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QUASI–CONVEX FUNCTIONS OF HIGHER ORDER

JACEK MROWIEC AND TERESA RAJBA*

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Abstract. We introduce and investigate the notions of n-quasi-convex as well as strongly n-quasi-convex functions with modulus $c>0$. We give characterizations of these functions, which are counterparts of those given for quasi-convex and strongly n-convex functions. We introduce and investigate the notions of n-quasi-concave and n-quasi-affine functions, as well as strongly n-quasi-concave and strongly n-quasi-affine functions. We also give a generalization of higher order quasi-convex functions introduced by E. Popoviciu (1982).

1. Introduction and preliminaries

Throughout this paper $\mathbb{N}$, $\mathbb{R}$, and I will denote the sets of all positive integers, real numbers, and a non-degenerate subinterval of $\mathbb{R}$ (an interval is degenerate if it is either empty or a singleton). In the whole paper we assume that $c \geqslant 0$ is a given number. By the standard definition (cf. [9, 17]), a real valued function $f\colon I \to \mathbb{R}$ is called *convex* if $f(tx+(1-t)y) \leqslant tf(x)+(1-t)f(y)$ for all $t \in (0,1)$ and $x,y \in I$. If c is a positive real number, f is called *strongly convex* with modulus c if $f(tx+(1-t)y) \leqslant tf(x)+(1-t)f(y)-ct(1-t)(x-y)^2$ for all $t \in (0,1)$ and $x,y \in I$. Strongly convex functions have been introduced by Polyak [11]. They play an important role in optimization theory and mathematical economics. In the classical theory of convex functions their natural generalization are convex functions of higher order. We recall the definition. Let $n \in \mathbb{N}$ and $x_0,\dots,x_n$ be the distinct points in I. Denote by $[x_0,\dots,x_n;f]$ the divided difference of f defined by the recurrence $[x_0;f]=f(x_0)$, $[x_0,\dots,x_n;f]=(x_n-x_0)^{-1}([x_1,\dots,x_n;f]-[x_0,\dots,x_{n-1};f])$. Following Hopf [5] and Popoviciu [14, 15] a function $f\colon I \to \mathbb{R}$ is called *convex of order n* (or *n-convex*) if $[x_0,\dots,x_{n+1};f] \geqslant 0$ for all $x_0<\dots<x_{n+1}$ in I. A function $f\colon I \to \mathbb{R}$ is called *strongly convex of order n* (or *strongly n-convex*) with modulus $c>0$ if $[x_0,\dots,x_{n+1};f] \geqslant c$ for all $x_0<\dots<x_{n+1}$ in I (cf. [3, 16]). The class of *quasi-convex* functions f on I is defined (cf. [1, 4, 12, 17]) as consisting of those functions which satisfy $f(tx+(1-t)y) \leqslant \max\{f(x),f(y)\}$ for all $t \in (0,1)$ and $x,y \in I$. This notion occurred to be very useful in mathematical economics (for more information and further references see [1]).

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* Corresponding author.

In this paper, we introduce and investigate the notion of strongly quasi-convex, strongly quasi-concave and strongly quasi-affine function. Let us note, that presented in this paper the notion of strongly quasi-convex function differs from that of Korablev [8] (see also [6]) and coincides with that given in [19]. However our notions of strongly quasi-concave and strongly quasi-affine function differ from that given in [19]. We introduce and investigate the notion of n-quasi-convex, n-quasi-concave and n-quasi-affine functions, as well as strongly n-quasi-convex, strongly n-quasi-concave and strongly n-quasi-affine functions. We give a characterization of these functions, which is a counterpart of that given for quasi-convex functions (cf. [1, 4]) and strongly convex functions of higher order (cf. [3, 16]). We also define and study (n,k)-quasi-convex functions as a generalization of higher order quasi-convex functions given by E. Popoviciu [13].

2. Strongly quasi-convex functions

The notion of quasi-convexity (cf. [1, 4, 12, 17]) is a generalization of the convexity.

DEFINITION 2.1. We say that a function $f\colon I\to\mathbb{R}$ is

(i) *quasi-convex* if $f(tx+(1-t)y)\leqslant\max\{f(x),f(y)\}$ for $t\in(0,1)$, $x,y\in I$,

(ii) *quasi-concave* if $f(tx+(1-t)y)\geqslant\min\{f(x),f(y)\}$ for $t\in(0,1)$, $x,y\in I$,

If the function $f\colon I\to\mathbb{R}$ is simultaneously quasi-convex and quasi-concave, then we say that f is *quasi-affine*. For functions $f\colon I\to\mathbb{R}$ quasi-affinity means monotonicity.

REMARK 2.1. A function f is quasi-concave if, and only if, $-f$ is quasi-convex. A function $f\colon I\to\mathbb{R}$ is quasi-convex, if and only, if $f(u)-f(x)\leqslant 0$ or $f(y)-f(u)\geqslant 0$, for all $x,y,u\in I$ such that $x<u<y$.

Given sets I_1, $I_2\subset\mathbb{R}$, we write $I_1<I_2$ if $x_1<x_2$ for all $x_1\in I_1$, $x_2\in I_2$. The following proposition can be found, e.g., in [1, Theorem 2.5.1]

PROPOSITION 2.1. *A function $f\colon I\to\mathbb{R}$ is*

(i) *quasi-convex if, and only if, there exist (possibly degenerate) intervals I_1, I_2, $I_1<I_2$ such that $I_1\cup I_2=I$, $f|_{I_1}$ is non-increasing and $f|_{I_2}$ is non-decreasing,*

(ii) *quasi-concave if, and only if, there exist (possibly degenerate) intervals I_1, I_2, $I_1<I_2$ such that $I_1\cup I_2=I$, $f|_{I_1}$ is non-decreasing and $f|_{I_2}$ is non-increasing,*

Remark 2.1 motivates the introduction of the following notion of strongly quasi-convex, quasi-concave and quasi-affine functions.

