(2k)}(a)\int_a^b w(t)\Lambda_k\left(\frac{b-t}{b-a}\right)dt\right.$$
$$\left.+f^{(2k)}(b)\int_a^b w(t)\Lambda_k\left(\frac{t-a}{b-a}\right)dt\right]$$
$$+(b-a)^{2n-1}\int_a^b\left(\int_a^b w(t)G_n\left(\frac{t-a}{b-a},\frac{s-a}{b-a}\right)dt\right)f^{(2n)}(s)ds. \tag{2.4}$$

If we subtract (2.4) from the identity (1.6) stated for variable x instead of t we get

$$f(x)=\int_a^b w(t)f(t)\,dt+\sum_{k=0}^{n-1}(b-a)^{2k}\left[f^{(2k)}(a)\left(\Lambda_k\left(\frac{b-x}{b-a}\right)-\int_a^b w(t)\Lambda_k\left(\frac{b-t}{b-a}\right)dt\right)\right.$$
$$\left.+f^{(2k)}(b)\left(\Lambda_k\left(\frac{x-a}{b-a}\right)-\int_a^b w(t)\Lambda_k\left(\frac{t-a}{b-a}\right)dt\right)\right] \tag{2.5}$$
$$+(b-a)^{2n-2}\int_a^b\left(G_n\left(\frac{x-a}{b-a},\frac{s-a}{b-a}\right)-\int_a^b w(t)G_n\left(\frac{t-a}{b-a},\frac{s-a}{b-a}\right)dt\right)f^{(2n)}(s)ds.$$

By using the weighted Montgomery identity (1.7) for functions $x\mapsto\Lambda_k\left(\frac{b-x}{b-a}\right)$, $x\mapsto\Lambda_k\left(\frac{x-a}{b-a}\right)$ and $x\mapsto G_n\left(\frac{x-a}{b-a},\frac{s-a}{b-a},\right)$ we get respectively

$$\Lambda_k\left(\frac{b-x}{b-a}\right)-\int_a^b w(t)\Lambda_k\left(\frac{b-t}{b-a}\right)dt=\frac{-1}{b-a}\int_a^b P_w(x,t)\Lambda_k'\left(\frac{b-t}{b-a}\right)dt, \tag{2.6}$$

$$\Lambda_k\left(\frac{x-a}{b-a}\right)-\int_a^b w(t)\Lambda_k\left(\frac{t-a}{b-a}\right)dt=\frac{1}{b-a}\int_a^b P_w(x,t)\Lambda_k'\left(\frac{t-a}{b-a}\right)dt, \tag{2.7}$$

and

$$G_n\left(\frac{x-a}{b-a},\frac{s-a}{b-a}\right)-\int_a^b w(t)G_n\left(\frac{t-a}{b-a},\frac{s-a}{b-a}\right)dt$$
$$=\frac{1}{b-a}\int_a^b P_w(x,t)\frac{\partial}{\partial t}G_n\left(\frac{t-a}{b-a},\frac{s-a}{b-a}\right)dt. \tag{2.8}$$

Applying (2.5) with (2.6), (2.7) and (2.8) we obtain (2.3). □

3. Weighted Ostrowski type inequalities

THEOREM 3. *Let $w : [a,b] \to [0,\infty\rangle$ be a normalized weight function, $f \in C^{(2n+1)}([a,b])$ and $p,q \in \mathbb{R}$, $1 \leqslant p,q \leqslant \infty$, $\frac{1}{p}+\frac{1}{q}=1$. Then the following inequality holds*

$$\left| f(x) - \int_a^b w(t)f(t)\,dt - \sum_{k=0}^{n-1}(b-a)^{2k}\left[f^{(2k+1)}(a)\int_a^b P_w(x,t)\Lambda_k\left(\frac{b-t}{b-a}\right)dt \right.\right.$$
$$\left.\left. + f^{(2k+1)}(b)\int_a^b P_w(x,t)\Lambda_k\left(\frac{t-a}{b-a}\right)dt\right]\right| \leqslant \|K_1(\cdot,x)\|_q\|f^{(2n+1)}\|_p, \quad (3.1)$$

where

$$K_1(s,x) = (b-a)^{2n-1}\int_a^b P_w(x,t)G_n\left(\frac{t-a}{b-a},\frac{s-a}{b-a}\right)dt.$$

