Mathematical
Inequalities
& Applications
Volume 22, Number 4 (2019), 1283–1301 doi:10.7153/mia-2019-22-88

NEW BOUNDS FOR SHANNON, RELATIVE AND MANDELBROT ENTROPIES VIA ABEL–GONTSCHAROFF INTERPOLATING POLYNOMIAL

SAAD IHSAN BUTT *, NASIR MEHMOOD, ĐILDA PEČARIĆ AND JOSIP PEČARIĆ

(*Communicated by J. Jakšetić*)

Abstract. The Jensen's inequality has tremendous implications in many fields of modern analysis. It helps computing useful upper bounds for several entropic measures used in information theory. We use discrete and continuous cyclic refinements of Jensen's inequality and extend them from convex to higher order convex function by using new Green functions and Abel-Gontscharoff interpolating polynomial. As an application of our work, we establish connection among new entropic bounds for Shanon, Relative and Mandelbrot entropies.

1. Introduction

Information theory is a mathematical representation of the conditions and parameters which govern the transmission and processing of information. It is an evolving discipline getting huge attention from experimentalists and theorists of various disciplines like computer science, physics, pattern recognition, ecology, accounting, fuzzy set theory etc.

Jensen's inequality for differentiable convex functions has significant applications in information theory. It is used to obtain upper bounds for several quantitative measures arising from information theory for continuous random variable. It also helps computing several useful bounds for joint entropy, conditional entropy and mutual information. It provides different counterpart inequalities of Shanon entropy which is one of the major tools used in information theory and hence helps solving many problems in accounting, economics, psychology, statistics, ecology, computer science etc.

We give some fundamental results regarding Abel-Gontscharoff interpolating polynomial.

Let $-\infty < \alpha_1 < \alpha_2 < \infty$ and let $\alpha_1 \leqslant \xi_1 < \xi_2 < \cdots \xi_n \leqslant \alpha_2$ be the given points. For $\phi \in C^n[\alpha_1, \alpha_2]$, Abel-Gontscharoff interpolating polynomial AP of degree $(n-1)$ satisfying Abel-Gontscharoff conditions

$$AP^{(\sigma)}(\xi_{\sigma+1}) = \phi^{(\sigma)}(\xi_{\sigma+1}), \quad 0 \leqslant \sigma \leqslant n-1$$

Mathematics subject classification (2010): 26A51, 26D15, 26E60, 94A17, 94A15.
Keywords and phrases: $n-$convex function, Abel-Gontscharoff interpolating polynomial, new Green functions, Shannon entropy, relative entropy, Zipf-Mandelbrot entropy.

* Corresponding author.

exists uniquely [8, 9]. This condition in particular includes two point right focal conditions.

$$AP_{(2)}^{(\sigma)}(\xi_1) = \phi^{(\sigma)}(\xi_1), \quad 0 \leqslant \sigma \leqslant t,$$
$$AP_{(2)}^{(\sigma)}(\xi_2) = \phi^{(\sigma)}(\xi_2), \quad t+1 \leqslant \sigma \leqslant n-1, \quad \alpha_1 \leqslant \xi_1 < \xi_2 \leqslant \alpha_2.$$

First we give representation of Abel-Gontscharoff interpolating polynomial:

THEOREM 1. [1] *Abel-Gontscharoff interpolating polynomial AP of function ϕ can be expressed as*

$$AP(z) = \sum_{\sigma=0}^{n-1} \Lambda_\sigma(z)\phi^{(\sigma)}(\xi_{\sigma+1}), \tag{1}$$

where $\Lambda_0(z) = 1$ and Λ_σ, $1 \leqslant \sigma \leqslant n-1$ is the unique polynomial of degree σ satisfying

$$\Lambda_\sigma^{(l)}(\xi_{l+1}) = 0, \quad 0 \leqslant l \leqslant \sigma - 1,$$
$$\Lambda_\sigma^{(\sigma)}(\xi_{\sigma+1}) = 1$$

and it can be written as

$$\Lambda_\sigma(z) = \frac{1}{1!2!\cdots\sigma!}\begin{vmatrix} 1 & \xi_1 & \xi_1^2 & \ldots & \xi_1^{\sigma-1} & \xi_1^\sigma \\ 0 & 1 & 2\xi_2 & \ldots & (\sigma-1)\xi_2^{\sigma-2} & \sigma\xi_2^{\sigma-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \\ 0 & 0 & 0 & \ldots & (\sigma-1)! & \sigma!\xi_\sigma \\ 1 & z & z^2 & \ldots & z^{\sigma-1} & z^\sigma \end{vmatrix}$$
$$= \int_{\xi_1}^{z}\int_{\xi_2}^{z_1}\int_{\xi_3}^{z_2}\cdots\int_{\xi_\sigma}^{z_{\sigma-1}} dz_\sigma dz_{\sigma-1}\cdots dz_1, \quad (z_0 = z). \tag{2}$$

In particular, we have

$$\Lambda_0(z) = 1,$$
$$\Lambda_1(z) = z - \xi_1,$$
$$\Lambda_2(z) = \frac{1}{2}[z^2 - 2\xi_2 z + \xi_1(2\xi_2 - \xi_1)].$$

COROLLARY 1. *The two point right focal interpolating polynomial $AP_{(2)}(z)$ of the function ϕ can be written as*

$$AP_{(2)}(z) = \sum_{\sigma=0}^{t}\frac{(z-\xi_1)^\sigma}{\sigma!}\phi^{(\sigma)}(\xi_1) + \sum_{w=0}^{n-t-2}\left[\sum_{\sigma=0}^{w}\frac{(z-\xi_1)^{t+1+\sigma}(\xi_1-\xi_2)^{w-\sigma}}{(t+1+\sigma)!(w-\sigma)!}\right]\phi^{(t+1+w)}(\xi_2). \tag{3}$$

The associate error $Error(z) = \phi(z) - AP(z)$ can be represented in terms of the Green function $AG(z,r;n)$ of the boundary value problem

$$y^{(n)} = 0,\ y^{(\sigma)}(\xi_{\sigma+1}) = 0,\ 0 \leqslant \sigma \leqslant n-1$$

and appears as (see [1]):

$$AG(z,r;n) = \begin{cases} \sum_{\sigma=0}^{l-1} \frac{\Lambda_\sigma(z)}{(n-\sigma-1)!}(\xi_{\sigma+1}-r)^{(n-\sigma-1)},\ \xi_l \leqslant r \leqslant z, \\ -\sum_{\sigma=l}^{n-1} \frac{\Lambda_\sigma(z)}{(n-\sigma-1)!}(\xi_{\sigma+1}-r)^{(n-\sigma-1)},\ z \leqslant r \leqslant \xi_{l+1}, \\ \qquad l = 0,1,\dots,n\ (\xi_0 = \alpha_1, \xi_{n+1} = \alpha_2). \end{cases} \tag{4}$$

