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HERMITE–HADAMARD INEQUALITY FOR A FRUSTUM OF A SIMPLEX

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Dedicated to Academician Josip Pečarić on the occasion of his 70^{th} *birthday*

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Abstract. In this paper, we establish a refinement of the Hermite-Hadamard inequality for convex functions of several variables defined on a frustum of a simplex.

1. Introduction

Let $f\colon [a,b] \to \mathbb{R}$ be a convex function, then the inequality

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \leqslant \frac{f(a)+f(b)}{2} \tag{1.1}$$

is known as the Hermite-Hadamard inequality. The paper [1] gives some generalization of (1.1). It says that if $\Delta \subset \mathbb{R}^n$ is a simplex with barycenter b_Δ and vertices $x_0,\dots,x_n$ and $f\colon \Delta \to \mathbb{R}$ is convex, then

$$f(b_\Delta) \leqslant \frac{1}{\operatorname{Vol}\Delta}\int_\Delta f(x)\,\mathrm{d}x \leqslant \frac{f(x_0)+\ldots+f(x_n)}{n+1}, \tag{1.2}$$

where $\operatorname{Vol}\Delta$ denotes the volume of Δ.
The Hermite-Hadamard inequality has been extended to many other convex (and not only convex) bodies. For more details see the monograph [3], paper [5] and the references therein. In this paper we establish two types of Hermite-Hadamard inequality for a frustum of a simplex.

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2. Definitions and notations

We begin with some definitions and notations. For $x_0,\dots,x_n \in \mathbb{R}^n$ in general position the set $\Delta = \operatorname{conv}\{x_0,\dots,x_n\}$ is called a *simplex*. Every point $x \in \Delta$ admits a unique representation of the form

$$x = \alpha_0 x_0 + \ldots + \alpha_n x_n, \quad \alpha_i \geqslant 0, \quad \alpha_0 + \ldots + \alpha_n = 1.$$

The coefficients $(\alpha_0,\dots,\alpha_n)$ are called *barycentric coordinates* of x. The point

$$b_\Delta = \frac{1}{n+1}(x_0 + \ldots + x_n)$$

is called the *barycenter* of Δ.

Define a one-to-one mapping from the standard simplex $E_n = \{(\alpha_1,\dots,\alpha_n) : \alpha_i \geqslant 0,\ \alpha_1 + \ldots + \alpha_n \leqslant 1\}$ to Δ given by

$$\varphi(\alpha_1,\dots,\alpha_n) = (1-\alpha_1-\ldots-\alpha_n)x_0 + \alpha_1 x_1 + \ldots + \alpha_n x_n.$$

The following lemma holds.

LEMMA 2.1. *If $f\colon \Delta \to \mathbb{R}$ is a Riemann-integrable function, then*

$$\frac{1}{\operatorname{Vol}\Delta}\int_\Delta f(x)\,\mathrm{d}x = n!\int_{E_n} f(\varphi(\alpha))\,\mathrm{d}\alpha.$$

Proof. It is easy to see that the absolute value of the Jacobi determinant of φ equals $n!\operatorname{Vol}\Delta$ and the lemma follows from the change of variables formula. □

Without loss of generality we assume $x_0 = 0$. For $0 < t \leqslant 1$ let $\Delta_t = \operatorname{conv}\{tx_1,\dots,tx_n\}$. Given $0 < A < B \leqslant 1$ we shall call a *frustum of a simplex* the set $\Delta_{AB} = \operatorname{conv}(\Delta_A \cup \Delta_B) = \bigcup_{A\leqslant t\leqslant B}\Delta_t$. The sets Δ_A and Δ_B will be called the *upper* and *lower bases* of a frustum, and the point x_0 its *apex*. If $\Sigma \subset \mathbb{R}^k$ and $f\colon \Sigma \to \mathbb{R}$ is a Riemann-integrable function, then by

$$\operatorname{Avg}(f,\Sigma) = \frac{1}{\operatorname{Vol}\Sigma}\int_\Sigma f(x)\,\mathrm{d}x$$

we shall denote its average value over Σ.

Let us recall some inequalities.

THEOREM 2.1. *(Chebyshev's inequality, see [2]) If $f,g\colon [a,b] \to \mathbb{R}$ are two monotonic functions of the opposite monotonicity, then*

$$\frac{1}{b-a}\int_a^b f(x)g(x)\,\mathrm{d}x \leqslant \left(\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right)\left(\frac{1}{b-a}\int_a^b g(x)\,\mathrm{d}x\right).$$

If f and g are of the same monotonicity, then the above inequality works in the reverse way.

