Mathematical
Inequalities
& Applications
Volume 22, Number 4 (2019), 1327–1333 doi:10.7153/mia-2019-22-91

A CRITERION FOR THE CONVERSE OF THE HERMITE–HADAMARD INEQUALITY ON SIMPLICES

ELEUTHERIUS SYMEONIDIS

(*Communicated by J. Jakšetić*)

Abstract. It is already known that if a function satisfies the left or the right Hermite-Hadamard inequality for all simplices in its domain of definition, then it is convex, provided that the density of the measure is continuous and does not vanish identically on any segment. Here we show that this condition can be relaxed.

1. Introduction

If f is a continuous convex function on an interval $[a,b] \subset \mathbb{R}$, then the following inequalities hold:

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \leqslant \frac{f(a)+f(b)}{2}.$$

This fact was published by Charles Hermite in 1883 ([2]). Independently, ten years later Jacques Hadamard published the left inequality ([1]). Nowadays, both inequalities are termed *Hermite-Hadamard inequalities*.

In [5] it is shown that the Hermite-Hadamard inequalities are a special case of certain inequalities for a convex function defined on a metrizable compact convex subset K (which plays the role of the interval $[a,b]$) of a locally convex Hausdorff space.

In this note we shall restrict ourselves to the case where K will be an n-simplex in $\mathbb{R}^n$, that is, the convex hull of $n+1$ points that do not lie on one and the same hyperplane. In the sequel, this simplex is denoted by Δ. The integration is taken with respect to a (Borel) measure μ on Δ such that $0 < \mu(\Delta) < \infty$. Let $x = (x_1,\ldots,x_n) \in \Delta$, and let $h = (h_1,\ldots,h_n) : \Delta \to \Delta$ be the identity function, that is, $h_j(x) = x_j$ for $1 \leqslant j \leqslant n$. The midpoint $(a+b)/2$ of the interval $[a,b]$ is replaced by the so-called barycenter

$$b_\mu := \frac{1}{\mu(\Delta)}\int_\Delta h(x)\,\mathrm{d}\mu(x) := \frac{1}{\mu(\Delta)}\left(\int_\Delta h_1(x)\,\mathrm{d}\mu(x),\ldots,\int_\Delta h_n(x)\,\mathrm{d}\mu(x)\right)$$

of the measure μ on the simplex Δ. The above result is generalized as follows.

Mathematics subject classification (2010): 26B25.
Keywords and phrases: Convexity, Hermite-Hadamard inequality, simplex.

THEOREM 1. *Let $P_1,\dots,P_{n+1}$ be the vertices of an n-simplex $\Delta\subset\mathbb{R}^n$ $(n\geqslant 1)$, let μ be a measure on Δ such that $0<\mu(\Delta)<\infty$ and b_μ belongs to the interior of Δ, and let $\lambda_1,\dots,\lambda_{n+1}$ be positive numbers with $\sum_{j=1}^{n+1}\lambda_j=1$ such that $b_\mu=\sum_{j=1}^{n+1}\lambda_jP_j$. If $f:\Delta\to\mathbb{R}$ is a continuous and convex function, then*

$$f(b_\mu)\leqslant\frac{1}{\mu(\Delta)}\int_\Delta f(x)\,\mathrm{d}\mu(x)\leqslant\sum_{j=1}^{n+1}\lambda_jf(P_j)\,.$$

Since one can hardly find an elementary proof in the literature, we give here one for the sake of completeness.

Proof. Since b_μ belongs to the interior of Δ, and f is convex, there is a supporting hyperplane of f at b_μ, that is, there exists a linear functional $h:\mathbb{R}^n\to\mathbb{R}$ such that

$$f(x)\geqslant f(b_\mu)+h(x-b_\mu)$$

for every $x\in\Delta$ (see [4, section 3.3]). Therefore,

$$\begin{aligned}\frac{1}{\mu(\Delta)}\int_\Delta f(x)\,\mathrm{d}\mu(x)&\geqslant f(b_\mu)+\frac{1}{\mu(\Delta)}\int_\Delta h(x-b_\mu)\,\mathrm{d}\mu(x)\\&=f(b_\mu)+\frac{1}{\mu(\Delta)}h\left(\int_\Delta(x-b_\mu)\,\mathrm{d}\mu(x)\right)\end{aligned}$$

because h is linear,

$$=f(b_\mu)+h\left(\frac{1}{\mu(\Delta)}\int_\Delta(x-b_\mu)\,\mathrm{d}\mu(x)\right)=f(b_\mu)+h(0)=f(b_\mu).$$

