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GENERALIZED STEFFENSEN'S INEQUALITY BY MONTGOMERY IDENTITIES AND GREEN FUNCTIONS

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Abstract. A new generalization of Steffensen's inequality and other inequalities related to Steffnesen's inequality have been proved. The contribution of these new generalizations has been presented to theory of $(n+1)$-convex functions and exponentially convex functions.

1. Introduction

Steffensen [14] proved the following inequality: if $f,h:[\alpha,\beta]\to\mathbb{R}$, $0\leqslant h\leqslant 1$ and f is decreasing, then

$$\int_\alpha^\beta f(t)h(t)\,dt\leqslant\int_\alpha^{\alpha+\gamma}f(t)\,dt,\qquad \text{where } \gamma=\int_\alpha^\beta h(t)\,dt. \tag{1}$$

Since then, generalization and improvement of Steffensen's inequality is a topic of interest of several Mathematicians, for example see [11], [12] and references therein. One recent generalization is given by Rabier [13].

THEOREM 1. *Let $\phi:[0,\infty)\to\mathbb{R}$ be convex and continuous with $\phi(0)=0$. If $b>0$ and $h\in L^\infty(0,b), h\geqslant 0$ and $\|h\|_\infty\leqslant 1$, then $h\phi'\in L^1(0,b)$ and*

$$\phi\Big(\int_0^b h(t)\,dt\Big)\leqslant\int_0^b h(t)\phi'(t)\,dt \tag{2}$$

Another generalization of Steffensen's inequality is given by Pečarić [9].

THEOREM 2. *Let $g:[a,b]\to\mathbb{R}$ be a non-decreasing and differentiable function and $f:I\to\mathbb{R}$ be a non-decreasing function (I is an interval in $\mathbb{R}$ such that $a,b,g(a),g(b)\in I$).*

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(a) *If* $g(x) \leqslant x$, *then*

$$\int_a^b f(t)g'(t)\,dt \geqslant \int_{g(a)}^{g(b)} f(t)\,dt. \tag{3}$$

(b) *If* $g(x) \geqslant x$, *then the reverse of the above inequality holds.*

REMARK 1. In the Theorem 2 one may take g as absolutely continuous function instead of differentiable function because if f is non-decreasing then the function $F(x) = \int_a^x f(t)\,dt$ is well defined and $F' = f$ holds almost everywhere on I. Then if g is any absolutely continuous and non-decreasing function then the substitution $z = g(t)$ in the integral is justified (see [6, Corollary 20.5]), so

$$F(g(b)) - F(g(a)) = \int_{g(a)}^{g(b)} f(z)\,dz = \int_a^b f(g(t))g'(t)\,dt \leqslant \int_a^b f(t)g'(t)\,dt, \tag{4}$$

where the last inequality holds when $g(x) \leqslant x$.

With suitable substitution in (4) one may get all (1), (2) and (3), see [3]. Recently, Fahad, Pečarić and Praljak proved generalization [3, 4] of Steffensen's inequality and related results by extending the results given in [9]. The following is a consequence of a Theorem proved in [3].

COROLLARY 1. *Let* $g : [a,b] \to \mathbb{R}$ *be non-decreasing and differentiable and let* $f : I \to \mathbb{R}$ *(where* I *is an interval such that* $a, b, g(a), g(b) \in I$*) be differentiable convex function.*

(a) *If* $g(x) \leqslant x$, *then*

$$f(g(b)) \leqslant f(g(a)) + \int_a^b f'(t)g'(t)\,dt. \tag{5}$$

(b) *If* $g(x) \geqslant x$, *then the reverse of the above inequality holds.*

The preceding corollary yields (4) and consequently (1), (2) and (3). Now, we include two more consequences of the results proved in [3].

COROLLARY 2. *Let* $f : [0,b] \to \mathbb{R}$ *be differentiable convex function with* $f(0) = 0$ *let* $h : [0,b] \to [0,+\infty)$ *be another function.*

(a) *If* $\int_0^x h(t)\,dt \leqslant x$ *for every* $x \in [0,b]$, *then*

$$f\left(\int_0^b h(t)\,dt\right) \leqslant \int_0^b f'(t)h(t)\,dt. \tag{6}$$

(b) *If* $x \leqslant \int_0^x h(t)\,dt$ *for every* $x \in [0,b]$, *then the reverse of the above inequality holds.*

COROLLARY 3. *Let h and f be as given in Corollary 2 and let $k : [0,b] \to [0,+\infty)$ and denote $K(t) = \int_t^b k(x)\,dx$.*

(a) *If $\int_0^x h(t)\,dt \leqslant x$ for every $x \in [0,b]$, then*

$$\int_0^b k(x) f\left(\int_0^x h(t)\,dt\right) dx \leqslant \int_0^b K(t) f'(t) h(t)\,dt. \tag{7}$$

(b) *If $x \leqslant \int_0^x h(t)\,dt$ for every $x \in [0,b]$, then the reverse of the above inequality holds.*

The main objective of this article is to establish generalization of (4) and ultimately produce the generalizations of (1), (2) and (3). The connection between Classical Hardy-type inequalities and inequalities (6) and (7) has been elaborated in [3]. Due to significance of (6) and (7), we prove their generalizations as well. As an application, we present contribution of new inequalities to theory of $(n+1)$-convex functions and exponentially convex functions. To achieve this objective, we use Montogomery identities, Taylor's interpolation and Green functions. Following lemma has been given in [8].