THEOREM 2.2. *(Steffensen's inequality, see [7]) Let $f,g:[a,b]\to\mathbb{R}$ be integrable functions such that f is decreasing and $0\leqslant g(x)\leqslant 1$ for $x\in[a,b]$. Then*

$$\int_{b-\lambda}^{b} f(x)\,\mathrm{d}x \leqslant \int_a^b f(x)g(x)\,\mathrm{d}x \leqslant \int_a^{a+\lambda} f(x)\,\mathrm{d}x,$$

where $\lambda=\int_a^b g(x)\,\mathrm{d}x$.

THEOREM 2.3. *(Grüss' inequality, see [4]) Let $f,g:[a,b]\to\mathbb{R}$ be integrable functions such that $\varphi\leqslant f(x)\leqslant\Phi$, $\gamma\leqslant g(x)\leqslant\Gamma$, for all $x\in[a,b]$, where $\varphi,\Phi,\gamma,\Gamma\in\mathbb{R}$. Then*

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)\,\mathrm{d}x-\frac{1}{(b-a)^2}\int_a^b f(x)\,\mathrm{d}x\int_a^b g(x)\,\mathrm{d}x\right|\leqslant\frac{(\Phi-\varphi)(\Gamma-\gamma)}{4}.$$

THEOREM 2.4. *(see [6]) Let f and g be real-valued, nonnegative and convex functions on $[a,b]$. Then*

$$\begin{aligned}2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)-\frac{1}{6}M(a,b)-\frac{1}{3}N(a,b)&\leqslant\frac{1}{b-a}\int_a^b f(x)g(x)\,\mathrm{d}x\\&\leqslant\frac{1}{3}M(a,b)+\frac{1}{6}N(a,b),\end{aligned}$$

where $M(a,b)=f(a)g(a)+f(b)g(b)$, $N(a,b)=f(a)g(b)+f(b)g(a)$.

3. Bounds by averages over upper and lower bases

The main theorem of this section is the following.

THEOREM 3.1. *If $f:\Delta_{AB}\to\mathbb{R}$ is convex, then*

$$\operatorname{Avg}(f,\Delta_{AB})\leqslant\alpha\operatorname{Avg}(f,\Delta_A)+(1-\alpha)\operatorname{Avg}(f,\Delta_B),$$

where $\alpha=\frac{1}{n+1}\frac{B}{B-A}-\frac{n}{n+1}\frac{A^n}{B^n-A^n}$.

Proof. Of course

$$\operatorname{Vol}\Delta_t=t^{n-1}\operatorname{Vol}\Delta_1 \text{ and } \operatorname{Vol}\Delta_{AB}=(B^n-A^n)\operatorname{Vol}\Delta. \tag{3.1}$$

For every point $x\in\Delta_{AB}$ the line passing through x_0 and x meets Δ_1 at the point $x_\alpha=\sum_{i=1}^n\alpha_i x_i$ and the bases of Δ_{AB} at points $A_\alpha=Ax_\alpha$ and $B_\alpha=Bx_\alpha$ respectively. So x is a convex combination of A_α and B_α that can be uniquely written as

$$x=tx_\alpha=\frac{B-t}{B-A}A_\alpha+\frac{t-A}{B-A}B_\alpha,$$

or equivalently

$$x=tx_\alpha=t\left(\alpha_1x_1+\ldots+\alpha_{n-1}x_{n-1}+(1-\alpha_1-\ldots-\alpha_{n-1})x_n\right),$$

where $A \leqslant t \leqslant B$, $\alpha_1, \ldots, \alpha_{n-1} > 0$ and $\alpha_1 + \ldots + \alpha_{n-1} \leqslant 1$.

Consider the mapping $\Phi\colon [A,B] \times E_{n-1} \to \Delta_{AB}$ given by the formula $\Phi(t,\alpha) = tx_\alpha$. The absolute value of its Jacobi determinant equals

$$\left\| \begin{matrix} x_n + \sum_{i=1}^{n-1} \alpha_i (x_i - x_n) \\ t(x_1 - x_n) \\ \vdots \\ t(x_{n-1} - x_n) \end{matrix} \right\| = t^{n-1} \left\| \begin{matrix} x_n \\ x_1 - x_n \\ \vdots \\ x_{n-1} - x_n \end{matrix} \right\| = t^{n-1} \left\| \begin{matrix} x_n \\ x_1 \\ \vdots \\ x_{n-1} \end{matrix} \right\| = n! t^{n-1} \operatorname{Vol} \Delta. \tag{3.2}$$