DEFINITION 2.2. Let $c>0$. We say that a function $f\colon I\to\mathbb{R}$ is

(i) *strongly quasi-convex with modulus* c if for all $t \in (0,1)$ and $x,y \in I$ such that $x < y$, $\quad f(tx+(1-t)y) \leqslant \max\{f(x) - c(1-t)(y-x), f(y) - ct(y-x)\}$,

(ii) *strongly quasi-concave with modulus* c if for all $t \in (0,1)$ and $x,y \in I$ such that $x < y$, $\quad f(tx+(1-t)y) \geqslant \min\{f(x) + c(1-t)(y-x), f(y) + ct(y-x)\}$,

(iii) *strongly quasi-affine with modulus* c if f is strongly quasi-convex and strongly quasi-concave with modulus c.

REMARK 2.2. It is easy to see that the function f is strongly quasi-concave with modulus c if, and only if, $-f$ is strongly quasi-convex with modulus c. The definition of strongly quasi-convex functions with $c = 0$ gives the concept of quasi-convex functions. Obviously, if f is strongly quasi-convex with $c > 0$, then f is quasi-convex.

It is not difficult to prove the following propositions.

PROPOSITION 2.2. *Let $f\colon I \to \mathbb{R}$ be a function and $c > 0$. Then the following statements are equivalent:*

(i) *f is strongly quasi-convex with modulus c,*

(ii) *for all $t \in (0,1)$ and $x,y \in I$ such that $x < y$*

$$\frac{f(tx+(1-t)y) - f(x)}{(1-t)(y-x)} \leqslant -c \quad \text{or} \quad \frac{f(y) - f(tx+(1-t)y)}{t(y-x)} \geqslant c,$$

(iii) *for all $x \in I$ and $h_1, h_2 > 0$ such that $x + h_1 + h_2 \in I$*

$$\frac{f(x+h_1) - f(x)}{h_1} \leqslant -c \quad \text{or} \quad \frac{f(x+h_1+h_2) - f(x+h_1)}{h_2} \geqslant c,$$

(iv) *for all $x_0, x_1, x_2 \in I$ such that $x_0 < x_1 < x_2$, $[x_0,x_1;f] \leqslant -c$ or $[x_1,x_2;f] \geqslant c$.*

PROPOSITION 2.3. *Let I_0 be a non-degenerate subinterval of I. If a function $f\colon I \to \mathbb{R}$ is quasi-convex (strongly quasi-convex with modulus c), then $f|_{I_0}$ is quasi-convex (strongly quasi-convex with modulus c).*

We define strongly increasing functions, which are a counterpart of strongly convex functions.

DEFINITION 2.3. Let $c > 0$. We say that a function $f\colon I \to \mathbb{R}$ is

(i) *strongly increasing with modulus* c, if $[x_0,x_1;f] \geqslant c$ ($x_0, x_1 \in I$, $x_0 < x_1$),

(ii) *strongly decreasing with modulus* c, if $[x_0,x_1;f] \leqslant -c$ ($x_0, x_1 \in I$, $x_0 < x_1$),

(iii) *strongly monotone with modulus* c, if f is strongly increasing or strongly decreasing with modulus c.

We will prove that strong quasi-affinity means strong monotonicity (Theorem 2.3).

REMARK 2.3. Let $c>0$. Obviously for the function $g(x)=cx$ $(x\in I)$, $[x_0,x_1;g]$ $=c$ for $x_0,x_1\in I$, $x_0<x_1$. Consequently, we have that the function f is strongly increasing with modulus c if, and only if, $[x_0,x_1;f-g]\geqslant 0$ for $x_0,x_1\in I$, $x_0<x_1$. This implies that, f is strongly increasing with modulus c if, and only if, the function $f-g$ is non-decreasing, in other words $f(x)=f_0(x)+cx$, where $f_0\colon I\to\mathbb{R}$ is a non-decreasing function. Similarly, we can prove, that f is strongly decreasing with modulus c if, and only if, $f(x)=f_0(x)-cx$, where $f_0\colon I\to\mathbb{R}$ is a non-increasing function.

THEOREM 2.1. *Let $f\colon(a,b)\to\mathbb{R}$ be a function and $c>0$. Then f is strongly increasing with modulus c if, and only if, f is increasing and $f'(x)\geqslant c$ for $x\in(a,b)$ λ a.e. (λ denotes the Lebesgue measure).*

Proof. Assume that $[x_0,x_1;f]\geqslant c$ for all $a<x_0<x_1<b$. For the function $g(x)=cx$ $(x\in(a,b))$, we have that $[x_0,x_1;g]=c$ for $a<x_0<x_1<b$ and $g'(x)=c$ $(x\in(a,b))$. Then we have $[x_0,x_1;f-g]\geqslant 0$ for all $a<x_0<x_1<b$, which implies that the function $f-g$ is non-decreasing on (a,b), and consequently $(f-g)'(x)\geqslant 0$ for $x\in(a,b)$ λ a.e. ([18]). Taking into account that $g'(x)=c$, we obtain $f'(x)\geqslant c$ for $x\in(a,b)$ λ a.e.