Proof. Apply Hölder inequality to (2.1). □

THEOREM 4. *Let $w : [a,b] \to [0,\infty\rangle$ be a normalized weight function, $f \in C^{(2n)}([a,b])$ and $p,q \in \mathbb{R}$, $1 \leqslant p,q \leqslant \infty$, $\frac{1}{p}+\frac{1}{q}=1$. Then the following inequality holds*

$$\left| f(x) - \int_a^b w(t)f(t)\,dt - \sum_{k=0}^{n-1}(b-a)^{2k-1}\left[-f^{(2k)}(a)\int_a^b P_w(x,t)\Lambda_k'\left(\frac{b-t}{b-a}\right)dt \right.\right.$$
$$\left.\left. + f^{(2k)}(b)\int_a^b P_w(x,t)\Lambda_k'\left(\frac{t-a}{b-a}\right)dt\right]\right| \leqslant \|K_2(\cdot,x)\|_q\|f^{(2n)}\|_p, \quad (3.2)$$

where

$$K_2(s,x) = (b-a)^{2n-2}\int_a^b P_w(x,t)\frac{\partial}{\partial t}G_n\left(\frac{t-a}{b-a},\frac{s-a}{b-a}\right)dt.$$

Proof. Apply Hölder inequality to (2.3). □

3.1. Case $f \in C^2([a,b])$

THEOREM 5. *Let $w : [a,b] \to [0,\infty\rangle$ be a normalized weight function, $f \in C^2([a,b])$ and $p,q \in \mathbb{R}$, $1 \leqslant p,q \leqslant \infty$, $\frac{1}{p}+\frac{1}{q}=1$. Then the following inequality holds*

$$\left| f(x) - \int_a^b w(t)f(t)\,dt + \frac{f(a)-f(b)}{b-a}\left(\int_a^b P_w(x,t)dt\right)\right| \leqslant \|A_w(\cdot,x)\|_q\|f''\|_p$$

where

$$A_w(s,x) = \int_a^b P_w(x,t)\frac{\partial}{\partial t}G_1\left(\frac{t-a}{b-a},\frac{s-a}{b-a}\right)dt.$$

Proof. We apply (3.2) with $n=1$ and $\Lambda_0(t)=t$. □

COROLLARY 1. *Let $f \in C^2([a,b])$ and $p,q \in \mathbb{R}$, $1 \leqslant p,q \leqslant \infty$, $\frac{1}{p}+\frac{1}{q}=1$. Then the following inequality holds*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt + \frac{f(a)-f(b)}{b-a}\left(x-\frac{a+b}{2}\right)\right| \leqslant \|A(\cdot,x)\|_q \|f''\|_p$$

where

$$A(s,x) = \frac{1}{2(b-a)^2}\cdot \begin{cases} (s-a)(2x-b-s), & \text{if } s \leqslant x, \\ (s-b)(2x-a-s), & \text{if } x \leqslant s. \end{cases}$$

Proof. We apply Theorem 5 with uniform weight function $w(t)=\frac{1}{b-a}$, $t \in [a,b]$

$$\int_a^b P_w(x,t)dt = \int_a^x \frac{t-a}{b-a}dt + \int_x^b \frac{t-b}{b-a}dt = x - \frac{a+b}{2}, \tag{3.3}$$

$$A_w(s,x) = \int_a^x \frac{t-a}{b-a}\frac{\partial}{\partial t}G_1\left(\frac{t-a}{b-a},\frac{s-a}{b-a}\right)dt + \int_x^b \frac{t-b}{b-a}\frac{\partial}{\partial t}G_1\left(\frac{t-a}{b-a},\frac{s-a}{b-a}\right)dt$$

and

$$G_1\left(\frac{t-a}{b-a},\frac{s-a}{b-a}\right) = \frac{1}{(b-a)^2}\begin{cases} (t-b)(s-a), & \text{if } s \leqslant t, \\ (s-b)(t-a), & \text{if } t \leqslant s. \end{cases},$$

$$\frac{\partial}{\partial t}G_1\left(\frac{t-a}{b-a},\frac{s-a}{b-a}\right) = \frac{1}{b-a}\cdot\begin{cases} s-a, & \text{if } s \leqslant t, \\ s-b, & \text{if } t \leqslant s. \end{cases}$$