Corresponding to the two point right focal conditions, Green function $AG_{(2)}(z,r;n)$ of the boundary value problem

$$y^{(n)} = 0,\ y^{(\sigma)}(\xi_1) = 0,\ 0 \leqslant \sigma \leqslant t, y^{(\sigma)}(\xi_2) = 0,\ t+1 \leqslant \sigma \leqslant n-1$$

is given by (see [1]):

$$AG_{(2)}(z,r;n) = \frac{1}{(n-1)!} \begin{cases} \sum_{\sigma=0}^{t} \binom{n-1}{\sigma} (z-\xi_1)^\sigma (\xi_1 - r)^{n-\sigma-1}, & \alpha_1 \leqslant r \leqslant z, \\ -\sum_{\sigma=t+1}^{n-1} \binom{n-1}{\sigma} (z-\xi_1)^\sigma (\xi_1 - r)^{n-\sigma-1}, & z \leqslant r \leqslant \alpha_2. \end{cases} \tag{5}$$

Further, for $\xi_1 \leqslant r,\quad z \leqslant \xi_2$ the following inequalities hold

$$(-1)^{n-t-1} \frac{\partial^\sigma AG_{(2)}(z,r;n)}{\partial z^\sigma} \geqslant 0, \quad 0 \leqslant \sigma \leqslant t, \tag{6}$$

$$(-1)^{n-\sigma} \frac{\partial^\sigma AG_{(2)}(z,r;n)}{\partial z^\sigma} \geqslant 0, \quad t+1 \leqslant \sigma \leqslant n-1. \tag{7}$$

THEOREM 2. *Let $\phi \in C^n[\alpha_1, \alpha_2]$, and let $AP(\cdot)$ be its Abel-Gontscharoff interpolating polynomial. Then*

$$\phi(z) = AP(z) + Error(z) = \sum_{\sigma=0}^{n-1} \Lambda_\sigma(z) \phi^{(\sigma)}(\xi_{\sigma+1}) + \int_{\alpha_1}^{\alpha_2} AG(z,r;n) \phi^{(n)}(r)dr, \tag{8}$$

where $\Lambda(\cdot)$ is defined by (2) and $AG(z,r;n)$ is defined by (4).

THEOREM 3. *Let $\phi \in C^n[\alpha_1,\alpha_2]$, and let $AP_{(2)}(\cdot)$ be its two points right focal Abel-Gontscharoff interpolating polynomial . Then*

$$\begin{aligned}\phi(z) =& AP_{(2)}(z)+Error(z)\\ =& \sum_{\sigma=0}^{t}\frac{(z-\xi_1)^\sigma}{\sigma!}\phi^{(\sigma)}(\xi_1)+\sum_{w=0}^{n-t-2}\left[\sum_{\sigma=0}^{w}\frac{(z-\xi_1)^{t+1+\sigma}(\xi_1-\xi_2)^{w-\sigma}}{(t+1+\sigma)!(w-\sigma)!}\right]\phi^{(t+1+w)}(\xi_2)\\ &+\int_{\alpha_1}^{\alpha_2}AG_{(2)}(z,r;n)\phi^{(n)}(r)dr,\end{aligned} \tag{9}$$

where $AG_{(2)}(z,r;n)$ is defined by (5).

For $j=1,\ldots,5$, consider the well known Lagrange Green function along with new Green functions $G_j:[\alpha_1,\alpha_2]\times[\alpha_1,\alpha_2]\to\mathbb{R}$ defined as

$$G_1(z,r)=\begin{cases}\frac{(\alpha_2-z)(\alpha_1-r)}{\alpha_2-\alpha_1}, & \alpha_1\leqslant r\leqslant z,\\ \frac{(\alpha_2-r)(\alpha_1-z)}{\alpha_2-\alpha_1}, & z\leqslant r\leqslant \alpha_2.\end{cases} \tag{10}$$

$$G_2(z,r)=\begin{cases}\alpha_1-r, & \alpha_1\leqslant r\leqslant z,\\ \alpha_1-z, & z\leqslant r\leqslant \alpha_2.\end{cases} \tag{11}$$

$$G_3(z,r)=\begin{cases}z-\alpha_2, & \alpha_1\leqslant r\leqslant z,\\ r-\alpha_2, & z\leqslant r\leqslant \alpha_2.\end{cases} \tag{12}$$

$$G_4(z,r)=\begin{cases}z-\alpha_1, & \alpha_1\leqslant r\leqslant z,\\ r-\alpha_1, & z\leqslant r\leqslant \alpha_2.\end{cases} \tag{13}$$

$$G_5(z,r)=\begin{cases}\alpha_2-r, & \alpha_1\leqslant r\leqslant z,\\ \alpha_2-z, & z\leqslant r\leqslant \alpha_2.\end{cases} \tag{14}$$

All these functions are convex and continuous w.r.t. both variables and the following Lemma holds:

LEMMA 1. [20] *Let $\phi\in C^2[\alpha_1,\alpha_2]$, then the following identities hold:*

$$\phi(z)=\frac{\alpha_2-z}{\alpha_2-\alpha_1}\phi(\alpha_1)+\frac{z-\alpha_1}{\alpha_2-\alpha_1}\phi(\alpha_2)+\int_{\alpha_1}^{\alpha_2}G_1(z,r)\phi''(r)dr, \tag{15}$$

$$\phi(z)=\phi(\alpha_1)+(z-\alpha_1)\phi'(\alpha_2)+\int_{\alpha_1}^{\alpha_2}G_2(z,r)\phi''(r)dr, \tag{16}$$

$$\phi(z)=\phi(\alpha_2)+(\alpha_2-z)\phi'(\alpha_1)+\int_{\alpha_1}^{\alpha_2}G_3(z,r)\phi''(r)dr, \tag{17}$$

$$\phi(z)=\phi(\alpha_2)-(\alpha_2-\alpha_1)\phi'(\alpha_2)+(z-\alpha_1)\phi'(\alpha_1)+\int_{\alpha_1}^{\alpha_2}G_4(z,r)\phi''(r)dr, \tag{18}$$

$$\phi(z) = \phi(\alpha_1) + (\alpha_2 - \alpha_1)\phi'(\alpha_1) - (\alpha_2 - z)\phi'(\alpha_2) + \int_{\alpha_1}^{\alpha_2} G_5(z,r)\phi''(r)dr. \tag{19}$$

REMARK 1. The Green function $G_1(\cdot,\cdot)$ is called Lagrange Green function (see [25]). The new Green functions $G_j(\cdot,\cdot)$, $(j = 2,3,4,5)$, were introduced by Pečarić et al. in [20]. The result (16) given in the Lemma 1 represents a special case of the representation of the function using the so-called 'two-point right focal' interpolating polynomial in case when n = 2 and p = 0 (see [1]).

The most important inequality concerning convex functions is the classical Jensen's inequality (see [12]). We present some recent work on cyclic refinements of classical and discrete Jensen's inequalities (see [11]). To make statements of that work simple, we need the following hypothesis:

(H_1) Let $2 \leqslant k \leqslant m$ be integers, and let $p_1,\ldots,p_m$ and $\lambda_1,\ldots,\lambda_k$ represent positive probability distributions.

(H_2) Let C be a convex subset of a real vector space Z and ϕ be a real valued convex function defined on C.

THEOREM 4. *Assume (H_1) and (H_2). If* $\mathbf{z} := z_1,\ldots,z_m \in C$, $\mathbf{p} := (p_1,\ldots,p_m)$ *and* $\lambda := (\lambda_1,\ldots,\lambda_k)$, *then*

$$\phi\left(\sum_{u=1}^{m} p_u z_u\right) \leqslant C_{dis} = C_{dis}(\phi,\mathbf{z},\mathbf{p},\lambda) \tag{20}$$

$$:= \sum_{u=1}^{m}\left(\sum_{v=0}^{k-1}\lambda_{v+1}p_{u+v}\right)\phi\left(\frac{\sum_{v=0}^{k-1}\lambda_{v+1}p_{u+v}z_{u+v}}{\sum_{v=0}^{k-1}\lambda_{v+1}p_{u+v}}\right) \leqslant \sum_{u=1}^{m} p_u\phi(z_u),$$

where $u+v$ *means* $u+v-m$ *in case of* $u+v>m$.