For the right inequality we express every $x\in\Delta$ in the form $x=\sum_{j=1}^{n+1}\alpha_j(x)P_j$ with continuous nonnegative functions $\alpha_1,\dots,\alpha_{n+1}$ such that $\sum_{j=1}^{n+1}\alpha_j(x)\equiv 1$. It then holds:

$$b_\mu=\frac{1}{\mu(\Delta)}\int_\Delta x\,\mathrm{d}\mu(x)=\frac{1}{\mu(\Delta)}\sum_{j=1}^{n+1}\int_\Delta\alpha_j(x)\,\mathrm{d}\mu(x)\cdot P_j.$$

The uniqueness of the coefficients of P_j gives $\lambda_j=\frac{1}{\mu(\Delta)}\int_\Delta\alpha_j(x)\,\mathrm{d}\mu(x)$ for $1\leqslant j\leqslant n+1$.

Now we can apply the Jensen inequality on f and write:

$$\begin{aligned}\frac{1}{\mu(\Delta)}\int_\Delta f(x)\,\mathrm{d}\mu(x)&=\frac{1}{\mu(\Delta)}\int_\Delta f\left(\sum_{j=1}^{n+1}\alpha_j(x)P_j\right)\mathrm{d}\mu(x)\\&\leqslant\frac{1}{\mu(\Delta)}\int_\Delta\left[\sum_{j=1}^{n+1}\alpha_j(x)f(P_j)\right]\mathrm{d}\mu(x)=\sum_{j=1}^{n+1}\lambda_jf(P_j).\qquad\square\end{aligned}$$

In the sequel we shall call these inequalities *left Hermite-Hadamard inequality* and *right Hermite-Hadamard inequality* and denote them by (LHH) and (RHH), respectively.

The converse of the Hermite-Hadamard inequalities consists in the question whether these inequalities characterize a convex function (if they hold for every simplex in the domain of definition of the function and the measure is inherited from a given measure on this domain). For measures other than the Lebesgue measure, this question was first studied in [3]. There, the following theorem was proved.

THEOREM 2. *Let $D \subseteq \mathbb{R}^n$ $(n \geqslant 1)$ be a nonempty open convex set and μ a Borel measure on D such that $\mathrm{d}\mu(x) = p(x)\mathrm{d}x$, where $p : D \to [0,\infty)$ is continuous and $\{x \in D : p(x) = 0\}$ does not contain any nontrivial segment. Let $f : D \to \mathbb{R}$ be continuous.*

1. *If f satisfies (LHH) for all simplices $\Delta \subset D$, then f is convex.*
2. *If f satisfies (RHH) for all simplices $\Delta \subset D$, then f is convex.*

In this note we show that the condition that there exist no segment in $p^{-1}(\{0\})$ can be considerably relaxed.

2. The criterion

Our result states as follows.

THEOREM 3. *The previous theorem remains valid, when the condition*

> $p^{-1}(\{0\})$ *contains no nontrivial segment*

is replaced by the condition

> *There is a dense subset $S \subseteq D$, such that for every two points $a,b \in S$, $p^{-1}(\{0\}) \cap [a,b]$ is of (one-dimensional) Lebesgue measure zero in $[a,b]$.*

(We denote by $[a,b]$ the segment of endpoints a,b.)