LEMMA 1. *For a function $f \in C^2([a,b])$ we have:*

$$f(x) = \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b) + \int_a^b G_{*,1}(x,s) f''(s)\,ds, \tag{8}$$

$$f(x) = f(a) + (x-a) f'(b) + \int_a^b G_{*,2}(x,s) f''(s)\,ds, \tag{9}$$

$$f(x) = f(b) + (b-x) f'(a) + \int_a^b G_{*,3}(x,s) f''(s)\,ds, \tag{10}$$

$$f(x) = f(b) - (b-a) f'(b) + (x-a) f'(a) + \int_a^b G_{*,4}(x,s) f''(s)\,ds, \tag{11}$$

$$f(x) = f(a) + (b-a) f'(a) - (b-x) f'(b) + \int_a^b G_{*,5}(x,s) f''(s)\,ds, \tag{12}$$

where

$$G_{*,1}(x,s) = \begin{cases} \frac{(x-b)(s-a)}{b-a}, & \text{if } a \leqslant s \leqslant x, \\ \frac{(s-b)(x-a)}{b-a}, & \text{if } x < s \leqslant b, \end{cases} \tag{13}$$

$$G_{*,2}(x,s) = \begin{cases} a-s, & \text{if } a \leqslant s \leqslant x, \\ a-x, & \text{if } x < s \leqslant b, \end{cases} \tag{14}$$

$$G_{*,3}(x,s) = \begin{cases} x-b, & \text{if } a \leqslant s \leqslant x, \\ s-b, & \text{if } x < s \leqslant b, \end{cases} \tag{15}$$

$$G_{*,4}(x,s) = \begin{cases} x-a, & \text{if } a \leqslant s \leqslant x, \\ s-a, & \text{if } x < s \leqslant b, \end{cases} \tag{16}$$

and

$$G_{*,5}(x,s) = \begin{cases} b-s, & \text{if } a \leqslant s \leqslant x, \\ b-x, & \text{if } x < s \leqslant b. \end{cases} \tag{17}$$

Following simple lemma has been proved in [5]

LEMMA 2. *For a function $f \in C^1[a,b]$, the following identities hold*

$$f(x) = \frac{1}{b-a}\int_a^b f(s)\,ds + \int_a^b p_1(x,s) f'(s)\,ds, \tag{18}$$

$$f(x) = f(b) + \int_a^b p_2(x,s) f'(s)\,ds \tag{19}$$

and

$$f(x) = f(a) + \int_a^b p_3(x,s) f'(s)\,ds, \tag{20}$$

where

$$p_1(x,s) = \begin{cases} \frac{s-a}{b-a}, & \text{if } a \leqslant s \leqslant x, \\ \frac{s-b}{b-a}, & \text{if } x < s \leqslant b, \end{cases} \tag{21}$$

$$p_2(x,s) = \begin{cases} 0, & \text{if } a \leqslant s \leqslant x, \\ -1, & \text{if } x < s \leqslant b, \end{cases} \tag{22}$$

and

$$p_3(x,s) = \begin{cases} 1, & \text{if } a \leqslant s \leqslant x, \\ 0, & \text{if } x < s \leqslant b. \end{cases} \tag{23}$$

Clearly,

$$p_i(x,s) = \left(G_{*,i}(x,s)\right)_x \text{ for all } i = 1,2,3, \tag{24}$$

$$p_2(x,s) = \left(G_{*,5}(x,s)\right)_x \text{ and } p_3(x,s) = \left(G_{*,4}(x,s)\right)_x.$$

During the proofs in the next section, we will use, $p_i(x,s)$ corresponding to $G_{*,i}(x,s)$ for $i = 1,2,3$ and for $G_4(x,s)$, and $G_5(x,s)$, $p_3(x,s)$ and $p_2(x,s)$ respectively. The next section contains the main results of this paper.

2. Generalized Steffensen's Inequality

Throughout the paper we use following notations,

$$S_1(f,g,a,b) = f(g(a)) - f(g(b)) + \int_a^b f'(t)g'(t)\,dt,$$

$$S_2(f,h,b) = \int_0^b f'(t)h(t)\,dt - f\left(\int_0^b h(t)\,dt\right)$$

and

$$S_3(f,h,k,b) = \int_0^b K(t)f'(t)h(t)\,dt - \int_0^b k(x) f\left(\int_0^x h(t)\,dt\right)dx.$$

Now, we prove following theorem which enables us to obtain generalization of (5).

THEOREM 3. *Let $n \in \mathbb{N}$ with $n \geqslant 3$ and $f : [a,b] \to \mathbb{R}$ be n times differentiable function. Let $g : [a,b] \to \mathbb{R}$ be a non-decreasing function with $g(a), g(b) \in [a,b]$ then:*

(a) *For* $j = 1,2,4,5$, *we have*

$$S_1(f,g,a,b) = \sum_{k=0}^{n-3} \frac{f^{(k+2)}(a)}{k!} \int_a^b S_1(G_{*,j}(.,s),g,a,b)(s-a)^k\,ds$$
$$+ \frac{1}{(n-3)!} \int_a^b S_1(G_{*,j}(.,s),g,a,b) \left(\int_a^s f^{(n)}(\xi)(s-\xi)^{n-3} d\xi \right) ds.$$

(b) *If* $f'(a) = 0$ *then*

$$S_1(f,g,a,b) = \sum_{k=0}^{n-3} \frac{f^{(k+2)}(a)}{k!} \int_a^b S_1(G_{*,3}(.,s),g,a,b)(s-a)^k\,ds$$
$$+ \frac{1}{(n-3)!} \int_a^b S_1(G_{*,3}(.,s),g,a,b) \left(\int_a^s f^{(n)}(\xi)(s-\xi)^{n-3} d\xi \right) ds,$$

where $G_{*,j}(x,s)$, *for* $j = 1,2,\ldots,5$, *is given by* $((13)-(17))$.