Using formula (3.2), the convexity of f and Lemma 2.1 we obtain

$$\begin{aligned} \frac{1}{n! \operatorname{Vol} \Delta} \int_{\Delta_{AB}} f(x) \,\mathrm{d}x = & \int_A^B \int_{E_{n-1}} f\left(\frac{B-t}{B-A} A_\alpha + \frac{t-A}{B-A} B_\alpha \right) t^{n-1} \,\mathrm{d}\alpha \,\mathrm{d}t \\ \leqslant & \frac{1}{B-A} \int_A^B (B-t) t^{n-1} \,\mathrm{d}t \int_{E_{n-1}} f(A_\alpha) \,\mathrm{d}\alpha \\ & + \frac{1}{B-A} \int_A^B (t-A) t^{n-1} \,\mathrm{d}t \int_{E_{n-1}} f(B_\alpha) \,\mathrm{d}\alpha \\ = & \frac{B(B^n - A^n) - n(B-A)A^n}{n(n+1)(B-A)} \cdot \frac{1}{(n-1)! \operatorname{Vol} \Delta_A} \int_{\Delta_A} f(x) \,\mathrm{d}x \\ & + \frac{n(B-A)B^n - A(B^n - A^n)}{n(n+1)(B-A)} \cdot \frac{1}{(n-1)! \operatorname{Vol} \Delta_B} \int_{\Delta_B} f(x) \,\mathrm{d}x. \end{aligned} \tag{3.3}$$

We complete the proof by dividing both sides of (3.3) by $B^n - A^n$ and taking into account (3.1). □

Setting $n = 2$ we obtain the result below.

COROLLARY 3.1. *If $PQRS$ is a trapezoid with $PQ \parallel RS$, $|PQ| = p$, $|RS| = r$ and f is a convex function, then*

$$\begin{aligned} \operatorname{Avg}(f, PQRS) & \leqslant \frac{1}{3} \left\{ \frac{2p+r}{p+r} \operatorname{Avg}(f, PQ) + \frac{p+2r}{p+r} \operatorname{Avg}(f, RS) \right\} \\ & \leqslant \frac{1}{6} \left\{ \frac{2p+r}{p+r} (f(P) + f(Q)) + \frac{p+2r}{p+r} (f(R) + f(S)) \right\}. \end{aligned}$$

4. Bounds by values on certain line segments

In this section we apply the Hermite-Hadamard inequalities to the sections of a frustum by hyperplanes parallel to its bases (that are also simplices). Applying (1.2) to Δ_t, we obtain

$$f(b_{\Delta_t}) \leqslant \operatorname{Avg}(f, \Delta_t) \leqslant \frac{1}{n} \sum_{i=1}^{n} f(tx_i). \tag{4.1}$$

Multiplying both sides of (4.1) by $\frac{\operatorname{Vol}\Delta_t}{\operatorname{Vol}\Delta_{AB}}$, integrating the resulting inequality over $[A,B]$, using (3.1) and taking into account that $\operatorname{Vol}\Delta = \frac{1}{n}h\operatorname{Vol}\Delta_1$, where h is the height of the simplex Δ from the apex x_0, we have

$$\frac{n}{(B^n-A^n)h}\int_A^B t^{n-1} f(b_{\Delta_t})\,\mathrm{d}t \leqslant \frac{\int_A^B \int_{\Delta_t} f(x)\,\mathrm{d}x\,\mathrm{d}t}{\operatorname{Vol}\Delta_{AB}} \leqslant \frac{1}{(B^n-A^n)h}\sum_{i=1}^n \int_A^B t^{n-1} f(tx_i)\,\mathrm{d}t. \tag{4.2}$$

Multiplying both sides of (4.2) by h, we get

$$\frac{n}{(B^n-A^n)}\int_A^B t^{n-1} f(b_{\Delta_t})\,\mathrm{d}t \leqslant \operatorname{Avg}(f,\Delta_{AB}) \leqslant \frac{1}{B^n-A^n}\sum_{i=1}^n \int_A^B t^{n-1} f(tx_i)\,\mathrm{d}t. \tag{4.3}$$

The factor t^{n-1} in both extreme integrals does not allow for a simple estimation of both sides by the mean values of f at the edges or on the line segment joining the barycenters of the bases. However under some additional assumptions on f we can obtain new upper bounds for the average value of f over the frustum of the simplex. In what follows the symbol Ax_iBx_i denotes the line segment between Ax_i and Bx_i.

COROLLARY 4.1. *Let f be a convex function defined on a simplex $\Delta = \operatorname{conv}\{x_0 = 0, x_1, , \ldots, x_n\}$. If additionally $f(tx_i)$ is a decreasing function for $t \in [A,B], i = 1,\ldots,n$, then*

$$\operatorname{Avg}(f,\Delta_{AB}) \leqslant \frac{1}{n}\sum_{i=1}^n \operatorname{Avg}(f, Ax_iBx_i). \tag{4.4}$$

Proof. Applying Chebyshev's inequality (Theorem 2.1) to the right-hand side of (4.3), we obtain

$$\operatorname{Avg}(f,\Delta_{AB}) \leqslant \frac{1}{(B-A)n}\sum_{i=1}^n \int_A^B f(tx_i)\,\mathrm{d}t. \tag{4.5}$$

Note that

$$\mathrm{d}t = \frac{(B-A)\,\mathrm{d}s_i}{|Ax_iBx_i|}, \tag{4.6}$$

where $\mathrm{d}s_i$ is an element of length of the i-th edge. From (4.5) and (4.6) we obtain (4.4). □

COROLLARY 4.2. *Under the assumptions of Corollary 4.1, we have*

$$\operatorname{Avg}(f,\Delta_{AB}) \leqslant \frac{1}{n}\sum_{i=1}^n \operatorname{Avg}(f, Ax_iCx_i),$$

where $C = A + \frac{B^n-A^n}{nB^{n-1}}$.