Proof. Let a continuous $f : D \to \mathbb{R}$ satisfy (LHH) or (RHH), the same for all simplices in D. It suffices to show that for every two points $a,b \in S$, $f|_{[a,b]}$ is convex. For if, namely, $x,y \in D$ and $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are sequences in S converging to x and y, respectively, then for every $\lambda \in [0,1]$,

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &= \lim_{n\to\infty} f(\lambda a_n + (1-\lambda)b_n) \leqslant \lim_{n\to\infty} (\lambda f(a_n) + (1-\lambda)f(b_n)) \\ &= \lambda f(x) + (1-\lambda)f(y). \end{aligned}$$

We proceed by reductio ad absurdum and assume that there exist $a,b \in S$ and $\varepsilon \in (0,1)$ such that

$$f((1-\varepsilon)a + \varepsilon b) > (1-\varepsilon)f(a) + \varepsilon f(b).$$

Let

$$\varepsilon_1 = \inf\{t \in [0,\varepsilon] : \forall \tau \in [t,\varepsilon]\ f((1-\tau)a + \tau b) > (1-\tau)f(a) + \tau f(b)\},$$

$$\varepsilon_2 = \sup\{t \in [\varepsilon, 1] : \forall \tau \in [\varepsilon, t]\ f((1-\tau)a + \tau b) > (1-\tau)f(a) + \tau f(b)\}.$$

Since f is continuous, $\varepsilon_1 < \varepsilon < \varepsilon_2$. Let $c = (1-\varepsilon_1)a + \varepsilon_1 b$, $d = (1-\varepsilon_2)a + \varepsilon_2 b$. For every $\lambda \in (0, 1)$ we then have:

$$\begin{aligned} f((1-\lambda)c + \lambda d) &= f[((1-\lambda)(1-\varepsilon_1) + \lambda(1-\varepsilon_2))a + ((1-\lambda)\varepsilon_1 + \lambda\varepsilon_2)b] \\ &= f[(1-\varepsilon_1 + \lambda\varepsilon_1 - \lambda\varepsilon_2)a + (\varepsilon_1 - \lambda\varepsilon_1 + \lambda\varepsilon_2)b] \\ &> (1-\varepsilon_1 + \lambda\varepsilon_1 - \lambda\varepsilon_2)f(a) + (\varepsilon_1 - \lambda\varepsilon_1 + \lambda\varepsilon_2)f(b) \\ &= (1-\lambda)f(c) + \lambda f(d), \end{aligned} \tag{1}$$

because $f(c) = (1-\varepsilon_1)f(a) + \varepsilon_1 f(b)$ and $f(d) = (1-\varepsilon_2)f(a) + \varepsilon_2 f(b)$. This means that $f|_{[c,d]}$ is strictly concave.

Now let $v_1, \ldots, v_{n-1} \in \mathbb{R}^n$ be such that $\{v_1, \ldots, v_{n-1}, d-c\}$ is a basis of $\mathbb{R}^n$. We assume that $v_1, \ldots, v_{n-1}$ are small enough, so that $c + v_i \in D$ for $1 \leqslant i \leqslant n-1$. Let $P_{i,m} := c + \frac{1}{m}v_i$ for $m \in \mathbb{N}$ and $1 \leqslant i \leqslant n-1$, let Δ_m be the simplex of vertices $c, P_{1,m}, \ldots, P_{n-1,m}, d$.

Let $\Sigma_n = \{(t_1, \ldots, t_n) \in [0, 1]^n : \sum_{i=1}^n t_i \leqslant 1\}$. This standard simplex parametrizes Δ_m by the map $T_m : \Sigma_n \to \Delta_m$,

$$T_m(t_1, \ldots, t_n) = (1 - t_1 - \ldots - t_n)c + t_1 P_{1,m} + \ldots + t_{n-1}P_{n-1,m} + t_n d.$$

For a continuous function $h : \Delta_m \to \mathbb{R}$ it holds:

$$\frac{1}{\mu(\Delta_m)} \int_{\Delta_m} h(x)\,\mathrm{d}\mu(x) = \frac{\int_{\Delta_m} h(x)p(x)\,\mathrm{d}x}{\int_{\Delta_m} p(x)\,\mathrm{d}x} = \frac{\int_{\Sigma_n} h(T_m(t))p(T_m(t))\,\mathrm{d}t}{\int_{\Sigma_n} p(T_m(t))\,\mathrm{d}t}$$