If $s \leqslant x$

$$\begin{aligned} A_w(s,x) &= \frac{1}{(b-a)^2}\left[\int_a^s (t-a)(s-b)\,dt + \int_s^x (t-a)(s-a)\,dt + \int_x^b (t-b)(s-a)\,dt\right] \\ &= \frac{(s-a)(2x-b-s)}{2(b-a)}. \end{aligned}$$

If $x \leqslant s$

$$\begin{aligned} A_w(s,x) &= \frac{1}{(b-a)^2}\left[\int_a^x (t-a)(s-b)\,dt + \int_x^s (t-b)(s-b)\,dt + \int_s^b (t-b)(s-a)\,dt\right] \\ &= \frac{(s-b)(2x-a-s)}{2(b-a)}. \end{aligned}$$

Thus, the proof is completed. □

COROLLARY 2. *Let $f \in C^2([0,1])$ and $p,q \in \mathbb{R}$, $1 \leqslant p,q \leqslant \infty$, $\frac{1}{p}+\frac{1}{q}=1$. Then the following inequality holds*

$$\left| f(x) - \frac{3}{2}\int_0^1 \sqrt{t} f(t)\,dt + (f(0)-f(1))\left(x-\frac{3}{5}\right)\right| \leqslant \|A_w(\cdot,x)\|_q \|f''\|_p$$

where

$$A_w(s,x) = \begin{cases} s\left(x-\frac{3}{5}\right) - \frac{2}{5}\sqrt{s^5}, & \text{if } s \leqslant x, \\ s\left(x+\frac{2}{5}\right) - x - \frac{2}{5}\sqrt{s^5}, & \text{if } x \leqslant s. \end{cases}$$

Proof. We apply Theorem 5 with normalized weight function $w(t) = \frac{3}{2}\sqrt{t}$, $t \in [0,1]$

$$\int_a^b P_w(x,t)dt = \int_0^x \sqrt{t^3}dt + \int_x^1 \left(\sqrt{t^3}-1\right)dt = x - \frac{3}{5},$$

$$A_w(s,x) = \int_0^x \sqrt{t^3}\frac{\partial}{\partial t}G_1(t,s)\,dt + \int_x^1 \left(\sqrt{t^3}-1\right)\frac{\partial}{\partial t}G_1(t,s)\,dt$$

and

$$G_1(t,s) = \begin{cases} (t-1)s, & \text{if } s \leqslant t, \\ (s-1)t, & \text{if } t \leqslant s. \end{cases}$$

$$\frac{\partial}{\partial t}G_1(t,s) = \begin{cases} s, & \text{if } s \leqslant t, \\ s-1, & \text{if } t \leqslant s. \end{cases}$$

If $s \leqslant x$

$$\begin{aligned} A_w(s,x) &= \int_0^s \sqrt{t^3}(s-1)dt + \int_s^x \sqrt{t^3}s\,dt + \int_x^1 \left(\sqrt{t^3}-1\right)s\,dt \\ &= s\left(x-\frac{3}{5}\right) - \frac{2}{5}\sqrt{s^5}. \end{aligned}$$

If $x \leqslant s$

$$\begin{aligned} A_w(s,x) &= \int_0^x \sqrt{t^3}(s-1)dt + \int_x^s \left(\sqrt{t^3}-1\right)(s-1)dt + \int_s^1 \left(\sqrt{t^3}-1\right)s\,dt \\ &= s\left(x+\frac{2}{5}\right) - x - \frac{2}{5}\sqrt{s^5}. \end{aligned}$$