Proof.

(a) We prove for the case when $j = 1$, other cases $j = 2,4,5$ are similar to this proof. By using (8) and (18) for f and f' respectively, we have

$$\begin{aligned}
S_1(f,g,a,b) &= f(g(a)) - f(g(b)) + \int_a^b f'(t)g'(t)\,dt \\
&= \frac{b-g(a)}{b-a} f(a) + \frac{g(a)-a}{b-a} f(b) + \int_a^b G_{*,1}(g(a),s) f''(s)\,ds \\
&\quad - \frac{b-g(b)}{b-a} f(a) - \frac{g(b)-a}{b-a} f(b) - \int_a^b G_{*,1}(g(b),s) f''(s)\,ds \\
&\quad + \int_a^b \left[\frac{f(b)-f(a)}{b-a} + \int_a^b p_1(t,s) f''(s)\,ds \right] g'(t)\,dt.
\end{aligned}$$

By simplifying and using Fubini's theorem, we have

$$\begin{aligned}
S_1(f,g,a,b) &= \frac{g(b)-g(a)}{b-a} f(a) - \frac{g(b)-g(a)}{b-a} f(b) \\
&\quad + \int_a^b [G_{*,1}(g(a),s) - G_{*,1}(g(b),s)] f''(s)\,ds \\
&\quad + \frac{f(b)-f(a)}{b-a} (g(b)-g(a)) + \int_a^b \int_a^b p_1(t,s) g'(t) f''(s)\,dt\,ds \\
&= \int_a^b S_1(G_{*,1}(.,s),g,a,b) f''(s)\,ds.
\end{aligned}$$

Further, the $(n-3)$-rd order Taylor approximation for f'' yields

$$\begin{aligned}
&S_1(f,g,a,b) \\
&= \int_a^b S_1(G_{*,1}(.,s),g,a,b) \left(\sum_{k=0}^{n-3} f^{(k+2)}(a) \frac{(s-a)^k}{k!} + \int_a^s f^{(n)}(\xi) \frac{(s-\xi)^{n-3}}{(n-3)!} d\xi \right) ds
\end{aligned}$$

which upon simplification gives required identity.

(b) The proof is similar to part (a) except the use of the assumption $f'(a) = 0$. □

Following theorem gives generalized Steffensen's inequality.

THEOREM 4. *Let $n \in \mathbb{N}$ with $n \geqslant 3$ and let $f : [a,b] \to \mathbb{R}$ be n times differentiable, $g : [a,b] \to \mathbb{R}$ be non-decreasing with $g(x) \leqslant x$ and $g(a), g(b) \in [a,b]$. Then*

(a) *If f is n-convex, then*

$$S_1(f,g,a,b) \geqslant \sum_{k=0}^{n-3} \frac{f^{(k+2)}(a)}{k!} \int_a^b S_1(G_{*,j}(.,s),g,a,b)(s-a)^k ds$$

for $j = 1,2,\ldots,5$, where $f'(a) = 0$ for $j = 3$.

(b) *If $-f$ is n-convex, then the reverse of inequality in part (a) holds.*

Proof. For fix s and any $j \in \{1,2,3,4,5\}$, the function $G_{*,j}(.,s)$ is convex and differentiable and since g is non-decreasing with $g(x) \leqslant x$, therefore Corollary 1 (a) gives $S_1(G_{*,j}(.,s),g,a,b) \geqslant 0$. Moreover, n-convexity of f implies $f^{(n)}(x) \geqslant 0$ for $x \in [a,b]$ and we get

$$\frac{1}{(n-3)!} \int_a^b S_1(G_{*,j}(.,s),g,a,b) \left(\int_a^s f^{(n)}(\xi)(s-\xi)^{n-3} d\xi \right) ds \geqslant 0.$$

Further, for each j, identity in Theorem 3 produces the desired inequality. □

In particular, the above theorem gives $S_1(f,g,a,b) \geqslant 0$ and $S_1(f,g,a,b) \leqslant 0$ which gives (5) and its reverse. Consequently, Theorem 4 produces generalization of (1), (2) and (3). Now, we prove following theorem which enables us to prove generalization of (6).

THEOREM 5. *Let $n \in \mathbb{N}$ with $n \geqslant 3$ and let $f : [0,b] \to \mathbb{R}$ be n times differentiable function with $f(0) = 0$. If $h : [0,b] \to [0,+\infty)$ is an integrable function then*

(a)

$$\begin{aligned} S_2(f,h,b) = & \sum_{k=0}^{n-3} \frac{f^{(k+2)}(0)}{k!} \int_0^b S_2(G_{*,j}(.,s),h,b) s^k ds \\ & + \frac{1}{(n-3)!} \int_0^b S_2(G_{*,j}(.,s),h,b) \left(\int_0^s f^{(n)}(\xi)(s-\xi)^{n-3} d\xi \right) ds \end{aligned}$$

for $j = 1,2$.