Theorem 4 can be considered as the weighted form of Theorem 2.1 in [2]. To refine the classical Jensen's inequality, we first introduce some hypotheses and notations.

(H_3) Let $(Z,\mathscr{B},\delta)$ be a probability space.

Let $l \geqslant 2$ be a fixed integer. For $j = 1,\cdots,l$, the σ-algebra in Z^l generated by the projection mappings $pr_j : Z^l \to Z$ defined by

$$pr_j(z_1,\ldots,z_l) := z_j$$

is denoted by $\mathscr{B}^l$. δ^l is the product measure on $\mathscr{B}^l$. This measure is uniquely (δ is σ-finite) specified by

$$\delta^l(B_1\times\ldots\times B_l) := \delta(B_1)\ldots\delta(B_l), \quad B_j\in\mathscr{B}, \quad j = 1,\ldots,l.$$

(H$_4$) Let f be a δ-integrable function on Z having values in an interval $I \subset \mathbb{R}$.

(H$_5$) Let ϕ be a convex function on I such that $\phi \circ f$ is δ-integrable on Z.

Under the conditions (H$_1$) and (H$_3$-H$_5$) we define

$$\begin{aligned} C_{int} &= C_{int}(\phi, f, \delta, \mathbf{p}, \lambda) \\ &:= \sum_{u=1}^{m} \left(\sum_{v=0}^{k-1} \lambda_{v+1} p_{u+v} \right) \int_{Z^m} \phi \left(\frac{\sum_{v=0}^{k-1} \lambda_{v+1} p_{u+v} f(z_{u+v})}{\sum_{v=0}^{k-1} \lambda_{v+1} p_{u+v}} \right) d\delta^m (z_1, \ldots, z_m), \end{aligned} \tag{21}$$

and for $t \in [0,1]$

$$\begin{aligned} & C_{par}(t) \\ =& C_{par}(t, \phi, f, \delta, \mathbf{p}, \lambda) \\ :=& \sum_{u=1}^{m} \left(\sum_{v=0}^{k-1} \lambda_{v+1} p_{u+v} \right) \cdot \int_{Z^m} \phi \left(t \frac{\sum_{v=0}^{k-1} \lambda_{v+1} p_{u+v} f(z_{u+v})}{\sum_{v=0}^{k-1} \lambda_{v+1} p_{u+v}} + (1-t) \int_Z f d\delta \right) d\delta^m (z_1, \ldots, z_m), \end{aligned} \tag{22}$$

where $u+v$ means $u+v-m$ in case of $u+v > m$.

REMARK 2. Lemma 2.1 (b) in [10] assures that the integrals in (21) and (22) exist and are finite.

THEOREM 5. *Assume (H$_1$) and (H$_3$-H$_5$). Also let* $\mathbf{p} := (p_1, \ldots, p_m)$ *and* $\lambda := (\lambda_1, \ldots, \lambda_k)$. *Then*

$$\phi \left(\int_Z f d\delta \right) \leqslant C_{par}(t) \leqslant C_{int} \leqslant \int_Z \phi \circ f d\delta, \quad t \in [0,1].$$

In order to achieve our goals, we consider the following hypotheses for next sections.

(M_1) Let $I \subset \mathbb{R}$ be an interval, $\mathbf{z} := (z_1, \cdots, z_m) \in I^m$ and let $p_1, \ldots, p_m$ and $\lambda_1, \ldots, \lambda_k$ represent positive probability distributions for $2 \leqslant k \leqslant m$.

REMARK 3. Under the conditions (M_1), we define

$$J_1(\phi) = J_1(\mathbf{z}, \mathbf{p}, \lambda; \phi) := \sum_{u=1}^{m} p_u \phi(z_u) - C_{dis}(\phi, \mathbf{z}, \mathbf{p}, \lambda),$$

$$J_2(\phi) = J_2(\mathbf{z}, \mathbf{p}, \lambda; \phi) := C_{dis}(\phi, \mathbf{z}, \mathbf{p}, \lambda) - \phi \left(\sum_{u=1}^{m} p_u z_u \right),$$

where $\phi : I \to \mathbb{R}$ is a function. The functionals $\phi \to J_i(\phi)$ are linear and Theorem 4 implies that

$$J_i(\phi) \geqslant 0, \quad i = 1,2$$

provided that ϕ is a convex function.

Assume (H_1) and (H_3-H_5). Then we have the following additional linear functionals

$$J_3(\phi) = J_3(\phi, f, \delta, \mathbf{p}, \lambda) := \int_Z \phi \circ f d\delta - C_{int}(\phi, f, \delta, \mathbf{p}, \lambda) \geqslant 0,$$

$$J_4(\phi) = J_4(t, \phi, f, \delta, \mathbf{p}, \lambda) := \int_Z \phi \circ f d\delta - C_{par}(t, \phi, f, \delta, \mathbf{p}, \lambda) \geqslant 0; \ \ t \in [0,1],$$

$$J_5(\phi) = J_5(t, \phi, f, \delta, \mathbf{p}, \lambda) := C_{int}(\phi, f, \delta, \mathbf{p}, \lambda) - C_{par}(t, \phi, f, \delta, \mathbf{p}, \lambda) \geqslant 0; \quad t \in [0,1],$$

$$J_6(\phi) = J_6(t, \phi, f, \delta, \mathbf{p}, \lambda) := C_{par}(t, \phi, f, \delta, \mathbf{p}, \lambda) - \phi\left(\int_Z f d\delta\right) \geqslant 0; \quad t \in [0,1].$$

2. Extensions of cyclic refinements of Jensen's inequality by Abel-Gontscharoff interpolation

To start for real weights, we need the following assumptions:

(A_1) For the linear functionals $J_i(\cdot)$ $(i = 1,2)$, assume further that $\dfrac{\sum_{v=0}^{k-1} \lambda_{v+1} p_{u+v} z_{u+v}}{\sum_{v=0}^{k-1} \lambda_{v+1} p_{u+v}} \in [\alpha_1, \alpha_2]$ for $u = 1, \ldots m$.

(A_2) For the linear functionals $J_i(\cdot)$ $(i = 3, \ldots, 6)$, assume further that $\dfrac{\sum_{v=0}^{k-1} \lambda_{v+1} p_{u+v} f(z_{u+v})}{\sum_{v=0}^{k-1} \lambda_{v+1} p_{u+v}} \in [\alpha_1, \alpha_2]$ for $u = 1, \ldots m$.

We consider discrete as well as continuous version of cyclic refinements of Jensen's inequality and construct the generalized new identities having real weights utilizing Abel-Gontscharoff interpolating polynomial.