Now let us assume that f is increasing on (a,b) and $f'(x)\geqslant c$ for $x\in(a,b)$ λ a.e. Since f is increasing on (a,b), it can be regarded as a distribution function corresponding to a σ-finite measure. Via Lebesgue's decomposition theorem and the decomposition of a singular measure, every σ-finite measure can be decomposed into a sum of an absolutely continuous measure (with respect to the Lebesgue measure), a singular continuous measure, and a discrete measure (these three measures are uniquely determined [18]). Then f can be written in the form $f=f_d+f_{sc}+f_{ac}$, where f_d, f_{sc}, f_{ac} are the distribution functions corresponding to the discrete part, the singular continuous part and the absolutely continuous part, respectively. Obviously, the distribution functions f_d, f_{sc}, f_{ac} are non-decreasing functions, $f_d'(x)=f_{sc}'(x)=0$ and $f'(x)=f_{ac}'(x)$ for $x\in(a,b)$ λ a.e. This implies that $f_{ac}'(x)\geqslant c$ and taking into account $g'(x)=c$, we obtain $f_{ac}'(x)-g'(x)\geqslant 0$ for $x\in(a,b)$ λ a.e. Then

$$\Delta_h(f_{ac}(x)-g(x))=\int_x^{x+h}(f_{ac}'(u)-g'(u))\,du\geqslant 0 \tag{2.1}$$

for all $a<x<x+h<b$. By (2.1), we have $\Delta_h(f_{ac}(x)-g(x))=[f_{ac}(x+h)-g(x+h)]-[f_{ac}(x)-g(x)]\geqslant 0$ $(a<x<x+h<b)$, or equivalently $[x_0,x_1;f_{ac}-g]\geqslant 0$ $(a<x_0<x_1<b)$, which implies $[x_0,x_1;f_{ac}]\geqslant[x_0,x_1;g]=c$ $(a<x_0<x_1<b)$. Since f_d and f_{sc} are non-decreasing functions, it follows $[x_0,x_1;f_d]\geqslant 0$ and $[x_0,x_1;f_{sc}]\geqslant 0$. Consequently, we obtain $[x_0,x_1;f]=[x_0,x_1;f_d]+[x_0,x_1;f_{sc}]+[x_0,x_1;f_{ac}]\geqslant c$ $(a<x_0<x_1<b)$. The theorem is proved. $\square$

COROLLARY 2.1. *Let $f\colon I\to\mathbb{R}$ be a function and $c>0$. Then f is strongly decreasing with modulus c if, and only if, f is non-increasing and $f'(x)\leqslant -c$ for $x\in I$ λ a.e.*

The following proposition follows immediately from the definition of strong quasi-convexity.

PROPOSITION 2.4. *Let $f: I \to \mathbb{R}$ be a function and $c > 0$. If f is non-decreasing, then f is strongly quasi-convex with modulus c if, and only if, f is strongly increasing with modulus c. If f is non-increasing, then f is strongly quasi-convex with modulus c if, and only if, f is strongly decreasing with modulus c.*

THEOREM 2.2. *A function $f: I \to \mathbb{R}$ is strongly quasi-convex with modulus c if, and only if, there exist (possibly degenerate) intervals I_1, I_2, $I_1 < I_2$, $I_1 \cup I_2 = I$, such that: (i) $f|_{I_1}$ is strongly decreasing with modulus c, (ii) $f|_{I_2}$ is strongly increasing with modulus c.*

Proof. Assume that f is strongly quasi-convex with modulus c. By Remark 2.2, f is quasi-convex. By Proposition 2.1, there exist (possibly degenerate) intervals I_1, I_2, $I_1 < I_2$ such that $I_1 \cup I_2 = I$, $f|_{I_1}$ is non-increasing and $f|_{I_2}$ is non-decreasing. From Propositions 2.4 and 2.3, it follows that the conditions (i) and (ii) are satisfied.

Since the converse follows immediately from the definition of strong quasi-convexity, the theorem is proved. □

COROLLARY 2.2. *A function $f: I \to \mathbb{R}$ is strongly quasi-concave with modulus c if, and only if, there exist (possibly degenerate) intervals J_1, J_2, $J_1 < J_2$, $J_1 \cup J_2 = I$, such that: (i) $f|_{J_1}$ is strongly increasing with modulus c, (ii) $f|_{J_2}$ is strongly decreasing with modulus c.*

COROLLARY 2.3. *Let $f: I \to \mathbb{R}$ be a quasi-convex function. Then f is strongly quasi-convex with modulus $c > 0$ if, and only if, $|f'(x)| \geqslant c$ for $x \in I$ λ a.e.*

THEOREM 2.3. *Let $c > 0$. A function $f: I \to \mathbb{R}$ is strongly quasi-affine with modulus c if, and only if, f is strongly monotone with modulus c.*

Proof. It follows from Theorem 2.2, Corollary 2.2 and Remark 2.3, that if the function $f: I \to \mathbb{R}$ is strongly monotone with modulus $c > 0$, then f is strongly quasi-affine with modulus $c > 0$.

Now let us assume that $f: I \to \mathbb{R}$ is strongly quasi-affine with modulus $c > 0$. Then f is quasi-affine, which implies that f is monotone. Then from Theorem 2.2 and Corollary 2.2 it follows, that f is strongly monotone with modulus c. The theorem is proved. □

As an immediate consequence of Theorem 2.3 and Remark 2.3, we obtain the following characterization of strong quasi-affinity.

THEOREM 2.4. *A function $f: I \to \mathbb{R}$ is strongly quasi-affine with modulus $c > 0$ if, and only if, it has one of the following forms: (i) $f(x) = f_0(x) + cx$, $(x \in I)$, where $f_0: I \to \mathbb{R}$ is a non-decreasing function, (ii) $f(x) = f_0(x) - cx$, $(x \in I)$, where $f_0: I \to \mathbb{R}$ is a non-increasing function.*

Based on Korablev's definition [8] (cf. also [6]), a function $f\colon I\to\mathbb{R}$ is said to be strongly quasi-convex (Korablev strongly quasi-convex), if for all $t\in(0,1)$ and $x,y\in I$

$$f(tx+(1-t)y)\leqslant \max\{f(x),f(y)\}-ct(1-t)(x-y)^2. \tag{2.2}$$

It is not difficult to prove the following proposition on the Korablev strongly quasi-convex functions.