(b) *If* $f'(0)=0$ *then*

$$\begin{aligned}
&S_2(f,h,b)+f(b)\\
&=\sum_{k=0}^{n-3}\frac{f^{(k+2)}(0)}{k!}\int_0^b S_2(G_{*,3}(.,s),h,b)s^k ds\\
&\quad+\frac{1}{(n-3)!}\int_0^b S_2(G_{*,3}(.,s),h,b)\left(\int_0^s f^{(n)}(\xi)(s-\xi)^{n-3}d\xi\right)ds.
\end{aligned}$$

(c)

$$\begin{aligned}
&S_2(f,h,b)+f(b)-bf'(b)\\
&=\sum_{k=0}^{n-3}\frac{f^{(k+2)}(0)}{k!}\int_0^b S_2(G_{*,4}(.,s),h,b)s^k ds\\
&\quad+\frac{1}{(n-3)!}\int_0^b S_2(G_{*,4}(.,s),h,b)\left(\int_0^s f^{(n)}(\xi)(s-\xi)^{n-3}d\xi\right)ds.
\end{aligned}$$

(d) *If* $f'(0)=0$ *then*

$$\begin{aligned}
&S_2(f,h,b)-bf'(b)\\
&=\sum_{k=0}^{n-3}\frac{f^{(k+2)}(0)}{k!}\int_0^b S_2(G_{*,5}(.,s),h,b)s^k ds\\
&\quad+\frac{1}{(n-3)!}\int_0^b S_2(G_{*,5}(.,s),h,b)\left(\int_0^s f^{(n)}(\xi)(s-\xi)^{n-3}d\xi\right)ds.
\end{aligned}$$

Proof. First, we prove for $j=1$, the proof of other cases is similar. By using (8) and (18) for f and f' respectively and using the assumption that $f(0)=0$, we have

$$\begin{aligned}
S_2(f,h,b) &= \int_0^b f'(t)h(t)\,dt - f\left(\int_0^b h(t)\,dt\right)\\
&= \int_0^b \frac{1}{b}f(b)h(t)\,dt + \int_0^b\left[\int_0^b G_{*,1_t}(t,s)f''(s)\,ds\right]h(t)\,dt\\
&\quad -\frac{\int_0^b h(t)\,dt}{b}f(b) - \int_0^b G_{*,1}\left(\int_0^b h(t)\,dt, s\right)f''(s)\,ds\\
&= \int_0^b S_2(G_{*,1}(.,s),h,b)f''(s)\,ds.
\end{aligned}$$

Further, by using $(n-3)$-rd order Taylor approximation for f'' and simplifying we get the required identities. □

In the next theorem we prove generalization of (6).

THEOREM 6. *Let* $n\in\mathbb{N}$ *with* $n\geqslant 3$ *and let* $f:[0,b]\to\mathbb{R}$ *be* n *times differentiable function with* $f(0)=0$ *and* h *be as in Corollary* 2 (a). *Then*

(a) *If f is n-convex, then*

(i) *For $j=1,2$, we have:*

$$S_2(f,h,b) \geqslant \sum_{k=0}^{n-3} \frac{f^{(k+2)}(0)}{k!} \int_0^b S_2(G_{*,j}(.,s),h,b)s^k ds.$$

(ii) *If $f'(0)=0$ then*

$$S_2(f,h,b)+f(b) \geqslant \sum_{k=0}^{n-3} \frac{f^{(k+2)}(0)}{k!} \int_0^b S_2(G_{*,3}(.,s),h,b)s^k ds.$$

(iii)

$$S_2(f,h,b)+f(b)-bf'(b) \geqslant \sum_{k=0}^{n-3} \frac{f^{(k+2)}(0)}{k!} \int_0^b S_2(G_{*,4}(.,s),h,b)s^k ds.$$

(iv) *If $f'(0)=0$ then*

$$S_2(f,h,b)-bf'(b) \geqslant \sum_{k=0}^{n-3} \frac{f^{(k+2)}(0)}{k!} \int_0^b S_2(G_{*,5}(.,s),h,b)s^k ds.$$

(b) *If $-f$ is n-convex, then for each j the reverse of inequality in part (a) holds.*

Proof. The proof can be obtained from Theorem 5 and Corollary 2 (a) on the same lines as Theorem 4 has been proved by using Theorem 3 and Corollary 1 (a). □

Now, we prove identities to obtain generalization of (7).

THEOREM 7. *Let $n \in \mathbb{N}$ with $n \geqslant 3$ and let $f:[0,b] \to \mathbb{R}$ be n times differentiable function with $f(0)=0$ and k and K be as in Corollary 3. If $h:[0,b] \to [0,+\infty)$ is integrable then:*

(a) *For $j=1,2$, we have*

$$S_3(f,h,k,b) = \sum_{k=0}^{n-3} \frac{f^{(k+2)}(0)}{k!} \int_0^b S_3(G_{*,j}(.,s),h,k,b)s^k\, ds + \frac{1}{(n-3)!} \int_0^b S_3(G_{*,j}(.,s),h,k,b) \left(\int_0^s f^{(n)}(\xi)(s-\xi)^{n-3} d\xi \right) ds.$$