THEOREM 6. *Let $m, k \in \mathbb{N}$, $p_1, \ldots, p_m$ and $\lambda_1, \ldots, \lambda_k$ be real tuples for $2 \leqslant k \leqslant m$, such that $\sum_{v=0}^{k-1} \lambda_{v+1} p_{u+v} \neq 0$ for $u = 1, \ldots m$ with $\sum_{u=1}^{m} p_u = 1$ and $\sum_{v=1}^{k} \lambda_v = 1$. Also let $z \in [\alpha_1, \alpha_2] \subset \mathbb{R}$ and $\mathbf{z} \in [\alpha_1, \alpha_2]^m$. Assume $\phi \in C^n[\alpha_1, \alpha_2]$ and consider interval with points $-\infty < \alpha_1 \leqslant \xi_1 < \xi_2 < \cdots \xi_n \leqslant \alpha_2 < \infty$, such that $\phi(\alpha_1) = \phi(\alpha_2)$, $\phi'(\alpha_1) = 0 = \phi'(\alpha_2)$, $\Lambda(\cdot)$ is defined by (2), $AG(\cdot, r; n)$ in (4) and G_j, $(j = 1, \ldots, 5)$ be the Green functions defined in (10)–(14), respectively.*

Then for $i = 1, \ldots, 6$ along with assumptions (A_1) and (A_2), we have the following generalized identities:

(a) For $n \geqslant 1$

$$J_i(\phi(z)) = \sum_{\sigma=1}^{n-1} \phi^{(\sigma)}(\xi_{\sigma+1}) J_i\Big(\Lambda_\sigma(z)\Big) + \int_{\alpha_1}^{\alpha_2} J_i\Big(AG(z,r;n)\Big)\phi^{(n)}(r)dr. \tag{23}$$

(b) For $n \geqslant 3$

$$\begin{aligned} J_i(\phi(z)) &= \int_{\alpha_1}^{\alpha_2} J_i\Big(G_j(z,r)\Big) \sum_{\sigma=0}^{n-3} \phi^{(\sigma+2)}(\xi_{\sigma+1})\Lambda_\sigma(r)dr \\ &\quad + \int_{\alpha_1}^{\alpha_2}\int_{\alpha_1}^{\alpha_2} J_i\Big(G_j(z,r)\Big) AG(r,v;n-2)\phi^{(n)}(v)dvdr. \end{aligned} \tag{24}$$

Proof. Fix $i = 1,\ldots,6$.

(a) Applying cyclic Jensen's type linear functionals $J_i(\cdot)$ on (8) and practicing properties of the functional, we get (23).

(b) For fix $j = 5$, testing (19) in cyclic Jensen's type functional $J_i(\cdot)$ and employing the properties of $J_i(\cdot)$ along with the assumed conditions, we have

$$\begin{aligned} J_i(\phi) &= J_i\Big(\phi(\alpha_1)\Big) + J_i\Big((\alpha_2-\alpha_1)\phi'(\alpha_1))\Big) - J_i\Big((\alpha_2-z)\phi'(\alpha_2))\Big) \\ &\quad + \int_{\alpha_1}^{\alpha_2} J_i\Big(G_5(z,r)\Big)\phi''(r)dr = \phi'(\alpha_2)J_i(z) + \int_{\alpha_1}^{\alpha_2} J_i\Big(G_5(z,r)\Big)\phi''(r)dr \\ &= \int_{\alpha_1}^{\alpha_2} J_i(G_5(z,r))\phi''(r)dr. \end{aligned} \tag{25}$$

By Theorem 2, $\phi''(r)$ can be expressed as:

$$\phi''(r) = \sum_{\sigma=0}^{n-3} \Lambda_\sigma(r)\phi^{(\sigma+2)}(\xi_{\sigma+1}) + \int_{\alpha_1}^{\alpha_2} AG(r,v;n-2)\phi^{(n)}(v)dv. \tag{26}$$

Putting (26) in (25), we get (24) respectively for $j = 5$ and $i = 1,\cdots,6$. The cases for $j = 1,2,3,4$ are treated analogously and are left for the reader interest. □

Now we obtain extensions and improvements of discrete and integral cyclic Jensen's inequalities with real weights.

THEOREM 7. *Consider ϕ be $n-$convex function along with the suppositions of Theorem 6. Then we conclude the following results:*

(a) If for all $i = 1,\ldots,6$,

$$J_i\Big(AG(z,r;n)\Big) \geqslant 0, \quad r \in [\alpha_1,\alpha_2] \tag{27}$$

holds, then we have

$$J_i(\phi(z)) \geqslant \sum_{\sigma=1}^{n-1} \phi^{(\sigma)}(\xi_{\sigma+1}) J_i\Big(\Lambda_\sigma(z)\Big) \tag{28}$$

for $i = 1,\ldots,6$.

(b) If for all $i = 1,\ldots,6$ *and* $j = 1,\ldots,5$

$$\int_{\alpha_1}^{\alpha_2} J_i\Big(G_j(z,r)\Big) AG(r,v;n-2)dr \geqslant 0, \qquad r \in [\alpha_1,\alpha_2] \tag{29}$$

holds then

$$J_i(\phi(z)) \geqslant \int_{\alpha_1}^{\alpha_2} J_i\Big(G_j(z,r)\Big) \sum_{\sigma=0}^{n-3} \phi^{(\sigma+2)}(\xi_{\sigma+1})\Lambda_\sigma(r)dr \tag{30}$$

for $i = 1,\ldots,6$.

Proof.

(a) Fix $i = 1,\ldots,6$. As the function $\phi \in C^n[\alpha_1,\alpha_2]$ is assumed to be $n-$convex, therefore using the characterization of $n-$convex function $\phi^{(n)}(z) \geqslant 0$ for all $z \in [\alpha_1,\alpha_2]$ (see [22], p. 16). Hence we can apply Theorem 6(a) to obtain (28).

(b) Fix $i = 1,\ldots,6$ and $j = 1,\ldots,5$. Following similar steps as above, we use Theorem 6(b) to get (30). □

In the next corollary, we give Theorem 7 by considering two points right focal Abel--Gontscharoff interpolating polynomial:

COROLLARY 2. *Assume* $\phi \in C^n[\alpha_1,\alpha_2]$ *on the interval with points* $\alpha_1 \leqslant \xi_1 < \xi_2 < \alpha_2$ *along with the suppositions of Theorem 6. Let* $AG_{(2)}(z,r;n)$ *be the Green function defined in (5). If* ϕ *be* $n-$*convex function, then we conclude the following results:*

(a) If for all $i = 1,\ldots,6$,

$$J_i\Big(AG_{(2)}(z,r;n)\Big) \geqslant 0, \quad r \in [\alpha_1,\alpha_2] \tag{31}$$

holds, then we have

$$J_i(\phi(z)) \geqslant \sum_{\sigma=1}^{t} \frac{\phi^{(\sigma)}(\xi_1)}{\sigma!} J_i\Big((z-\xi_1)^{\sigma}\Big) + \sum_{w=0}^{n-t-2}\left[\sum_{\sigma=0}^{w} \frac{\phi^{(t+1+w)}(\xi_2)(\xi_1-\xi_2)^{w-\sigma}}{(t+1+\sigma)!(w-\sigma)!}\right] J_i\Big((z-\xi_1)^{t+1+\sigma}\Big) \quad (32)$$

for $i = 1,\dots,6$.