Thus, the proof is completed. □

COROLLARY 3. *Let* $f \in C^2([0,1])$ *and* $p,q \in \mathbb{R}$, $1 \leqslant p,q \leqslant \infty$, $\frac{1}{p}+\frac{1}{q}=1$. *Then the following inequality holds*

$$\left| f(x) - \frac{1}{2}\int_0^1 \frac{f(t)}{\sqrt{t}}dt + (f(0)-f(1))\left(x-\frac{1}{3}\right)\right| \leqslant \|A_w(\cdot,x)\|_q \|f''\|_p$$

where

$$A_w(s,x) = \begin{cases} s\left(x-\frac{1}{3}\right) - \frac{2}{3}\sqrt{s^3}, & \text{if } s \leqslant x, \\ s\left(x+\frac{2}{3}\right) - x - \frac{2}{3}\sqrt{s^3}, & \text{if } x \leqslant s. \end{cases}$$

Proof. We apply Theorem 5 with normalized weight function $w(t) = \frac{1}{2\sqrt{t}}$, $t \in \langle 0,1]$

$$\int_a^b P_w(x,t)dt = \int_0^x \sqrt{t}dt + \int_x^1 \left(\sqrt{t}-1\right)dt = x - \frac{1}{3}, \tag{3.4}$$

$$A_w(s,x) = \int_0^x \sqrt{t}\frac{\partial}{\partial t}G_1(t,s)\,dt + \int_x^1 \left(\sqrt{t}-1\right)\frac{\partial}{\partial t}G_1(t,s)\,dt$$

and again

$$\frac{\partial}{\partial t}G_1(t,s) = \begin{cases} s, & \text{if } s \leqslant t, \\ s-1, & \text{if } t \leqslant s. \end{cases}$$

If $s \leqslant x$

$$\begin{aligned} A_w(s,x) &= \int_0^s \sqrt{t}(s-1)dt + \int_s^x \sqrt{t}sdt + \int_x^1 \left(\sqrt{t}-1\right)sdt \\ &= s\left(x-\frac{1}{3}\right) - \frac{2}{3}\sqrt{s^3}. \end{aligned}$$

If $x \leqslant s$

$$\begin{aligned} A_w(s,x) &= \int_0^x \sqrt{t}(s-1)dt + \int_x^s \left(\sqrt{t}-1\right)(s-1)dt + \int_s^1 \left(\sqrt{t}-1\right)sdt \\ &= s\left(x+\frac{2}{3}\right) - x - \frac{2}{3}\sqrt{s^3}. \end{aligned}$$

Thus, the proof is completed. □

3.2. Case $f \in C^3([a,b])$

THEOREM 6. *Let $w:[a,b] \to [0,\infty\rangle$ be a normalized weight function, $f \in C^{(3)}([a,b])$ and $p,q \in \mathbb{R}$, $1 \leqslant p,q \leqslant \infty$, $\frac{1}{p}+\frac{1}{q}=1$. Then the following inequality holds*

$$\left| f(x) - \int_a^b w(t)f(t)\,dt - \frac{1}{b-a}\left[f'(a)\int_a^b (b-t)P_w(x,t)dt \right.\right.$$
$$\left.\left. + f'(b)\int_a^b (t-a)P_w(x,t)dt\right]\right| \leqslant \|B_w(\cdot,x)\|_q \|f'''\|_p$$

where

$$B_w(s,x) = (b-a)\int_a^b P_w(x,t)G_1\left(\frac{t-a}{b-a},\frac{s-a}{b-a}\right)dt.$$