(b) *If $f'(0)=0$ then*

$$\begin{aligned} &S_3(f,h,k,b)+f(b)\int_0^b k(x)\,dx \\ &= \sum_{k=0}^{n-3} \frac{f^{(k+2)}(0)}{k!} \int_0^b S_3(G_{*,3}(.,s),h,k,b)s^k\, ds \\ &+ \frac{1}{(n-3)!} \int_0^b S_3(G_{*,3}(.,s),h,k,b) \left(\int_0^s f^{(n)}(\xi)(s-\xi)^{n-3} d\xi \right) ds. \end{aligned}$$

(c)

$$\begin{aligned}
&S_3(f,h,k,b)+(f(b)-bf'(b))\int_0^b k(x)\,dx\\
=&\sum_{k=0}^{n-3}\frac{f^{(k+2)}(0)}{k!}\int_0^b S_3(G_{*,4}(.,s),h,k,b)s^k\,ds\\
&+\frac{1}{(n-3)!}\int_0^b S_3(G_{*,4}(.,s),h,k,b)\left(\int_0^s f^{(n)}(\xi)(s-\xi)^{n-3}d\xi\right)ds.
\end{aligned}$$

(d) *If* $f'(0)=0$ *then*

$$\begin{aligned}
&S_3(f,h,k,b)-bf'(b)\int_0^b k(x)\,dx\\
=&\sum_{k=0}^{n-3}\frac{f^{(k+2)}(0)}{k!}\int_0^b S_3(G_{*,5}(.,s),h,k,b)s^k\,ds\\
&+\frac{1}{(n-3)!}\int_0^b S_3(G_{*,5}(.,s),h,k,b)\left(\int_0^s f^{(n)}(\xi)(s-\xi)^{n-3}d\xi\right)ds.
\end{aligned}$$

Proof. We prove the result for $j=1$. The proof of other parts is similar. By using (8) and (18) for f and f' respectively, we have:

$$\begin{aligned}
S_3(f,h,k,b)=&\int_0^b K(t)f'(t)h(t)\,dt-\int_0^b k(x)f\left(\int_0^x h(t)\,dt\right)dx\\
=&\int_0^b K(t)h(t)\left[\frac{1}{b}f(b)+\int_0^b G_{*,1_t}(t,s)f''(s)\,ds\right]dt\\
&-\int_0^b k(x)\left[\frac{1}{b}f(b)\int_0^x h(t)\,dt+\int_0^b G_{*,1}\left(\int_0^x h(t)\,dt,s\right)f''(s)\,ds\right]dx\\
=&\frac{1}{b}f(b)\left[\int_0^b K(t)h(t)\,dt-\int_0^b k(x)\int_0^x h(t)\,dt\,dx\right]\\
&+\int_0^b K(t)h(t)\int_0^b G_{*,1_t}(t,s)f''(s)\,ds\,dt\\
&-\int_0^b k(x)\int_0^b G_{*,1}\left(\int_0^x h(t)\,dt,s\right)f''(s)\,ds\,dx.
\end{aligned}$$

Since $\int_0^b k(x)\int_0^x h(t)\,dt\,dx=\int_0^b h(t)\left(\int_t^b k(x)\,dx\right)dt=\int_0^b K(t)h(t)dt$, therefore $\int_0^b k(x)\int_0^x h(t)\,dt\,d$ $\int_0^b K(t)h(t)\,dt$

$$\begin{aligned}
S_3(f,h,k,b)=&\int_0^b\left[\int_0^b K(t)h(t)G_{*,1_t}(t,s)\,dt-\int_0^b k(x)G_{*,1}\left(\int_0^x h(t)\,dt,s\right)dx\right]f''(s)\,ds\\
=&\int_0^b S_3(G_{*,1}(.,s),h,k,b)f''(s)\,ds.
\end{aligned}$$

Rest follows from $(n-3)$-rd order Taylor approximation. □

Following theorem gives generalization of (7).

THEOREM 8. *Let $n\in\mathbb{N}$ with $n\geqslant 3$ and let $f:[0,b]\to\mathbb{R}$ be n times differentiable function with $f(0)=0$. Let k, K and h be as in Corollary 3 (a). Then*

(a) *If f is n-convex, then*

(i) *For $j=1,2$, we have*

$$S_3(f,h,k,b)\geqslant\sum_{k=0}^{n-3}\frac{f^{(k+2)}(0)}{k!}\int_0^b S_3(G_{*,j}(.,s),h,k,b)s^k\,ds.$$

(ii) *If $f'(0)=0$ then*

$$S_3(f,h,k,b)+f(b)\int_0^b k(x)\,dx\geqslant\sum_{k=0}^{n-3}\frac{f^{(k+2)}(0)}{k!}\int_0^b S_3(G_{*,3}(.,s),h,k,b)s^k\,ds.$$

(iii)

$$S_3(f,h,k,b)+(f(b)-bf'(b))\int_0^b k(x)\,dx\geqslant\sum_{k=0}^{n-3}\frac{f^{(k+2)}(0)}{k!}\int_0^b S_3(G_{*,4}(.,s),h,k,b)s^k\,ds.$$

(iv) *If $f'(0)=0$ then*

$$S_3(f,h,k,b)-bf'(b)\int_0^b k(x)\,dx\geqslant\sum_{k=0}^{n-3}\frac{f^{(k+2)}(0)}{k!}\int_0^b S_3(G_{*,5}(.,s),h,k,b)s^k\,ds.$$

(b) *If $-f$ is n-convex, then for each j the reverse of inequality in part (a) holds.*

Proof. Follows from Theorem 7 and Corollary 3 (a) in the similar way as Theorem 4 has been proved by using Theorem 3 and Corollary 1 (a). □

The next section contains the applications of these results to the theory of $(n+1)$-convex functions at a point.