PROPOSITION 2.5. *Let $f\colon I\to\mathbb{R}$ be a function. Then the following statements are equivalent:*

(i) $f(tx+(1-t)y)\leqslant \max\{f(x),f(y)\}-ct(1-t)(x-y)^2$, *for all* $t\in(0,1)$, $x,y\in I$,

(ii) *for all* $t\in(0,1)$ *and* $x,y\in I$ *such that* $x<y$

$$\frac{f(tx+(1-t)y)-f(x)}{(1-t)(y-x)t(y-x)}\leqslant -c \quad or \quad \frac{f(y)-f(tx+(1-t)y)}{t(y-x)(1-t)(y-x)}\geqslant c,$$

(iii) *for all* $x\in I$ *and* $h_1,h_2>0$ *such that* $x+h_1+h_2\in I$

$$\frac{f(x+h_1)-f(x)}{h_1h_2}\leqslant -c \quad or \quad \frac{f(x+h_1+h_2)-f(x+h_1)}{h_2h_1}\geqslant c.$$

Let us note, that introduced in Definition 2.2 the notion of strongly quasi-convex function differs from that of Korablev [8] (the formula (2.2)).

EXAMPLE 2.1. Let $c>0$. As an immediate corollary from Propositions 2.2 and 2.5, we obtain that the function $f(x)=c|x|$ $(x\in\mathbb{R})$ is strongly quasi-convex with modulus c (in the sense of Definition 2.2) and it is not Korablev strongly quasi-convex with modulus c.

EXAMPLE 2.2. Let $c>0$. The function $f(x)=\frac{c}{2}x^2$ $(x\in(-1,1))$ is not strongly quasi-convex with modulus c (Definition 2.2) and it is Korablev strongly quasi-convex with modulus c. Indeed, since $f(x)$ is quasi-convex and $|f'(x)|<c$ $(x\in(-1,1))$, by Proposition 2.2, it follows that $f(x)$ is not strongly quasi-convex with modulus c. Obviously, since $f''(x)=c$ $(x\in(-1,1))$, f is strongly convex (see [3, 16]). Further, taking into account that, if the function g is a quadratic function, then g is strongly convex if, and only if, it is Korablev strongly quasi-convex (see [7]), we conclude that f is Korablev strongly quasi-convex.

A similar approach to strong quasi-convexity was applied in [19], where the notion of ω-quasi-convexity was introduced.

Let $\omega\geqslant 0$ be a given number. A function $f\colon I\to\mathbb{R}$ is

(i) ω-quasi-convex if for all $t\in(0,1)$ and $x,y\in I$, $x\neq y$

$$f(tx+(1-t)y)\leqslant \max\{f(x),f(y)\}-\omega\min(t,1-t)|x-y|, \tag{2.3}$$

(ii) ω-quasi-concave if for all $t \in (0,1)$ and $x,y \in I$, $x \neq y$

$$f(tx+(1-t)y) \geqslant \max\{f(x), f(y)\} - \omega \max(t, 1-t)|x-y|, \tag{2.4}$$

(iii) ω-quasi-affine if f is ω-quasi-convex and ω-quasi-concave.

In [19], the authors proved (among others), that ω-quasi-convex functions can be separated from ω-quasi-concave by ω-quasi-affine ones.

It is not difficult to prove the following proposition on ω-quasi-convexity.

PROPOSITION 2.6. *Let $f: I \to \mathbb{R}$ be a function. Then the following statements are equivalent:*

(i) *for all $t \in (0,1)$ and $x,y \in I$, $x \neq y$*

$$f(tx+(1-t)y) \leqslant \max\{f(x), f(y)\} - \omega \min(t, 1-t)|x-y|,$$

(ii) *for all $t \in (0,1)$ and $x,y \in I$, $x < y$*

$$\frac{f(tx+(1-t)y)-f(x)}{\min(t,(1-t))(y-x)} \leqslant -\omega \quad or \quad \frac{f(y)-f(tx+(1-t)y)}{\min(t,(1-t))(y-x)} \geqslant \omega,$$

(iii) *for all $x \in I$ and $h_1, h_2 > 0$ such that $x+h_1+h_2 \in I$*

$$\frac{f(x+h_1)-f(x)}{\min(h_1,h_2)} \leqslant -\omega \quad or \quad \frac{f(x+h_1+h_2)-f(x+h_1)}{\min(h_1,h_2)} \geqslant \omega.$$

It can be proven that the strong quasi-convexity with modulus c is equivalent to the ω-quasi-convexity (for $c=\omega$). The notions of strongly quasi-concave and strongly quasi-affine functions with modulus c, given in this paper, differ from that given in [19]. In particular, we have that the function f is strongly quasi-concave with modulus c if, and only if, $-f$ is strongly quasi-convex with modulus c. On the other hand, if f is ω-quasi-concave, then $-f$ is not necessarily ω-quasi-convex. If the inequality (2.4) defining ω-quasi-concavity would be replaced by the following inequality

$$f(tx+(1-t)y) \geqslant \min\{f(x), f(y)\} + \omega \min(t, 1-t)|x-y|, \tag{2.5}$$

then we have that if f would be ω-quasi-concave, then $-f$ would be ω-quasi-convex. However, if we replace the inequality (2.4) by (2.5), the separation type result is no longer true.

In our opinion, the properties of the strong quasi-convexity can be proved simpler and shorter using the definition of strong quasi-convexity with modulus c than ω-quasi-convexity (see Proposition 2.4, Theorem 2.2). Moreover, our definition enable us to generalize the strong-quasi convexity to the strong-quasi-convexity of higher order.

Note, that the condition (2.3) for $t=\frac{1}{2}$ was studied in [20]. It follows from Theorem 2.2 [20] that there are no ω-quasi-convex functions with $\omega > 0$ on convex domain of dimension greater then one (obviously in multidimensional case "$|\ |$" is replaced by "$\|\ \|$").

3. Quasi-convex and quasi-concave functions of higher order

We begin with some notations. Let I_0 be a subinterval of I. We denote $I_0^- = \{x \in I \colon \{x\} < I_0\}$, $I_0^+ = \{x \in I \colon I_0 < \{x\}\}$. Then $I_0^- < I_0 < I_0^+$ and $I_0^- \cup I_0 \cup I_0^+ = I$. Proposition 2.2 motivates the introduction of quasi-convex functions of higher order.