Proof. We apply (3.1) with $n=1$ and $\Lambda_0(t) = t$. □

COROLLARY 4. *Let $f \in C^{(3)}([a,b])$ and $p,q \in \mathbb{R}$, $1 \leqslant p,q \leqslant \infty$, $\frac{1}{p}+\frac{1}{q}=1$. Then the following inequality holds*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt - f'(a)\left(\frac{b-a}{6} - \frac{(b-x)^2}{2(b-a)}\right) - f'(b)\left(\frac{b-a}{6} - \frac{(x-a)^2}{2(b-a)}\right)\right| \leqslant \|B(\cdot,x)\|_q \|f'''\|_p$$

where

$$B(s,x) = \begin{cases} \frac{(s-a)}{(b-a)^2}\left[\frac{(s-b)(s-a)^2}{3} + \left[\frac{x^3-s^3}{3} - (a+b)\frac{x^2-s^2}{2} + ab(x-s)\right] - \frac{(x-b)^3}{3}\right], & \text{if } s \leqslant x, \\ \frac{(s-b)}{(b-a)^2}\left[\frac{(x-a)^3}{3} + \left[\frac{s^3-x^3}{3} - (a+b)\frac{s^2-x^2}{2} + ab(s-x)\right] - \frac{(s-a)(s-b)^2}{3}\right], & \text{if } s \geqslant x. \end{cases}$$

Proof. We apply Theorem 6 with uniform weight function $w(t) = \frac{1}{b-a}$, $t \in [a,b]$

$$\int_a^b (b-t)P_w(x,t)dt = \frac{1}{b-a}\left(\int_a^x (t-a)(b-t)\,dt + \int_x^b (t-b)(b-t)\,dt\right) = \frac{1}{6}(b-a)^2 - \frac{1}{2}(b-x)^2,$$

$$\int_a^b (t-a)P_w(x,t)dt = \frac{1}{b-a}\left(\int_a^x (t-a)^2\,dt + \int_x^b (t-a)(t-b)\,dt\right) = \frac{1}{6}(b-a)^2 - \frac{1}{2}(x-a)^2,$$

$$B_w(s,x) = (b-a)\left[\int_a^x \frac{t-a}{b-a}G_1\left(\frac{t-a}{b-a},\frac{s-a}{b-a}\right)dt + \int_x^b \frac{t-b}{b-a}G_1\left(\frac{t-a}{b-a},\frac{s-a}{b-a}\right)dt\right].$$

If $s \leqslant x$

$$\begin{aligned} B_w(s,x) &= \int_a^s (t-a)\left(\frac{(s-b)(t-a)}{(b-a)^2}\right)dt \\ &\quad + \int_s^x (t-a)\left(\frac{(t-b)(s-a)}{(b-a)^2}\right)dt + \int_x^b (t-b)\left(\frac{(t-b)(s-a)}{(b-a)^2}\right)dt \\ &= \frac{(s-a)}{(b-a)^2}\left[(s-b)\frac{(s-a)^2}{3} + \left[\frac{x^3-s^3}{3} - (a+b)\frac{x^2-s^2}{2} + ab(x-s)\right] - \frac{(x-b)^3}{3}\right]. \end{aligned}$$

If $x \leqslant s$

$$\begin{aligned} B_w(s,x) &= \int_a^x (t-a)\left(\frac{(s-b)(t-a)}{(b-a)^2}\right)dt \\ &\quad + \int_x^s (t-b)\left(\frac{(s-b)(t-a)}{(b-a)^2}\right)dt + \int_s^b (t-b)\left(\frac{(t-b)(s-a)}{(b-a)^2}\right)dt \\ &= \frac{(s-b)}{(b-a)^2}\left[\frac{(x-a)^3}{3} + \left[\frac{s^3-x^3}{3} - (a+b)\frac{s^2-x^2}{2} + ab(s-x)\right] - (s-a)\frac{(s-b)^2}{3}\right]. \end{aligned}$$