3. Application to $(n+1)$-convex function at a point

The notion of $(n+1)$-convex function at a point was introduced in [10]. In the current section, we define some linear functionals by taking the difference of the left hand side and the right hand side of the inequalities in above section. By proving monotonicity of these functionals, we obtain new inequalities which contribute to theory of $(n+1)$-convex functions at a point. Following is the definition of $(n+1)$-convex function at point, see [10].

DEFINITION 1. Let $I \subseteq \mathbb{R}$ be an interval, $c \in I^0$ and $n \in \mathbb{N}$. A function $f : I \to \mathbb{R}$ is said to be $(n+1)$-convex at point c if there exists a constant K_c such that the function $F(x) = f(x) - K_c \frac{x^n}{n!}$ is n-concave on $I \cap (-\infty, c]$ and n-convex on $I \cap [c, \infty)$. A function f is said to be $(n+1)$ -concave at point c if the function $-f$ is $(n+1)$-convex at point c.

A function is $(n+1)$-convex on an interval if and only if it is $(n+1)$ -convex at each point of the interval (see [10]). Pečarić, Praljak and Witkowski in [10] studied necessary and sufficient conditions on two linear functionals $\Omega : C([\delta_1, c]) \to \mathbb{R}$ and $\Gamma : C([c, \delta_2]) \to \mathbb{R}$ so that the inequality $\Omega(f) \leqslant \Gamma(f)$ holds for every function f that is $(n+1)$-convex at point c. In this section, we define linear functionals from the inequalities proved in previous section and obtain such (as in [10]) results for these functionals.

Let $n \in \mathbb{N}$ with $n \geqslant 3$ and $f : [a,b] \to \mathbb{R}$ be n times differentiable function. Let $a_1 a_2 \in [a,b]$ and c be an interior point of $[a,b]$ such that $a_1 < c < a_2$. Let $g_1 : [a_1, c] \to \mathbb{R}$ and $g_2 : [c, a_2] \to \mathbb{R}$ be non-decreasing with $g_i(x) \leqslant x$ for $i = 1, 2$. For $j = 1, 2, \ldots, 5$, we define

$$\Omega_{1,j}(f) = S_1(f, g_1, a_1, c) - \sum_{k=0}^{n-3} \frac{f^{(k+2)}(a_1)}{k!} \int_{a_1}^{c} S_1(G_{*,j}(.,s), g_1, a_1, c)(s-a_1)^k \, ds$$

and

$$\Gamma_{1,j}(f) = S_1(f, g_2, c, a_2) - \sum_{k=0}^{n-3} \frac{f^{(k+2)}(c)}{k!} \int_{c}^{a_2} S_1(G_{*,j}(.,s), g_2, c, a_2)(s-c)^k \, ds.$$

Similarly let $c \in (0,b)$ and $b_1 \in (0,b]$ where $c < b_1$. Let $h_1 : [0,c] \to [0,+\infty)$ and $h_2 : [c, b_1] \to [0,+\infty)$ be as defined in Corollary 2 (a) (w.l.o.g. we may assume h_2 on $[0, b_1]$ by taking $h_2(t) = 0$ when $t \in [0,c]$). We define following pair of functionals:

(a)

$$\Omega_{2,j}(f) = S_2(f, h_1, c) - \sum_{k=0}^{n-3} \frac{f^{(k+2)}(0)}{k!} \int_0^c S_2(G_{*,j}(.,s), h_1, c) s^k ds$$

and

$$\Gamma_{2,j}(f) = S_2(f, h_2, b_1) - \sum_{k=0}^{n-3} \frac{f^{(k+2)}(0)}{k!} \int_c^{b_1} S_2(G_{*,j}(.,s), h_2, b_1) s^k ds$$

where $j = 1, 2$.

(b)

$$\Omega_{2,3}(f) = S_2(f, h_1, c) + f(c) - \sum_{k=0}^{n-3} \frac{f^{(k+2)}(0)}{k!} \int_0^c S_2(G_{*,3}(.,s), h_1, c) s^k ds$$

and

$$\Gamma_{2,3}(f) = S_2(f, h_2, b_1) + f(b_1) - \sum_{k=0}^{n-3} \frac{f^{(k+2)}(0)}{k!} \int_c^{b_1} S_2(G_{*,3}(.,s), h_2, b_1) s^k ds,$$

(c)

$$\Omega_{2,4}(f) = S_2(f,h_1,c) + f(c) - cf'(c) - \sum_{k=0}^{n-3} \frac{f^{(k+2)}(0)}{k!} \int_0^c S_2(G_{*,4}(.,s),h_1,c) s^k ds$$

and

$$\Gamma_{2,4}(f) = S_2(f,h_2,b_1) + f(b_1) - b_1 f'(b_1) - \sum_{k=0}^{n-3} \frac{f^{(k+2)}(0)}{k!} \int_c^{b_1} S_2(G_{*,4}(.,s),h_2,b_1) s^k ds,$$