(b) *If for all* $i = 1,\dots,6$ *and* $j = 1,\dots,5$

$$J_i\Big(G_j(z,r)\Big) \geqslant 0, \qquad r \in [\alpha_1,\alpha_2] \quad (33)$$

holds, provided that $(n = even,\ t = odd)$ *or* $(n = odd,\ t = even)$*, then*

$$J_i(\phi(z)) \geqslant \sum_{\sigma=0}^{t} \frac{\phi^{(\sigma+2)}(\xi_1)}{\sigma!} \int_{\alpha_1}^{\alpha_2} J_i\Big(G_j(z,r)\Big)(r-\xi_1)^{\sigma} dr + \sum_{w=0}^{n-t-4}\left[\sum_{\sigma=0}^{w} \frac{\phi^{(t+3+w)}(\xi_2)(\xi_1-\xi_2)^{w-\sigma}}{(t+1+\sigma)!(w-\sigma)!}\right] \int_{\alpha_1}^{\alpha_2} J_i\Big(G_j(z,r)\Big)(r-\xi_1)^{t+1+\sigma} dr \quad (34)$$

for $i = 1,\dots,6$.

Proof. Fix $i = 1,\dots,6$.

(a) Applying cyclic Jensen's type linear functionals $J_i(\cdot)$ on (9) and practicing properties of the functional, we get

$$J_i(\phi(z)) = \sum_{\sigma=1}^{t} \frac{\phi^{(\sigma)}(\xi_1)}{\sigma!} J_i\Big((z-\xi_1)^{\sigma}\Big) + \sum_{w=0}^{n-t-2}\left[\sum_{\sigma=0}^{w} \frac{\phi^{(t+1+w)}(\xi_2)(\xi_1-\xi_2)^{w-\sigma}}{(t+1+\sigma)!(w-\sigma)!}\right] J_i\Big((z-\xi_1)^{t+1+\sigma}\Big) + \int_{\alpha_1}^{\alpha_2} J_i\Big(AG_{(2)}(z,r;n)\Big)\phi^{(n)}(r)dr. \quad (35)$$

Now using (31) and $n-$convexity of the function ϕ, we get (32).

(b) Fix $i = 1,\dots,6$ and $j = 1,\dots,5$. By Theorem 6(b), we already proved

$$J_i(\phi) = \int_{\alpha_1}^{\alpha_2} J_i(G_j(z,r))\phi''(r)dr. \quad (36)$$

By Theorem 3, $\phi''(r)$ can be expressed as:

$$\begin{aligned}\phi''(r) = &\sum_{\sigma=0}^{t} \frac{(r-\xi_1)^{\sigma}}{\sigma!}\phi^{(\sigma+2)}(\xi_1)\\ &+\sum_{w=0}^{n-t-4}\left[\sum_{\sigma=0}^{w}\frac{(r-\xi_1)^{t+1+\sigma}(\xi_1-\xi_2)^{w-\sigma}}{(t+1+\sigma)!(w-\sigma)!}\right]\phi^{(t+3+w)}(\xi_2)\\ &+\int_{\alpha_1}^{\alpha_2} AG_{(2)}(r,v;n-2)\phi^{(n)}(v)dv. \end{aligned} \tag{37}$$

Putting (37) in (36), we get the following identity

$$\begin{aligned} J_i(\phi(z)) = &\sum_{\sigma=0}^{t}\frac{\phi^{(\sigma+2)}(\xi_1)}{\sigma!}\int_{\alpha_1}^{\alpha_2} J_i\Big(G_j(z,r)\Big)(r-\xi_1)^{\sigma}dr\\ &+\sum_{w=0}^{n-t-4}\left[\sum_{\sigma=0}^{w}\frac{\phi^{(t+3+w)}(\xi_2)(\xi_1-\xi_2)^{w-\sigma}}{(t+1+\sigma)!(w-\sigma)!}\right]\int_{\alpha_1}^{\alpha_2} J_i\Big(G_j(z,r)\Big)(r-\xi_1)^{t+1+\sigma}dr\\ &+\int_{\alpha_1}^{\alpha_2}\int_{\alpha_1}^{\alpha_2} J_i\Big(G_j(z,r)\Big)AG_{(2)}(r,v;n-2)\phi^{(n)}(v)dvdr. \end{aligned} \tag{38}$$

Now from (6), we have $(-1)^{n-t-3}AG_{(2)}(r,v;n-2)\geqslant 0$. Therefore utilizing our assumptions $(n=even, t=odd)$ or $(n=odd, t=even)$, we get $AG_{(2)}(r,v;n-2)\geqslant 0$. Now employing (33) alongside with $n-$convexity of ϕ yields (34). $\square$

We will finish the present section by the following generalizations of cyclic refinements of Jensen's inequalities by two points right focal Abel-Gontscharoff interpolating polynomial:

THEOREM 8. *If the assumptions of Corollary 2 are fulfilled with additional conditions that $p_1,\ldots,p_m$ and $\lambda_1,\ldots,\lambda_k$ are non negative tuples for $2\leqslant k\leqslant m$, such that $\sum_{u=1}^{m} p_u = 1$ and $\sum_{v=1}^{k}\lambda_v = 1$. Then for $\phi:[\alpha_1,\alpha_2]\to\mathbb{R}$ being $n-$convex function, we conclude the following results:*

(a) (32) *holds for the cases when $(n=even,\ t=odd)$ or $(n=odd,\ t=even)$. If* (32) *is valid along with the function*

$$\begin{aligned}\Gamma(z) := &\sum_{\sigma=0}^{t}\frac{(z-\xi_1)^{\sigma}}{\sigma!}\phi^{(\sigma)}(\xi_1)\\ &+\sum_{w=0}^{n-t-2}\left[\sum_{\sigma=0}^{w}\frac{(z-\xi_1)^{t+1+\sigma}(\xi_1-\xi_2)^{w-\sigma}}{(t+1+\sigma)!(w-\sigma)!}\right]\phi^{(t+1+w)}(\xi_2)\end{aligned} \tag{39}$$

to be convex, the right side of (32) *is non negative, means*

$$J_i(\phi)\geqslant 0, \qquad i=1,\ldots,6. \tag{40}$$

(b) For $(n = even,\ t = odd)$ *or* $(n = odd,\ t = even)$, (34) *holds. Further*

$$\sum_{\sigma=0}^{t} \frac{(r-\xi_1)^{\sigma}}{\sigma!}\phi^{(\sigma+2)}(\xi_1) + \sum_{w=0}^{n-t-4}\left[\sum_{\sigma=0}^{w} \frac{(r-\xi_1)^{t+1+\sigma}(\xi_1-\xi_2)^{w-\sigma}}{(t+1+\sigma)!(w-\sigma)!}\right]\phi^{(t+3+w)}(\xi_2) \geqslant 0 \tag{41}$$

the right side of (34) *is non negative, particularly* (40) *is established for all* $i = 1,\dots,6$ *and* $j = 1,\dots,5$.

Proof.

(a) Fix $i = 1,\dots,6$. Using (6), for $\xi_1 \leqslant r,\quad z \leqslant \xi_2$,

$$(-1)^{n-t-1}\frac{\partial^2 AG_{(2)}(z,r;n)}{\partial z^2} \geqslant 0 \tag{42}$$

ensures the convexity of $AG_{(2)}(z,r;n)$ w.r.t. first variable for the cases when $(n = even,\ t = odd)$ or $(n = odd,\ t = even)$. So (31) holds by virtue of Remark 3 on account of given weights to be positive. Hence (32) is established by taking into account Corollary 2 (a). Moreover, the R.H.S. of (32) can be written in the functional form $J_i(\Gamma)$ for all $(i = 1,\dots,6)$, after reorganizing this side. Employing Remark 3 the non negativity of R.H.S. of (32) is secured, especially (40) is established.