Thus, the proof is completed. □

COROLLARY 5. *Let $f \in C^{(3)}([0,1])$ and $p,q \in \mathbb{R}$, $1 \leqslant p,q \leqslant \infty$, $\frac{1}{p}+\frac{1}{q}=1$. Then the following inequality holds*

$$\left| f(x) - \frac{3}{2}\int_0^1 \sqrt{t} f(t)\,dt - \left[f'(0)\left(-\frac{x^2}{2}+x-\frac{27}{70}\right) + f'(1)\left(\frac{x^2}{2}-\frac{3}{14}\right)\right]\right| \leqslant \|B_w(\cdot,x)\|_q \|f'''\|_p$$

where

$$B_w(s,x) = \begin{cases} \frac{27}{70}s + \frac{4}{35}\sqrt{s^7} + s\left(\frac{x^2}{2}-x\right), & \text{if } s \leqslant x, \\ (s-1)\frac{x^2}{2} + \frac{4}{35}\sqrt{s^7} + \frac{27}{70}s - \frac{s^2}{2}, & \text{if } s \geqslant x. \end{cases}.$$

Proof. We apply Theorem 6 with normalized weight function $w(t) = \frac{3}{2}\sqrt{t}$, $t \in [0,1]$. Thus $W(t) = \sqrt{t^3}$, $t \in [0,1]$

$$\begin{aligned} \int_a^b (b-t) P_w(x,t)dt &= \left(\int_0^x (1-t)\sqrt{t^3}dt + \int_x^1 (1-t)\left(\sqrt{t^3}-1\right)dt \right) \\ &= -\frac{x^2}{2} + x - \frac{27}{70}, \end{aligned}$$

$$\begin{aligned} \int_a^b (t-a) P_w(x,t)dt &= \left(\int_0^x t\sqrt{t^3}dt + \int_x^1 t\left(\sqrt{t^3}-1\right)dt \right) \\ &= \frac{x^2}{2} - \frac{3}{14}, \end{aligned}$$

$$B_w(s,x) = \left[\int_0^x \sqrt{t^3} G_1(t,s)\,dt + \int_x^1 \left(\sqrt{t^3}-1\right) G_1(t,s)\,dt \right].$$

If $s \leqslant x$

$$\begin{aligned} B_w(s,x) &= \int_0^s \sqrt{t^3}(s-1)t\,dt + \int_s^x \sqrt{t^3}(t-1)s\,dt + \int_x^1 \left(\sqrt{t^3}-1\right)(t-1)s\,dt \\ &= \frac{27}{70}s + \frac{4}{35}\sqrt{s^7} + s\left(\frac{x^2}{2}-x\right). \end{aligned}$$

If $x \leqslant s$

$$\begin{aligned} B_w(s,x) &= \int_0^x \sqrt{t^3}(s-1)t\,dt + \int_x^s \left(\sqrt{t^3}-1\right)(s-1)t\,dt + \int_s^1 \left(\sqrt{t^3}-1\right)(t-1)s\,dt \\ &= (s-1)\frac{x^2}{2} + \frac{4}{35}\sqrt{s^7} + \frac{27}{70}s - \frac{s^2}{2}. \end{aligned}$$

Thus, the proof is completed. □

COROLLARY 6. *Let $f \in C^{(3)}([a,b])$ and $p,q \in \mathbb{R}$, $1 \leqslant p,q \leqslant \infty$, $\frac{1}{p}+\frac{1}{q}=1$. Then the following inequality holds*

$$\left| f(x) - \frac{1}{2}\int_0^1 \frac{f(t)}{\sqrt{t}}dt - \left[f'(0)\left(-\frac{x^2}{2}+x-\frac{7}{30}\right) + f'(1)\left(\frac{x^2}{2}-\frac{1}{10}\right)\right]\right| \leqslant \|B_w(\cdot,x)\|_q \|f'''\|_p$$

where

$$B_w(s,x) = \begin{cases} \frac{7}{30}s + \frac{4}{15}\sqrt{s^5} + s\left(\frac{x^2}{2}-x\right), & \text{if } s \leqslant x, \\ (s-1)\frac{x^2}{2} + \frac{4}{15}\sqrt{s^5} + \frac{7}{30}s - \frac{s^2}{2}, & \text{if } s \geqslant x. \end{cases}.$$