(d)

$$\Omega_{2,5}(f) = S_2(f,h_1,c) - cf'(c) - \sum_{k=0}^{n-3} \frac{f^{(k+2)}(0)}{k!} \int_0^c S_2(G_{*,5}(.,s),h_1,c) s^k ds$$

and

$$\Gamma_{2,5}(f) = S_2(f,h_2,b_1) - b_1 f'(b_1) - \sum_{k=0}^{n-3} \frac{f^{(k+2)}(0)}{k!} \int_c^{b_1} S_2(G_{*,5}(.,s),h_2,b_1) s^k ds.$$

Lastly, we define

(a)

$$\Omega_{3,j}(f) = S_3(f,h_1,k,c) - \sum_{k=0}^{n-3} \frac{f^{(k+2)}(0)}{k!} \int_0^c S_3(G_{*,j}(.,s),h_1,k,c) s^k\, ds$$

and

$$\Gamma_{3,j}(f) = S_3(f,h_2,k,b_1) - \sum_{k=0}^{n-3} \frac{f^{(k+2)}(0)}{k!} \int_c^{b_1} S_3(G_{*,j}(.,s),h_2,k,b_1) s^k\, ds,$$

where $j = 1,2$.

(b)

$$\Omega_{3,3}(f) = S_3(f,h_1,k,c) + f(c) \int_0^c k(x)\,dx - \sum_{k=0}^{n-3} \frac{f^{(k+2)}(0)}{k!} \int_0^c S_3(G_{*,3}(.,s),h_1,k,c) s^k\, ds$$

and

$$\begin{aligned} \Gamma_{3,3}(f) &= S_3(f,h_2,k,b_1) + f(b_1) \int_c^{b_1} k(x)\,dx \\ &\quad - \sum_{k=0}^{n-3} \frac{f^{(k+2)}(0)}{k!} \int_c^{b_1} S_3(G_{*,3}(.,s),h_2,k,b_1) s^k\, ds. \end{aligned}$$

(c)

$$\Omega_{3,4}(f) = S_3(f,h_1,k,c) + (f(c) - cf'(c)) \int_0^c k(x)\,dx - \sum_{k=0}^{n-3} \frac{f^{(k+2)}(0)}{k!} \int_0^c S_3(G_{*,4}(.,s),h_1,k,c)s^k\,ds$$

and

$$\Gamma_{3,4}(f) = S_3(f,h_2,k,b_1) + (f(b_1) - b_1 f'(b_1)) \int_c^{b_1} k(x)\,dx - \sum_{k=0}^{n-3} \frac{f^{(k+2)}(0)}{k!} \int_c^{b_1} S_3(G_{*,4}(.,s),h_2,k,b_1)s^k\,ds.$$

(d)

$$\Omega_{3,5}(f) = S_3(f,h_1,k,c) - cf'(c) \int_0^c k(x)\,dx - \sum_{k=0}^{n-3} \frac{f^{(k+2)}(0)}{k!} \int_0^c S_3(G_{*,5}(.,s),h_1,k,c)s^k\,ds$$

and

$$\Gamma_{3,5}(f) = S_3(f,h_2,k,b_1) - b_1 f'(b_1) \int_c^{b_1} k(x)\,dx - \sum_{k=0}^{n-3} \frac{f^{(k+2)}(0)}{k!} \int_c^{b_1} S_3(G_{*,5}(.,s),h_2,k,b_1)s^k\,ds$$

where k is as defined in Corollary 3. If f is n-convex (and $f'(0) = 0$ for $j = 3$) then Theorem 4 (a), Theorem 6 (a) and Theorem 8 (a) implies $\Gamma_{1,j}(f) \geqslant 0$, $\Gamma_{2,j}(f) \geqslant 0$ and $\Gamma_{3,j}(f) \geqslant 0$ for $j = 1,2,\ldots,$ respectively. Moreover, if $-f$ is n-convex (and $f'(0) = 0$ for $j = 3$) then Theorem 4 (b), Theorem 6 (b) and Theorem 8 (b) implies $\Omega_{1,j}(f) \leqslant 0$, $\Omega_{2,j}(f) \leqslant 0$ and $\Omega_{3,j}(f) \leqslant 0$ for $j = 1,2,\ldots,5$, respectively.

THEOREM 9. *Let $n \in \mathbb{N}$ with $n \geqslant 3$ and let $f : [a,b] \to \mathbb{R}$ be $(n+1)$-convex at a point c in $[a,b]$. Let $g_1 : [a_1,c] \to \mathbb{R}$ and $g_2 : [c,a_2] \to \mathbb{R}$, where $a_1 \leqslant c \leqslant a_2$, be non-decreasing and differentiable functions. If $\Omega_{1,j}(\phi_0) = \Gamma_{1,j}(\phi_0)$, for all $j = 1,2,\ldots,5$ (and $f'(a) = 0$ for $j = 3$), where $\phi_0(x) = x^n$ then*

$$\Omega_{1,j}(f) \leqslant \Gamma_{1,j}(f),$$

for $j = 1,2,\ldots,5$.