(b) Fix $i = 1,\dots,6$. We have assumed positive weights and for all $j = 1,\dots,5$, $G_j(z,r)$ is convex. Thus by practicing Remark 3, $J_i(G_j(z,r)) \geqslant 0$. Since ϕ is $n-$convex, hence by following Corollary 2 (b), we obtain (34). Now taking into account the positivity of $J_i(G_j(z,r))$ and (41), we get (40). □

3. Applications to entropic bounds

Let $\phi : (0,\infty) \to (0,\infty)$ be a convex function with $\mathbf{p} := (p_1,...,p_m)$ and $\mathbf{q} := (q_1,...,q_m)$ be positive probability distributions. Then ϕ-divergence functional is defined in [7] as follows:

$$I_\phi(\mathbf{p},\mathbf{q}) = \sum_{u=1}^{m} q_u \phi\left(\frac{p_u}{q_u}\right).$$

Surveying the classical Csiszár divergence functional, we propose a new functional:

DEFINITION 1. Let $\phi : I \to \mathbb{R}$ be a function with I an interval in $\mathbb{R}$. Let $\mathbf{p} := (p_1,\dots,p_m) \in \mathbb{R}^m$ and $\mathbf{q} := (q_1,\dots,q_m) \in (0,\infty)^m$ such that

$$\frac{p_u}{q_u} \in I, \quad u = 1,\dots,m.$$

Then let

$$\widetilde{I}_\phi(\mathbf{p},\mathbf{q}) = \sum_{u=1}^{m} q_u \phi\left(\frac{p_u}{q_u}\right). \tag{43}$$

REMARK 4. Now as a consequence of Theorem 7 we consider the discrete extensions of cyclic refinements of Jensen's inequalities for $(i=1)$, from (28) with respect to $n-$convex function ϕ in the explicit form:

$$\sum_{u=1}^{m} p_u \phi(z_u) - \sum_{u=1}^{m}\left(\sum_{v=0}^{k-1}\lambda_{v+1}p_{u+v}\right)\phi\left(\frac{\sum_{v=0}^{k-1}\lambda_{v+1}p_{u+v}z_{u+v}}{\sum_{v=0}^{k-1}\lambda_{v+1}p_{u+v}}\right)$$
$$\geqslant \sum_{\sigma=1}^{n-1}\phi^{(\sigma)}(\xi_{\sigma+1})\times\left(\sum_{u=1}^{m}p_u\Lambda_\sigma(z_u) - \sum_{u=1}^{m}\left(\sum_{v=0}^{k-1}\lambda_{v+1}p_{u+v}\right)\Lambda_\sigma\left(\frac{\sum_{v=0}^{k-1}\lambda_{v+1}p_{u+v}z_{u+v}}{\sum_{v=0}^{k-1}\lambda_{v+1}p_{u+v}}\right)\right), \tag{44}$$

where $\Lambda(\cdot)$ is defined by (2).

THEOREM 9. *Let $m,k\in\mathbb{N}$ $(2\leqslant k\leqslant m)$, $\lambda_1,\ldots,\lambda_k$ be positive probability distributions. Let $\mathbf{p}:=(p_1,\ldots,p_m)\in\mathbb{R}^m$ and $\mathbf{q}:=(q_1,\ldots,q_m)\in(0,\infty)^m$ such that*

$$\frac{p_u}{q_u}\in[\alpha_1,\alpha_2],\quad u=1,\ldots,m.$$

Also let $\phi\in C^n[\alpha_1,\alpha_2]$ and consider interval with points $-\infty<\alpha_1\leqslant\xi_1<\xi_2<\cdots\xi_n\leqslant\alpha_2<\infty$ such that ϕ is $n-$convex function. Then the following inequality holds:

$$\widetilde{I}_\phi(\mathbf{p},\mathbf{q})$$
$$\geqslant \sum_{u=1}^{m}\left(\sum_{v=0}^{k-1}\lambda_{v+1}q_{u+v}\right)\phi\left(\frac{\sum_{v=0}^{k-1}\lambda_{v+1}p_{u+v}}{\sum_{v=0}^{k-1}\lambda_{v+1}q_{u+v}}\right)$$
$$+\sum_{\sigma=1}^{n-1}\phi^{(\sigma)}(\xi_{\sigma+1})\times\left(\sum_{u=1}^{m}q_u\Lambda_\sigma\left(\frac{p_u}{q_u}\right) - \sum_{u=1}^{m}\left(\sum_{v=0}^{k-1}\lambda_{v+1}q_{u+v}\right)\Lambda_\sigma\left(\frac{\sum_{v=0}^{k-1}\lambda_{v+1}p_{u+v}}{\sum_{v=0}^{k-1}\lambda_{v+1}q_{u+v}}\right)\right). \tag{45}$$

Proof. Replacing p_u with q_u and z_u with $\frac{p_u}{q_u}$ for $(u=1,\ldots,m)$ in (44), we get (45). □

We now explore two special cases of the previous result. One corresponds to the entropy of a discrete probability distribution. For positive m-tuple $\mathbf{q}=(q_1,\ldots,q_m)$ such that $\sum_{u=1}^{m}q_u=1$, the **Shannon entropy** [24] is defined by

$$S(\mathbf{q}) = -\sum_{u=1}^{m}q_u\ln q_u. \tag{46}$$

Shannon entropy and related measures are increasingly used in molecular ecology, population genetics, information theory, dynamical systems and statistical physics (see [19]).

COROLLARY 3. *Let* $m,k \in \mathbb{N}$ $(2 \leqslant k \leqslant m)$, $\lambda_1, \dots, \lambda_k$ *be positive probability distributions.*

(a) If $\mathbf{q} := (q_1, \dots, q_m) \in (0,\infty)^m$ *and* $(n = even)$, *then*

$$\sum_{u=1}^{m} q_u \ln q_u \geqslant \sum_{u=1}^{m} \left(\sum_{v=0}^{k-1} \lambda_{v+1} q_{u+v} \right) \ln \left(\sum_{v=0}^{k-1} \lambda_{v+1} q_{u+v} \right) + \left(\sum_{\sigma=1}^{n-1} \frac{(-1)^{\sigma}(\sigma-1)!}{(\xi_{\sigma+1})^{\sigma}} \right. \\ \left. \times \left(\sum_{u=1}^{m} q_u \Lambda_\sigma \left(\frac{1}{q_u} \right) - \sum_{u=1}^{m} \left(\sum_{v=0}^{k-1} \lambda_{v+1} q_{u+v} \right) \Lambda_\sigma \left(\frac{1}{\sum_{v=0}^{k-1} \lambda_{v+1} q_{u+v}} \right) \right) \right). \tag{47}$$

(b) If $\mathbf{q} := (q_1, \dots, q_m)$ *is a positive probability distribution and* $(n = even)$, *then we get the bounds for the Shannon entropy of* $\mathbf{q}$.

$$S(\mathbf{q}) \leqslant - \sum_{u=1}^{m} \left(\sum_{v=0}^{k-1} \lambda_{v+1} q_{u+v} \right) \ln \left(\sum_{v=0}^{k-1} \lambda_{v+1} q_{u+v} \right) - \left(\sum_{\sigma=1}^{n-1} \frac{(-1)^{\sigma}(\sigma-1)!}{(\xi_{\sigma+1})^{\sigma}} \right. \\ \left. \times \left(\sum_{u=1}^{m} q_u \Lambda_\sigma \left(\frac{1}{q_u} \right) - \sum_{u=1}^{m} \left(\sum_{v=0}^{k-1} \lambda_{v+1} q_{u+v} \right) \Lambda_\sigma \left(\frac{1}{\sum_{v=0}^{k-1} \lambda_{v+1} q_{u+v}} \right) \right) \right). \tag{48}$$

If $(n = odd)$, *then* (47) *and* (48) *hold in reverse directions.*

Proof.