(the Jacobian of T_m is constant and can be divided out)

$$\begin{aligned} &\xrightarrow{m\to\infty} \frac{\int_{\Sigma_n} h(c + t_n(d-c))p(c + t_n(d-c))\,\mathrm{d}t}{\int_{\Sigma_n} p(c + t_n(d-c))\,\mathrm{d}t} \\ &= \int_0^1 h(c + t_n(d-c)) \cdot \frac{p(c + t_n(d-c))(1-t_n)^{n-1}\,\mathrm{d}t_n}{\int_0^1 p(c + \tau_n(d-c))(1-\tau_n)^{n-1}\,\mathrm{d}\tau_n} \end{aligned}$$

(the integration over $t_1, \ldots, t_{n-1}$ comprises the homothetical image $(1-t_n)\Sigma_{n-1}$ and leads to the factor $(1-t_n)^{n-1} \cdot \mathrm{vol}(\Sigma_{n-1})$, whereas the volume $\mathrm{vol}(\Sigma_{n-1})$ of Σ_{n-1} is eventually divided out). Let ν be the push-forward (image) by $s \mapsto c + s(d-c)$ of the measure

$$p(c + s(d-c))(1-s)^{n-1}\mathrm{d}s$$

on $[0, 1]$. Then, the last integral can be expressed as

$$\frac{1}{\nu([c,d])} \int_{[c,d]} h(x)\,\mathrm{d}\nu(x).$$

If we replace h by the components $h_1, \ldots, h_n$ of the identity function on Δ_m (as in the introduction for Δ), we see that the barycenter $b_{\mu,m}$ of μ on Δ_m converges to the barycenter b_ν of ν (on $[c,d]$). From this point onwards we treat the two cases in the theorem separately.

— Case 1, f satisfies (LHH).

Letting $m\to\infty$ in

$$f(b_{\mu,m}) \leqslant \frac{1}{\mu(\Delta_m)} \int_{\Delta_m} f(x)\,\mathrm{d}\mu(x)$$

we obtain

$$f(b_\nu) \leqslant \frac{1}{\nu([c,d])} \int_{[c,d]} f(x)\,\mathrm{d}\nu(x)\,. \tag{2}$$

On the other hand, since $f|_{[c,d]}$ is concave, there exists a linear functional h such that

$$f(x) \leqslant f(b_\nu) + h(x-b_\nu)$$

for $x\in[c,d]$ (see [4, section 3.3]). Due to the strict concavity, this inequality holds strictly for an $x_0\in[c,d]$ and then for a whole neighborhood of x_0. As in the proof of theorem 1, we therefore obtain

$$\frac{1}{\nu([c,d])} \int_{[c,d]} f(x)\,\mathrm{d}\nu(x) < f(b_\nu)\,,$$

which contradicts (2).

— Case 2, f satisfies (RHH).

This part of the proof follows almost word-for-word the corresponding part in [3]. We include it here for the sake of completeness.

Let $\lambda_1^{(m)},\dots,\lambda_{n+1}^{(m)}$ be nonnegative numbers such that $\left(\sum_{j=1}^{n-1}\lambda_j^{(m)}\right)+\lambda_n^{(m)}+\lambda_{n+1}^{(m)}=1$ and

$$b_{\mu,m} = \left(\sum_{j=1}^{n-1}\lambda_j^{(m)}P_{j,m}\right) + \lambda_n^{(m)}c + \lambda_{n+1}^{(m)}d$$

for $m\in\mathbb{N}$. Let $(m_k)_{k\in\mathbb{N}}$ be a sequence, for which all $\lambda_j^{(m_k)}$ converge, and set

$$\lambda_j^\infty = \lim_{k\to\infty}\lambda_j^{(m_k)} \qquad \text{for } 1\leqslant j\leqslant n+1\,.$$

It then holds:

$$\begin{aligned} b_\nu = \lim_{k\to\infty} b_{\mu,m_k} &= \lim_{k\to\infty}\left[\left(\sum_{j=1}^{n-1}\lambda_j^{(m_k)}P_{j,m_k}\right) + \lambda_n^{(m_k)}c + \lambda_{n+1}^{(m_k)}d\right] \\ &= \left(\sum_{j=1}^{n-1}\lambda_j^\infty c\right) + \lambda_n^\infty c + \lambda_{n+1}^\infty d = (1-\lambda_{n+1}^\infty)c + \lambda_{n+1}^\infty d\,. \end{aligned}$$