DEFINITION 3.1. We say that a function $f \colon I \to \mathbb{R}$ is

(i) *n-quasi-convex*, if $[y_n,\dots,y_0;f] \leqslant 0$ or $[x_0,\dots,x_n;f] \geqslant 0$,

(ii) *n-quasi-concave*, if $[y_n,\dots,y_0;f] \geqslant 0$ *or* $[x_0,\dots,x_n;f] \leqslant 0$

for all $y_n,\dots,y_0,x_0,\dots,x_n \in I$, $y_n < \dots < y_0 = x_0 < \dots < x_n$. The function f is *n-quasi-affine*, if it is simultaneously n-quasi-convex and n-quasi-concave.

The following theorem gives a characterization of quasi-convex functions of higher order, which generalizes that given in Proposition 2.1 for quasi-convex functions.

THEOREM 3.1. *Let $n \in \mathbb{N}$. Let $f \colon (a,b) \to \mathbb{R}$ $(a<b)$ be a function. Then f is $(n+1)$-quasi-convex on (a,b) if, and only if, one of the following conditions holds:*

(a) *f is n-convex or f is n-concave,*

(b) *there exists $x_0 \in (a,b)$, such that $f|_{(a,x_0]}$ is n-concave and $f|_{(x_0,b)}$ is n-convex,*

(c) *there exists $x_0 \in (a,b)$, such that $f|_{(a,x_0)}$ is n-concave and $f|_{[x_0,b)}$ is n-convex,*

(d) *there exists a non-degenerate interval $I_0 \subset (a,b)$ such that f is n-affine on I_0 and*

(W1) *f is n-concave on I_0^-,*

(W2) *f is n-convex on I_0^+,*

(W3) *$[y_{n+1},\dots,y_0;f] \leqslant 0$ or $[x_0,\dots,x_{n+1};f] \geqslant 0$ for all $\xi \in I_0$, $y_{n+1},\dots,y_0,x_0,\dots,x_{n+1} \in I$, $y_{n+1} < \dots < y_0 = \xi = x_0 < \dots < x_{n+1}$.*

Proof. Aiming for a contradiction, we suppose that $\neg[(a)\vee(b)\vee(c)\vee(d)]$, which is equivalent to $\neg(a)\wedge\neg(b)\wedge\neg(c)\wedge\neg(d)$.

Assume $\neg(b)$ (the other cases are analogous), which is equivalent to the condition

$$\forall \xi \in (a,b)\, \neg\,(f \text{ is } n-\text{concave on } (a,\xi]\,) \vee \neg\,(f \text{ is } n-\text{convex on } (\xi,b)\,). \tag{3.1}$$

By (3.1), we conclude that, for any $\xi \in (a,b)$, one of the following two conditions is satisfied: $\exists\ a < y_{n+1} < \dots < y_0 \leqslant \xi \quad [y_{n+1},\dots,y_0;f] > 0$, or $\exists\ \xi < x_0 < \dots < x_{n+1} < b \quad [x_0,\dots,x_{n+1};f] < 0$. We put

$$L_f = \{\xi \colon \exists\ a < y_{n+1} < \dots < y_0 \leqslant \xi \quad [y_{n+1},\dots,y_0;f] > 0\}, \tag{3.2}$$

$$R_f = \{\xi \colon \exists\ \xi < x_0 < \dots < x_{n+1} < b \quad [x_0,\dots,x_{n+1};f] < 0\}. \tag{3.3}$$

By (3.1), $L_f \cup R_f = (a,b)$. There are four possible cases: (A) $L_f = (a,b) \wedge R_f = \emptyset$, (B) $L_f = \emptyset \wedge R_f = (a.b)$, (C) $L_f \neq \emptyset \wedge R_f \neq \emptyset \wedge L_f \cap R_f = \emptyset$, (D) $L_f \cap R_f \neq \emptyset$.

We consider the case (A). Then we have: $\xi \in (a,b) \Longrightarrow \{ \xi \in L_f \wedge \xi \notin R_f \} \Longrightarrow \{ \exists\ a < y_{n+1} < \ldots < y_0 \leqslant \xi \quad [y_{n+1},\ldots,y_0;f] > 0 \ \wedge\ \forall\, \xi < x_0 < \ldots < x_{n+1} < b \quad [x_0,\ldots,x_{n+1};f] \geqslant 0 \} \Longrightarrow \{ f$ is n-convex on $(\xi,b) \}$. Consequently, we obtain, that for all $\xi \in (a,b)$, f is n-convex on (ξ,b), which implies that f is n-convex on (a,b). This contradicts the assumption $\neg$(a). Similarly, in the case (B), we obtain that f is n-concave on (a,b), contrary to the assumption $\neg$(a).

Now consider the case (C), i.e. $L_f \neq \emptyset$, $R_f \neq \emptyset$ and $L_f \cap R_f = \emptyset$. By the definitions of L_f and R_f, (3.2), (3.3), it follows that, if $\xi_1 \in L_f$, then $\xi_1' \in L_f$ for any $\xi_1' \geqslant \xi_1$, and if $\xi_2 \in R_f$, then $\xi_2' \in R_f$ for any $\xi_2' \leqslant \xi_2$. This implies

$$\xi_1 \in L_f \Longrightarrow [\xi_1,b) \subset L_f, \quad \xi_2 \in R_f \Longrightarrow (a,\xi_2] \subset R_f. \tag{3.4}$$