(a) Using $\phi(x) := -\ln x$, and $\mathbf{p} := (1,1,\dots,1)$ in Theorem 9, we get the required results.

(b) It is a special case of (a). □

The second case corresponds to the relative entropy or Kullback–Leibler divergence between two probability distributions. One of the best known distance function used in mathematical statistics, information theory and signal processing is Kullback-Leibler distance. The **Kullback-Leibler** distance [17, 18] between the positive probability distributions $\mathbf{p} = (p_1, ..., p_m)$ and $\mathbf{q} = (q_1, ..., q_m)$ is defined by

$$D(\mathbf{q} \parallel \mathbf{p}) = \sum_{u=1}^{m} q_u \ln \left(\frac{q_u}{p_u} \right). \tag{49}$$

COROLLARY 4. *Let $m,k \in \mathbb{N}$ $(2 \leqslant k \leqslant m)$, $\lambda_1,\ldots,\lambda_k$ be positive probability distributions.*

(a) If $\mathbf{q} := (q_1,\ldots,q_m), \mathbf{p} := (p_1,\ldots,p_m) \in (0,\infty)^m$ and $(n = even)$, then

$$\sum_{u=1}^{m} q_u \ln\left(\frac{q_u}{p_u}\right) \geqslant \sum_{u=1}^{m}\left(\sum_{v=0}^{k-1}\lambda_{v+1}q_{u+v}\right)\ln\left(\frac{\sum_{v=0}^{k-1}\lambda_{v+1}q_{u+v}}{\sum_{v=0}^{k-1}\lambda_{v+1}p_{u+v}}\right) + \left(\sum_{\sigma=1}^{n-1}\frac{(-1)^{\sigma}(\sigma-1)!}{(\xi_{\sigma+1})^{\sigma}}\right)$$
$$\times\left(\sum_{u=1}^{m} q_u \Lambda_\sigma\left(\frac{p_u}{q_u}\right) - \sum_{u=1}^{m}\left(\sum_{v=0}^{k-1}\lambda_{v+1}q_{u+v}\right)\Lambda_\sigma\left(\frac{\sum_{v=0}^{k-1}\lambda_{v+1}p_{u+v}}{\sum_{v=0}^{k-1}\lambda_{v+1}q_{u+v}}\right)\right). \tag{50}$$

(b) If $\mathbf{q} := (q_1,\ldots,q_m), \mathbf{p} := (p_1,\ldots,p_m)$ are positive probability distributions and $(n = even)$, then we have

$$D(\mathbf{q} \parallel \mathbf{p}) \geqslant \sum_{u=1}^{m}\left(\sum_{v=0}^{k-1}\lambda_{v+1}q_{u+v}\right)\ln\left(\frac{\sum_{v=0}^{k-1}\lambda_{v+1}q_{u+v}}{\sum_{v=0}^{k-1}\lambda_{v+1}p_{u+v}}\right) + \left(\sum_{\sigma=1}^{n-1}\frac{(-1)^{\sigma}(\sigma-1)!}{(\xi_{\sigma+1})^{\sigma}}\right)$$
$$\times\left(\sum_{u=1}^{m} q_u \Lambda_\sigma\left(\frac{p_u}{q_u}\right) - \sum_{u=1}^{m}\left(\sum_{v=0}^{k-1}\lambda_{v+1}q_{u+v}\right)\Lambda_\sigma\left(\frac{\sum_{v=0}^{k-1}\lambda_{v+1}p_{u+v}}{\sum_{v=0}^{k-1}\lambda_{v+1}q_{u+v}}\right)\right). \tag{51}$$

If $(n = odd)$, then (50) and (51) hold in reverse directions.

Proof.

(a) Using $\phi(x) := -\ln x$ in Theorem 9, we get the desired results.

(b) It is special case of (a). □

Zipf's law is one of the basic laws in information science and is extensively applied in linguistics. Apart from the use of this law in information science and linguistics, Zipf's law has a mythical impact in economics.
For $m \in \{1,2,\ldots\}$, $c \geqslant 0$ and $d > 0$ the **Zipf-Mandelbrot** law (probability mass function) is stated as

$$\phi(u;m,c,d) = \frac{1}{((u+c)^d H_{m,c,d})}, \quad u = (1,2,\ldots,m), \tag{52}$$

where

$$H_{m,c,d} = \sum_{s=1}^{m} \frac{1}{(s+c)^d}.$$

The probability mass function can be given as in the (52) and $H_{m,c,d}$ which may be thought of as a generalization of a harmonic number.Application of Zipf-Mandelbrot law can be found in linguistics, information sciences and also is often applicable in ecological field studies. Some recent results related to Zipf-Mandelbrot law can be seen in [13, 15, 16].

Let $m \in \{1,2,\ldots\}$, $c \geqslant 0$, $d > 0$, then **Zipf-Mandelbrot entropy** can be given as:

$$Z(H,c,d) = \frac{d}{H_{m,c,d}} \sum_{u=1}^{m} \frac{\ln(u+c)}{(u+c)^d} + \ln(H_{m,c,d}). \tag{53}$$

Consider

$$q_u = \phi(u;m,c,d) = \frac{1}{((u+c)^d H_{m,c,d})}. \tag{54}$$

Now we state our results involving entropy introduced by Mandelbrot Law:

THEOREM 10. *Let $m,k \in \mathbb{N}$ $(2 \leqslant k \leqslant m)$, $\lambda_1,\ldots,\lambda_k$ be positive probability distributions and $\mathbf{q}$ be as defined in (54) by Zipf-Mandelbrot law with parameters $m \in \{1,2,\ldots\}$, $c \geqslant 0$, $d > 0$. For $(n = even)$, the following holds*

$$\begin{aligned}
&S(\mathbf{q})\\
=&Z(H,c,d)\\
\leqslant& -\sum_{u=1}^{m}\left(\sum_{v=0}^{k-1}\frac{\lambda_{v+1}}{((u+v+c)^d H_{m,c,d})}\right)\ln\left(\frac{1}{H_{m,c,d}}\sum_{v=0}^{k-1}\frac{\lambda_{v+1}}{((u+v+c)^d)}\right)\\
&-\left(\sum_{\sigma=1}^{n-1}\frac{(-1)^{\sigma}(\sigma-1)!}{(\xi_{\sigma+1})^{\sigma}}\right)\left(\sum_{u=1}^{m}\frac{1}{((u+c)^d H_{m,c,d})}\Lambda_{\sigma}\left(((u+c)^d H_{m,c,d})\right)\right)\\
&+\left(\sum_{\sigma=1}^{n-1}\frac{(-1)^{\sigma}(\sigma-1)!}{(\xi_{\sigma+1})^{\sigma}}\right)\left(\sum_{u=1}^{m}\left(\sum_{v=0}^{k-1}\frac{\lambda_{v+1}}{((u+v+c)^d H_{m,c,d})}\right)\Lambda_{\sigma}\left(\frac{1}{\sum_{v=0}^{k-1}\frac{\lambda_{v+1}}{((u+v+c)^d H_{m,c,d})}}\right)\right).
\end{aligned} \tag{55}$$

If $(n = odd)$, then (55) holds in reverse direction.