Proof. Since f is $(n+1)$-convex at c so there exist K_c such that $F(x) = f(x) - \frac{K_c x^n}{n!}$ is n-concave (or $-F$ is n-convex) on $[a_1,c]$ and n-convex on $[c,a_2]$. Therefore for each $j = 1,2,\ldots,5$, we have $0 \geqslant \Omega_{1,j}(F) = \Omega_{1,j}(f) - \frac{K_c}{n!}\Omega_{1,j}(\phi_0)$. Moreover, since F is n-convex on $[c,a_2]$ therefore $0 \leqslant \Gamma_{1,j}(F) = \Gamma_{1,j}(f) - \frac{K_c}{n!}\Gamma_{1,j}(\phi_0)$. Since $\Omega_{1,j}(\phi_0) = \Gamma_{1,j}(\phi_0)$, therefore $\Omega_{1,j}(f) \leqslant \Gamma_{1,j}(f)$, which completes the proof. $\square$

THEOREM 10. *Let $n \in \mathbb{N}$ with $n \geqslant 3$ and let $h_1 : [0,c] \to [0,+\infty)$, $h_2 : [c,b_1] \to [0,+\infty)$, k and K be as defined in Corollary 3 (a). If $f : [0,b] \to \mathbb{R}$ is $(n+1)$-convex at a point c in $[0,b]$ and $\Omega_{l,j}(\phi_0) = \Gamma_{l,j}(\phi_0)$ then $\Omega_{l,j}(f) \leqslant \Gamma_{l,j}(f)$ for all $j = 1,2,\ldots,5$ and $l = 2,3$, where $f'(0) = 0$ for $j = 3$.*

Proof. Since f is $(n+1)$-convex at c so there exist K_c such that $F(x) = f(x) - \frac{K_c x^n}{n!}$ is n-concave (or $-F$ is n-convex) on $[0,c]$ and n-convex on $[c,b_1]$. Therefore, $0 \geqslant \Omega_{l,j}(F) = \Omega_{l,j}(f) - \frac{K_c}{n!}\Omega_{l,j}(\phi_0)$. On the other hand, since F is n-convex on $[c,b_1]$, therefore $0 \leqslant \Gamma_{l,j}(F) = \Gamma_{l,j}(f) - \frac{K_c}{n!}\Gamma_{l,j}(\phi_0)$. Since $\Omega_{l,j}(\phi_0) = \Gamma_{l,j}(\phi_0)$ therefore $\Omega_{l,j}(f) \leqslant \Gamma_{l,j}(f)$, which completes the proof. □

4. Further refinements

Theorem 4 can be refined further for some classes of functions, using exponential convexity (for details see [1, 2]). First, we use linear functional $\Omega_{1,j}$ define in previous section. Under assumptions of Theorem 4 (a), we conclude that, for any $n \in \mathbb{N}$ with $n \geqslant 3$ and for any $j \in \{1,2,\ldots,5\}$, $\Omega_{1,j}$ acts non-negatively on the class of n-convex functions.

Further, let us introduce a family of n-convex functions on $[0,\infty)$ with

$$\varphi_t(x) = \begin{cases} \frac{x^t}{t(t-1)\cdots(t-n+1)}, & t \notin \{0,1,\ldots,n-1\}; \\ \frac{x^j \ln x}{(-1)^{n-1-j} j!(n-1-j)!}, & t = j \in \{0,1,\ldots,n-1\}. \end{cases} \tag{25}$$

This is indeed family of n-convex functions since $\frac{d^n}{dx^n}\varphi_t(x) = x^{t-n} \geqslant 0$.

Since $t \mapsto x^{t-n} = e^{(t-n)\ln x}$ is exponentially convex function, the quadratic form

$$\sum_{i,k=1}^{l} \xi_i \xi_k \frac{d^n}{dx^n} \varphi_{\frac{p_i+p_k}{2}}(x) \tag{26}$$

is positively semi-definite. According Theorem 4 (a),

$$\sum_{i,k=1}^{s} \xi_i \xi_k \Omega_{1,j} \varphi_{\frac{p_i+p_k}{2}} \tag{27}$$

is also positively semi-definite, for any $s \in \mathbb{N}$, $\xi_i \in \mathbb{R}$ and $p_i \in \mathbb{R}$, concluding exponential convexity of the mapping $p \mapsto \Omega_{1,j}\varphi_p$. Specially, if we take $s = 2$ in (27) we have additionally that $p \mapsto \Omega_{1,j}\varphi_p$ is also log-convex mapping, property that we will need in the next theorem.

THEOREM 11. *Under assumptions of Theorem 4 (a) the following statements hold:*

(i) The mapping $p \mapsto \Omega_{1,j}\varphi_p$ is exponentially convex on $\mathbb{R}$.

(ii) For $p,q,r \in \mathbb{R}$ such that $p<q<r$, we have

$$\left(\Omega_{1,j}\varphi_q\right)^{r-p} \leqslant \left(\Omega_{1,j}\varphi_p\right)^{r-q}\left(\Omega_{1,j}\varphi_r\right)^{q-p}. \tag{28}$$

REMARK 2. We have outlined proof of the theorem in lines above. Second part of Theorem 11 is known as Lyapunov inequality, it follows from log -convexity, and it refines lower (upper) bound for action of the functional on the class of functions given in (25). This conclusion is a simple consequence of the fact that exponentially convex mappings are non-negative and if exponentially convex mapping attains zero value at some point it is zero everywhere (see [7]).

Similar estimation technique can be applied for classes of n -convex functions given in the paper [7]. Lastly, similar construction can be done for the linear functionals $\Omega_{2,j}$ and $\Omega_{3,j}$ to obtain inequalities given in Theorem 11 for these functionals.

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