Since $L_f \cap R_f = \emptyset$, it follows that if $\xi \in L_f$, then $\xi \notin R_f$. This implies that, if $\xi \in L_f$, then for any $\xi < x_0 < \ldots < x_{n+1} < b \quad [x_0,\ldots,x_{n+1};f] \geqslant 0$, i.e. f is n-convex on (ξ,b). Similarly, if $\xi \in R_f$, then $\xi \notin L_f$, which implies that, if $\xi \in R_f$, then for any $a < y_{n+1} < \ldots < y_0 \leqslant \xi \quad [y_{n+1},\ldots,y_0;f] \leqslant 0$, i.e. f is n-concave on $(a,\xi]$. We have

$$\forall \xi_1,\xi_2 \in (a,b)\ \xi_1 \in L_f \Rightarrow f|_{(\xi_1,b)} \text{ is } n\text{-convex}, \xi_2 \in R_f \Rightarrow f|_{(a,\xi_2]} \text{ is } n\text{-concave}. \tag{3.5}$$

Because $L_f \neq \emptyset$ and $R_f \neq \emptyset$, there exist $\xi_1 \in L_f$ and $\xi_2 \in R_f$. We put

$$\alpha = \inf\{ \xi : \ \xi \in L_f \}, \quad \beta = \sup\{ \xi : \ \xi \in R_f \}. \tag{3.6}$$

By (3.5), we conclude that

$$f \text{ is } n\text{-concave on } (a,\beta) \quad \text{and} \quad f \text{ is } n\text{-convex on } (\alpha,b). \tag{3.7}$$

By (3.4),

$$(a,\beta) \subset R_f, \quad (\alpha,b) \subset L_f. \tag{3.8}$$

Moreover, we have $\alpha = \beta$. Indeed, suppose that $\alpha \neq \beta$. If $\alpha < \beta$, then $(\alpha,\beta) \subset L_f \cap R_f$, contrary to $L_f \cap R_f = \emptyset$. If $\alpha > \beta$, then by (3.4) and (3.6), we obtain that $(\beta,\alpha) \subset (a,b) \setminus [L_f \cup R_f]$, which contradicts the assumption $L_f \cup R_f = (a,b)$. Consequently, we have $\alpha = \beta$. In view of (3.7) we obtain

$$f \text{ is } n\text{-concave on } (a,\alpha) \quad \text{and} \quad f \text{ is } n\text{-convex on } (\alpha,b). \tag{3.9}$$

If $\alpha \in R_f$, then by (3.5), f is n-concave on $(a,\alpha]$ and f is n-convex on (α,b), this contradicts the assumption $\neg$(b). We conclude that $\alpha \in L_f$, i.e. there exist $a < y_{n+1} < \ldots < y_0 \leqslant \alpha$ such that $[y_{n+1},\ldots,y_0;f] > 0$, which implies that f is not n- concave on $(a,\alpha]$. Since f is n-concave on (a,α), it follows that $y_0 = \alpha$.

Suppose, that f is not n-convex on $[\alpha,b)$. Taking into account, that by (3.9), f is n-convex on (α,b), it follows that there exist $\alpha = x_0 < \ldots < x_{n+1} < b$, such that $[x_0,\ldots,x_{n+1};f] < 0$. Because $a < y_{n+1} < \ldots < y_0 = \alpha = x_0 < \ldots < x_{n+1} < b$, $[y_{n+1},\ldots,y_0;f] > 0$ and $[x_0,\ldots,x_{n+1};f] < 0$, this contradicts the assumption that f is

$(n+1)$-quasi-convex on (a,b). Thus f is n-convex on $[\alpha,b)$. Taking into account that by (3.9) f is n-concave on (a,α), we obtain a contradiction with $\neg(c)$.

We consider the case (D), i.e. $L_f \cap R_f \neq \emptyset$. It suffices to prove that

$$\begin{aligned} &\exists\ \xi_0 \in (a,b)\ \exists\ a < \lambda_{n+1} < \ldots < \lambda_0 = \xi_0 = \eta_0 < \ldots < \eta_{n+1} < b \\ &\qquad [\lambda_{n+1},\ldots,\lambda_0;f] > 0\ \wedge\ [\eta_0,\ldots,\eta_{n+1};f] < 0, \end{aligned} \tag{3.10}$$

because (3.10) contradicts the assumption that f is $(n+1)$-quasi-convex on (a,b).

By the equality $L_f \cap R_f \neq \emptyset$, we obtain that

$$\begin{aligned} &\exists\ \xi_1 \in (a,b)\ \exists\ a < y_{n+1} < \ldots < y_0 \leqslant \xi_1 < x_0 < \ldots < x_{n+1} < b \\ &\qquad [y_{n+1},\ldots,y_0;f] > 0\ \ \wedge\ \ [x_0,\ldots,x_{n+1};f] < 0. \end{aligned} \tag{3.11}$$