Proof. Substituting this $q_u = \frac{1}{((u+c)^d H_{m,c,d})}$ in Corollary 3(b), we get the desired result. Since it is interesting to see that $\sum_{u=1}^{m} q_u = 1$. Moreover using above q_u in Shannon entropy (46), we get Mandelbrot entropy (53)

$$S(q) = -q_u \ln q_u = -\sum_{u=1}^{m} \frac{1}{((u+c)^d H_{m,c,d})} \ln \frac{1}{((u+c)^d H_{m,c,d})}$$

$$
\begin{aligned}
&= \frac{-1}{(H_{m,c,d})} \sum_{u=1}^{m} \frac{1}{(u+c)^d} \ln \frac{1}{(u+c)^d H_{m,c,d}} \\
&= \frac{-1}{(H_{m,c,d})} \sum_{u=1}^{m} \frac{1}{(u+c)^d} \Big(\ln(1) - d\ln(u+c) - \ln(H_{m,c,d}) \Big) \\
&= \frac{1}{(H_{m,c,d})} \sum_{u=1}^{m} \frac{1}{(u+c)^d} \Big(d\ln(u+c) + \ln(H_{m,c,d}) \Big) \\
&= \frac{d}{(H_{m,c,d})} \sum_{u=1}^{m} \frac{\ln(u+c)}{(u+c)^d} + \ln(H_{m,c,d}). \quad \square
\end{aligned} \tag{56}
$$

COROLLARY 5. *Let* $m,k \in \mathbb{N}$ $(2 \leqslant k \leqslant m)$, $\lambda_1, \dots, \lambda_k$ *be positive probability distributions and for* $c_1, c_2 \in [0,\infty)$, $d_1, d_2 > 0$, *let* $H_{m,c_1,d_1} = \sum_{s=1}^{m} \frac{1}{(s+c_1)^{d_1}}$ *and* $H_{m,c_2,d_2} = \sum_{s=1}^{m} \frac{1}{(s+c_2)^{d_2}}$. *Now using* $q_u = \dfrac{1}{(u+c_1)^{d_1} H_{m,c_1,d_1}}$ *and* $p_u = \dfrac{1}{(u+c_2)^{d_2} H_{m,c_2,d_2}}$ *in Corollary* 4*(b), with* $(n = even)$, *then the following holds:*

$$
\begin{aligned}
&D(\mathbf{q} \,\|\, \mathbf{p}) \\
&= \sum_{u=1}^{m} \frac{1}{(u+c_1)^{d_1} H_{m,c_1,d_1}} \ln \left(\frac{(u+c_2)^{d_2} H_{m,c_2,d_2}}{(u+c_1)^{d_1} H_{m,c_1,d_1}} \right) \\
&\geqslant \sum_{u=1}^{m} \left(\sum_{v=0}^{k-1} \frac{\lambda_{v+1}}{(u+v+c_1)^{d_1} H_{m,c_1,d_1}} \right) \ln \left(\frac{\sum_{v=0}^{k-1} \lambda_{v+1} \frac{1}{(u+v+c_1)^{d_1} H_{m,c_1,d_1}}}{\sum_{v=0}^{k-1} \lambda_{v+1} \frac{1}{(u+v+c_2)^{d_2} H_{m,c_2,d_2}}} \right) \\
&\quad + \left(\sum_{\sigma=1}^{n-1} \frac{(-1)^{\sigma} (\sigma-1)!}{(\xi_{\sigma+1})^{\sigma}} \right) \left(\sum_{u=1}^{m} \frac{1}{((u+c_1)^{d_1} H_{m,c_1,d_1})} \Lambda_{\sigma} \left(\frac{((u+c_1)^{d_1} H_{m,c_1,d_1})}{((u+c_2)^{d_2} H_{m,c_2,d_2})} \right) \right) \\
&\quad - \left(\sum_{\sigma=1}^{n-1} \frac{(-1)^{\sigma} (\sigma-1)!}{(\xi_{\sigma+1})^{\sigma}} \right) \\
&\quad \times \left(\sum_{u=1}^{m} \left(\sum_{v=0}^{k-1} \frac{\lambda_{v+1}}{(u+v+c_1)^{d_1} H_{m,c_1,d_1}} \right) \Lambda_{\sigma} \left(\frac{\sum_{v=0}^{k-1} \lambda_{v+1} \frac{1}{(u+v+c_2)^{d_2} H_{m,c_2,d_2}}}{\sum_{v=0}^{k-1} \lambda_{v+1} \frac{1}{(u+v+c_1)^{d_1} H_{m,c_1,d_1}}} \right) \right).
\end{aligned} \tag{57}
$$

If $(n = odd)$, *then* (57) *holds in reverse direction.*

REMARK 5. It is interesting to note that, in the similar passion we are able to construct different estimations of ϕ-divergences along with their applications to Shannon, Relative and Mandelbrot entropies using the other inequalities for $n-$convex functions constructed in Theorem 7 for discrete case of cyclic refinements of Jensen's inequality.

REMARK 6. We left for reader interest to construct upper bounds for Shannon, Relative and Mandelbrot entropies by considering two points right focal Abel-Gontscharoff interpolating polynomial in the above results.

4. Concluding remarks

It is refreshing to note that obtained inequalities for $n-$convex functions in the first section are worth more as they enable us to give variety of new and sharp upper bounds for Grüss and Ostrowski type inequalities (see [3]) as an application of the results obtained by Dragomir et al. in [6]. We can also give related inequalities for $n-$convex function at a point (see [23]), that is the more general class of $n-$convex functions. Furthermore, we can construct variety of functionals from the inequalities introduced in the Theorem 7 and present Cauchy and Lagrange type mean value theorems for $n-$convex functions. More than that, taking into account $n-$exponentially convex approach in [14] and [21](see also [4] and [5]), a new collection of non trivial examples of $n-$exponentially and exponentially convex functions can be established. Finally we are also able to construct monotonic Cauchy means.

Acknowledgement. The research of the 1st and 2nd author has been fully supported by H.E.C. Pakistan under NRPU project 7906. The research of 4th author was supported by the Ministry of Education and Science of the Russian Federation (the Agreement number No. 02.a03.21.0008.).

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(Received October 2, 2018)

Saad Ihsan Butt
COMSATS University Islamabad
Lahore Campus
Pakistan
e-mail: saadihsanbutt@gmail.com

Nasir Mehmood
COMSATS University Islamabad
Lahore Campus
Pakistan
e-mail: nasir.gcburewala@yahoo.com

Đilda Pečarić
Catholic University of Croatia
Ilica 242, Zagreb, Croatia
e-mail: gildapeca@gmail.com

Josip Pečarić
RUDN University
Miklukho-Maklaya str.6, 117198 Moscow, Russia
e-mail: pecaric@mahazu.hazu.hr