Proof. The result follows from the Steffensen's inequality (Theorem 2.2) applied to (4.3) with $g(t) = (t/B)^{n-1}$. $\square$

The Grüss inequality (Theorem 2.3) leads to.

COROLLARY 4.3. *Let f be a convex function defined on a simplex Δ with vertices $x_0 = 0, x_1, \ldots, x_n$. Then if $\varphi_i \leqslant f(tx_i) \leqslant \Phi_i$ for all $t \in [A,B]$ and $i = 1, \ldots, n$, then*

$$\operatorname{Avg}(f, \Delta_{AB}) \leqslant \sum_{i=1}^{n} \left(\frac{(\Phi_i - \varphi_i)(B^{n-1} - A^{n-1})(B-A)}{4(B^n - A^n)} + \frac{1}{n} \operatorname{Avg}(f, Ax_iBx_i) \right).$$

And finally by Theorem 2.4 we deduce the following result.

COROLLARY 4.4. *Let $f\colon \Delta = \operatorname{conv}\{x_0 = 0, x_1, \ldots, x_n\} \to \mathbb{R}$ be a convex function which is nonnegative on the line segment $Ax_iBx_i, i = 1, \ldots, n$. Under those conditions the following inequality is valid*

$$\operatorname{Avg}(f, \Delta_{AB}) \leqslant \frac{(B-A)(2A^{n-1} + B^{n-1})}{6(B^n - A^n)} \sum_{i=1}^{n} f(Ax_i) + \frac{(B-A)(A^{n-1} + 2B^{n-1})}{6(B^n - A^n)} \sum_{i=1}^{n} f(Bx_i).$$

We can use similar mechanisms to obtain the left-hand side bounds. The Chebyshev inequality gives.

COROLLARY 4.5. *Let f be a convex function defined on a simplex $\Delta = \operatorname{conv}\{x_0 = 0, x_1, \ldots, x_n\}$. If the function $t \mapsto f(tb_{\Delta_1})$ increases for $t \in [A,B]$, then*

$$\operatorname{Avg}(f, b_{\Delta_A} b_{\Delta_B}) \leqslant \operatorname{Avg}(f, \Delta_{AB}).$$

Applying the Steffensen inequality to the left-hand side of (4.3) we get the following.

COROLLARY 4.6. *Let f be a convex function defined on a simplex $\Delta = \operatorname{conv}\{x_0 = 0, x_1, \ldots, x_n\}$. If the function $t \mapsto f(tb_{\Delta_1})$ decreases for $t \in [A,B]$, then*

$$\operatorname{Avg}(f, b_{\Delta_D} b_{\Delta_B}) \leqslant \operatorname{Avg}(f, \Delta_{AB}),$$

where $D = B - \frac{B^n - A^n}{nB^{n-1}}$.

COROLLARY 4.7. *Let f be a convex function defined on a simplex $\Delta = \operatorname{conv}\{x_0 = 0, x_1, \ldots, x_n\}$ such that $\varphi \leqslant f(tb_{\Delta_1}) \leqslant \Phi$ for all $t \in [A,B]$, then*

$$\operatorname{Avg}\left(f, b_{\Delta_A} b_{\Delta_B}\right) - \frac{n(\Phi - \varphi)(B^{n-1} - A^{n-1})(B-A)}{4(B^n - A^n)} \leqslant \operatorname{Avg}(f, \Delta_{AB})$$

holds.

Theorem 2.4 applied to the left-hand side of (4.3) gives the following.

COROLLARY 4.8. *Let f be a convex function defined on a simplex $\Delta = \operatorname{conv}\{x_0 = 0, x_1, \dots, x_n\}$. Moreover, let f be a nonnegative function on the line segment $b_{\Delta_A} b_{\Delta_B}$. Then*

$$\frac{n(B-A)}{B^n - A^n}\left[2f\left(b_{\Delta_{\frac{A+B}{2}}}\right)\left(\frac{A+B}{2}\right)^{n-1} - \frac{A^{n-1}+2B^{n-1}}{6} f\left(b_{\Delta_A}\right) - \frac{2A^{n-1}+B^{n-1}}{6} f\left(b_{\Delta_B}\right)\right]$$

$$\leqslant \operatorname{Avg}(f, \Delta_{AB}).$$

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