Then, there are three possible cases: (D1) $\exists\, y_0 = \tau_0 < \ldots < \tau_{n+1} = x_0 \quad [\tau_0,\ldots,\tau_{n+1};f] > 0$, (D2) $\exists\, y_0 = \tau_0 < \ldots < \tau_{n+1} = x_0 \quad [\tau_0,\ldots,\tau_{n+1};f] < 0$, (D3) $\forall\, y_0 = \tau_0 < \ldots < \tau_{n+1} = x_0 \quad [\tau_0,\ldots,\tau_{n+1};f] = 0$. In the case (D1), we have $a < \tau_0 < \ldots < \tau_{n+1} = x_0 < \ldots < x_{n+1} < b \quad [\tau_0,\ldots,\tau_{n+1};f] > 0\ \wedge\ [x_0,\ldots,x_{n+1};f] < 0$, which implies that (3.10) is satisfied with $\xi_0 = \tau_{n+1} = x_0$. In the case (D2), we have $a < y_{n+1} < \ldots < y_0 = \tau_0 < \ldots < \tau_{n+1} < b \ \ [y_{n+1},\ldots,y_0;f] > 0 \ \ \wedge \ \ [\tau_0,\ldots,\tau_{n+1};f] < 0$, consequently, (3.10) is satisfied with $\xi_0 = \tau_0 = y_0$. In the case (D3), we have that f is n-affine on $[y_0,x_0]$ $(y_0 < x_0)$, which means that f is n-convex and n-concave on $[y_0,x_0]$. Suppose that $L_f \cap R_f$ is a one-point set, say $L_f \cap R_f = \{\xi_2\}$. Then, by (3.4), $\xi_2 = \alpha = \beta$, where α, β are defined by (3.6). Thus, $L_f = [\alpha,b)$, $R_f = (a,\alpha]$. Then, by (3.5), f is n-convex on (α,b) and f is n-concave on $(a,\alpha]$, which contradicts the assumption $\neg$(b). In the remaining cases, when $L_f \cap R_f$ is not a one-point set, by (3.8), $L_f \cap R_f$ is a non-degenerate interval. Without loss of generality, we may assume that $L_f \cap R_f = (\alpha,\beta)$, and consequently $L_f = (\alpha,b)$, $R_f = (a,\beta)$. Then f is n-concave on $R_f \setminus L_f = (a,\alpha]$ and f is n-convex on $L_f \setminus R_f = [\beta,b)$. Let $I_0 \supset [y_0,x_0]$ be the largest interval, on which f is n-affine. Without loss of generality we may assume that $I_0 = (a_1,b_1)$. Note, that now the condition $\neg$(d) is equivalent to $\neg(W1) \vee \neg(W2) \vee \neg(W3)$. If the condition $\neg$(W1) holds, then f is not n-concave on $(a,a_1]$. Taking into account that f is n-concave on $(a,\alpha]$, we conclude that $\alpha < a_1$. This implies that there exist $\alpha < y_0' \leqslant a_1$ and $y_{n+1}' < \ldots < y_0'$ such that $[y_{n+1}',\ldots,y_0';f] > 0$. Moreover, there exist $y_0' = x_0' < \ldots < x_{n+1}' = x_0$ such that $[x_0',\ldots,x_{n+1}';f] \neq 0$. Indeed, if for all $y_0' = x_0' < \ldots < x_{n+1}' = x_0\ [x_0',\ldots,x_{n+1}';f] = 0$, then f is n-affine on $[y_0',x_0]$, consequently also on $[y_0',b_1)$. Since $y_0' \leqslant a_1$, this contradicts the assumption that $I_0 = (a_1,b_1)$ is the largest interval on which f is n affine. Consequently, there exist $y_0' = x_0' < \ldots < x_{n+1}' = x_0$ such that $[x_0',\ldots,x_{n+1}';f] \neq 0$. If $[x_0',\ldots,x_{n+1}';f] > 0$, then we have $x_0' < \ldots < x_{n+1}' = x_0 < \ldots < x_{n+1}$, $[x_0',\ldots,x_{n+1}';f] > 0$ and in view of (3.11) $[x_0,\ldots,x_{n+1};f] < 0$, which implies that (3.10) is satisfied. If $[x_0',\ldots,x_{n+1}';f] < 0$, then we have $y_{n+1}' < \ldots < y_0' = x_0' < \ldots < x_{n+1}'$, $[y_{n+1}',\ldots,y_0';f] > 0$ and $[x_0',\ldots,x_{n+1}';f] < 0$, consequently, (3.10) is satisfied. Analogously we prove, that assuming $\neg$(W2), we obtain (3.10). Now consider $\neg$(W3). Then there exist $\xi_2 \in I_0$ and $y_{n+1}' < \ldots < y_0' = \xi_2 = x_0' < \ldots < x_{n+1}'$ such that $[y_{n+1}',\ldots,y_0';f] > 0$ and $[x_0',\ldots,x_{n+1}';f] < 0$, thus (3.10) is satisfied. Since the converse is obvious, the theorem is proved. $\square$

Theorem 3.1 can be rewritten as follows.

THEOREM 3.2. *A function* $f\colon I\to\mathbb{R}$ *is* $(n+1)$*-quasi-convex if, and only if there exist (possibly degenerate) intervals* I_1, I_0, I_2, $I_1<I_0<I_2$ *such that* $I_1\cup I_0\cup I_2=I$, $f|_{I_1}$ *is* n*-concave,* $f|_{I_2}$ *is* n*-convex,* $f|_{I_0}$ *is* n*-affine and* $[y_{n+1},\dots,y_0;f]\leqslant 0$ *or* $[x_0,\dots,x_{n+1};f]\geqslant 0$ *for all* $\xi\in I_0$, $y_{n+1},\dots,y_0,x_0,\dots,x_{n+1}\in I$, $y_{n+1}<\dots<y_0=\xi=x_0<\dots<x_{n+1}$.

THEOREM 3.3. *A function* $f\colon I\to\mathbb{R}$ *is* $(n+1)$*-quasi-concave if, and only if, there exist (possibly degenerate) intervals* J_1, J_0, J_2, $J_1<J_0<J_2$ *such that* $J_1\cup J_0\cup J_2=I$, $f|_{J_1}$ *is* n*-convex,* $f|_{J_2}$ *is* n*-concave,* $f|_{J_0}$ *is* n*-affine and* $[y_{n+1},\dots,y_0;f]\geqslant 0$ *or* $[x_0,\dots,x_{n+1};f]\leqslant 0$ *for all* $\xi\in I_0$, $y_{n+1},\dots,y_0,x_0,\dots,x_{n+1}\in I$, $y_{n+1}<\dots<y_0=\xi=x_0<\dots<x_{n+1}$.

Let $\chi_B(x)=1$ if $x\in B$ and $\chi_B(x)=0$ if $x\notin B$ $(B\subset\mathbb{R})$.

EXAMPLE 3.1. The function $f(x)=(x+2)\,\chi_{(-\infty,-2)}(x)+(-x)\,\chi_{[-2,2]}(x)+(x-2)\,\chi_{(2,\infty)}(x)\quad(x\in\mathbb{R})$ is 2-quasi-convex. The function $g(x)=(-x-2)\,\chi_{(-\infty,-2)}(x)+x\,\chi_{[-2,2]}(x)+(-x+2)\,\chi_{(2,\infty)}(x)\quad(x\in\mathbb{R})$ is 2-quasi-concave.