Proof. We apply Theorem 6 with normalized weight function $w(t) = \frac{1}{2\sqrt{t}}$, $t \in \langle 0,1]$. Thus $W(t) = \sqrt{t}$, $t \in [0,1]$

$$\int_a^b (b-t)P_w(x,t)dt = \left(\int_0^x (1-t)\sqrt{t}dt + \int_x^1 (1-t)\left(\sqrt{t}-1\right)dt\right) = -\frac{x^2}{2}+x-\frac{7}{30},$$

$$\int_a^b (t-a)P_w(x,t)dt = \left(\int_0^x t\sqrt{t}dt + \int_x^1 t\left(\sqrt{t}-1\right)dt\right) = \frac{x^2}{2}-\frac{1}{10},$$

$$B_w(s,x) = \left[\int_0^x \sqrt{t}G_1(t,s)\,dt + \int_x^1 \left(\sqrt{t}-1\right)G_1(t,s)\,dt\right].$$

If $s \leqslant x$

$$B_w(s,x) = \int_0^s \sqrt{t}(s-1)t\,dt + \int_s^x \sqrt{t}(t-1)s\,dt + \int_x^1 \left(\sqrt{t}-1\right)(t-1)s\,dt = \frac{7}{30}s + \frac{4}{15}\sqrt{s^5} + s\left(\frac{x^2}{2}-x\right).$$

If $x \leqslant s$

$$B_w(s,x) = \int_0^x \sqrt{t}(s-1)t\,dt + \int_x^s \left(\sqrt{t}-1\right)(s-1)t\,dt + \int_s^1 \left(\sqrt{t}-1\right)(t-1)s\,dt = (s-1)\frac{x^2}{2} + \frac{4}{15}\sqrt{s^5} + \frac{7}{30}s - \frac{s^2}{2}.$$

Thus, the proof is completed. □

Acknowledgement. The publication was supported by the Ministry of Education and Science of the Russian Federation (the Agreement number No. 02.a03.21.0008.)

REFERENCES

[1] R. P. AGARWAL, P. J. Y. WONG, *Error Inequalities in Polynomial Interpolation and Their Applications*, Kluwer Academic Publishers, Dordrecht/Boston/London, 1993.
[2] A. AGLIĆ ALJINOVIĆ, A. ČIVLJAK, S. KOVAČ, J. PEČARIĆ, M. RIBIČIĆ PENAVA, *General Integral Identities and Related inequalities*, Element, Zagreb, 2013.
[3] G. ARAS-GAZIĆ, V. ČULJAK, J. PEČARIĆ, A. VUKELIĆ, *Generalization of Jensen's inequality by Lidstone's polynomial and related results*, Math. Inequal. Appl. **16** (2013), 1243–1267.
[4] G. J. LIDSTONE, *Notes on the extensions of Aitken's teorem (for polynomial interpolation) to the Everett types*, Proc. Edinburgh Math. Soc. **2**, 2 (1929), 16–19.
[5] D. S. MITRINOVIĆ, J. E. PEČARIĆ, AND A. M. FINK, *Inequalities for functions and their Integrals and Derivatives*, Kluwer Academic Publishers, Dordrecht, 1994.
[6] A. OSTROWSKI, *Über die Absolutabweichung einer differentiebaren Funktion von ihrem Integralmittelwert*, Comment. Math. Helv. **10** (1938), 226–227.
[7] J. PEČARIĆ, *On the Čebyšev inequality*, Bul. Inst. Politehn. Timisoara **25** (39) (1980), 10–11.
[8] D. V. WIDDER, *Completly convex function and Lidstone series*, Trans. Am. Math. Soc. **51** (1942), 387–398.

(Received September 30, 2018)

Andrea Aglić Aljinović
University of Zagreb
Faculty of Electrical Engineering and Computing
Unska 3, 10 000 Zagreb, Croatia
e-mail: andrea.aglic@fer.hr

Ljiljanka Kvesić
University of Mostar
Faculty of Science and Education
Matice hrvatske bb, 88 000 Mostar, Bosnia and Herzegovina
e-mail: ljkvesic@gmail.com

Josip Pečarić
RUDN University
Miklukho-Maklaya str. 6, 117198 Moscow, Russia
e-mail: pecaric@hazu.hr