Now, since f satisfies (RHH), we have

$$\frac{1}{\mu(\Delta_m)} \int_{\Delta_m} f(x)\,\mathrm{d}\mu(x) \leqslant \left(\sum_{j=1}^{n-1}\lambda_j^{(m)}f(P_{j,m})\right) + \lambda_n^{(m)}f(c) + \lambda_{n+1}^{(m)}f(d)$$

for $m \in \mathbb{N}$. For $m = m_k \to \infty$ we obtain

$$\begin{aligned}\frac{1}{\nu([c,d])}\int_{[c,d]} f(x)\,\mathrm{d}\nu(x) &\leqslant \left(\sum_{j=1}^{n-1}\lambda_j^\infty f(c)\right)+\lambda_n^\infty f(c)+\lambda_{n+1}^\infty f(d)\\ &=(1-\lambda_{n+1}^\infty)f(c)+\lambda_{n+1}^\infty f(d)\,.\end{aligned}\tag{3}$$

On the other hand we have

$$\begin{aligned}(1-\lambda_{n+1}^\infty)c+\lambda_{n+1}^\infty d = b_\nu &= \frac{1}{\nu([c,d])}\int_{[c,d]} x\,\mathrm{d}\nu(x)\\ &=\int_0^1 [(1-s)c+sd]\cdot\frac{p((1-s)c+sd)(1-s)^{n-1}}{\int_0^1 p((1-\tau)c+\tau d)(1-\tau)^{n-1}\mathrm{d}\tau}\,\mathrm{d}s,\end{aligned}$$

so $\lambda_{n+1}^\infty = \int_0^1 \frac{s\cdot p((1-s)c+sd)(1-s)^{n-1}}{\int_0^1 p((1-\tau)c+\tau d)(1-\tau)^{n-1}\mathrm{d}\tau}\,\mathrm{d}s$. Thus, due to the assumption on p, it follows from (1) that

$$\begin{aligned}&\frac{1}{\nu([c,d])}\int_{[c,d]} f(x)\,\mathrm{d}\nu(x)\\ =&\int_0^1 f((1-s)c+sd)\cdot\frac{p((1-s)c+sd)(1-s)^{n-1}}{\int_0^1 p((1-\tau)c+\tau d)(1-\tau)^{n-1}\mathrm{d}\tau}\,\mathrm{d}s\\ >&\int_0^1 [(1-s)f(c)+sf(d)]\cdot\frac{p((1-s)c+sd)(1-s)^{n-1}}{\int_0^1 p((1-\tau)c+\tau d)(1-\tau)^{n-1}\mathrm{d}\tau}\,\mathrm{d}s\\ =&(1-\lambda_{n+1}^\infty)f(c)+\lambda_{n+1}^\infty f(d)\,,\end{aligned}$$

which contradicts (3). □

A case in which theorem 3 applies, but not theorem 2, is when the density p vanishes on (the intersection of D with) finitely many affine hyperplanes. Yet, even the weaker condition of theorem 3 does not seem to be the weakest possible. And in any case the question of the characterization of those measures μ on D (not necessarily absolutely continuous with respect to the Lebesgue measure) that allow the statements (1) and (2) of theorem 2 remains open.

REFERENCES

[1] J. HADAMARD, *Étude sur les propriétés des fonctions entières et en particulier d' une fonction considérée par Riemann*, Journ. de Math., **(4) IX** (1893), 171–215.

[2] CH. HERMITE, *Sur deux limites d' une intégrale définie*, Mathesis, **3** (1883), 82.

[3] F. - C. MITROI, E. SYMEONIDIS, *The converse of the Hermite-Hadamard inequality on simplices*, Expo. Math., **30** (2012), 389–396.

[4] C. P. NICULESCU, L. - E. PERSSON, *Convex Functions and Their Applications. A Contemporary Approach. Second Edition*, CMS Books in Mathematics (Canadian Mathematical Society), Springer, 2018.

[5] R. R. PHELPS, *Lectures on Choquet's Theorem*, D. van Nostrand Company Inc., Princeton, 1966.

(Received October 15, 2018)

Eleutherius Symeonidis
Katholische Universität Eichstätt-Ingolstadt
Mathematisch-Geographische Fakultät
85071 Eichstätt, Germany
e-mail: e.symeonidis@ku.de