4. Quasi-affine functions of higher order

Applying characterizations of $(n+1)$-quasi-convex and $(n+1)$-quasi-concave functions, we can give a characterization of $(n+1)$-quasi-affine functions.

THEOREM 4.1. *A function* $f\colon I\to\mathbb{R}$ *is (n+1)-quasi-affine if, and only if, one of the following conditions holds:*

(i) f *is* n*-convex or* n*-concave or*

(ii) *there exist intervals* I_1, I_0, I_2, $I_1<I_0<I_2$, $I_1\cup I_0\cup I_2=I$, *such that* $f|_{I_1}$, $f|_{I_2}$, $f|_{I_0}$ *are* n*-affine, and*

- $[y_{n+1},\dots,y_0;f]=0$ *or* $[x_0,\dots,x_{n+1};f]=0$ *or*
- $([y_{n+1},\dots,y_0;f]<0$ *and* $[x_0,\dots,x_{n+1};f]<0)$ *or*
- $([y_{n+1},\dots,y_0;f]>0$ *and* $[x_0,\dots,x_{n+1};f]>0)$

for all $\xi\in I_0$, $y_{n+1},\dots,y_0,x_0,\dots,x_{n+1}\in I$, $y_{n+1}<\dots<y_0=\xi=x_0<\dots<x_{n+1}$.

Proof. It follows from the definition of $(n+1)$-quasi-affine functions, that functions satisfying the above conditions are $(n+1)$-quasi-affine. Assume, that the function $f\colon I\to\mathbb{R}$ is $(n+1)$-quasi-affine. Then by Theorems 3.2 and 3.3 there exist intervals $\widetilde{I}_1$, $\widetilde{I}_0$, $\widetilde{I}_2$, $\widetilde{I}_1<\widetilde{I}_0<\widetilde{I}_2$, $\widetilde{I}_1\cup\widetilde{I}_0\cup\widetilde{I}_2=I$ and intervals J_1, J_0, J_2, $J_1<J_0<J_2$, $J_1\cup J_0\cup J_2=I$, that satisfy the conditions given in Theorems 3.2 and 3.3, respectively. If $\widetilde{I}_1=\widetilde{I}_0=J_0=J_2=\emptyset$ or $\widetilde{I}_0=\widetilde{I}_2=J_1=J_0=\emptyset$, which means that f is n-convex or n-concave, respectively, then the condition (i) is satisfied. If $\widetilde{I}_0=J_0=\emptyset$,

then three cases may occur. In the first case $\widetilde{I_1} = J_1$ and $\widetilde{I_2} = J_2$. Since $f|_{\widetilde{I_1}}$ is n-concave and $f|_{J_1}$ is n-convex, it follows that $f|_{\widetilde{I_1}}$ is n-affine. Similarly, we conclude that $f|_{\widetilde{I_2}}$ is n-affine. We obtain, that the condition (ii) is satisfied with $I_0 = \emptyset$, $I_1 = \widetilde{I_1} = J_1$ and $I_2 = \widetilde{I_2} = J_2$. In the second case ($\widetilde{I_1} \subset J_1$ and $J_1 \setminus \widetilde{I_1} = \{\xi\}$) or ($J_1 \subset \widetilde{I_1}$ and $\widetilde{I_1} \setminus J_1 = \{\xi\}$). In this case the condition (ii) is satisfied with $I_0 = \{\xi\}$, $I_1 = J_1$ and $I_2 = \widetilde{I_2}$. In the third case ($\widetilde{I_1} \subset J_1$ and $int(J_1 \setminus \widetilde{I_1}) \neq \emptyset$) or ($J_1 \subset \widetilde{I_1}$ and $int(\widetilde{I_1} \setminus J_1) \neq \emptyset$). In this case the condition (i) is satisfied. Now let us assume that all the intervals $\widetilde{I_1}$, $\widetilde{I_0}$, $\widetilde{I_2}$, J_1, J_0, J_2 are non-degenerate. It can be proved that $int(\widetilde{I_0} \cap J_0) \neq \emptyset$ (we leave the proof to the reader). Hence, without loss of generality, we may assume that $\widetilde{I_0} = J_0$ (taking $\widetilde{I_0} \cup J_0$ in place of $\widetilde{I_0}$ and J_0 if necessary), which implies $\widetilde{I_1} = J_1$ and $\widetilde{I_2} = J_2$. Put $I_i = \widetilde{I_i} = J_i$ for $i = 0,1,2$. Since f is $(n+1)$-quasi-affine, f is $(n+1)$-quasi-convex and $(n+1)$-quasi-concave, hence $f|_{I_1}$ and $f|_{I_2}$ are n-convex and n-concave, which means that $f|_{I_1}$ and $f|_{I_2}$ are n-affine. Taking into account that f is $(n+1)$-quasi-convex and $(n+1)$-quasi-concave, we have $\big([y_{n+1},\dots,y_0;f] \leqslant 0 \;\; or \;\; [x_0,\dots,x_{n+1};f] \geqslant 0\big) \;\; and \;\; \big([y_{n+1},\dots,y_0;f] \geqslant 0 \;\; or \;\; [x_0,\dots,x_{n+1};f] \leqslant 0\big)$ for all $\xi \in I_0$, $y_{n+1},\dots,y_0,x_0,\dots,x_{n+1} \in I$, $y_{n+1} < \dots < y_0 = \xi = x_0 < \dots < x_{n+1}$. Because these conditions are equivalent to those that appear in the condition (ii), the theorem is proved. $\square$

EXAMPLE 4.1. $f(x) = \chi_{\{0\}}(x) + |x|\,\chi_{(-\infty,0)\cup(0,\infty)}(x)$ $(x \in \mathbb{R})$ is 2-quasi-affine.

We will state a result that will be of use in the next theorem. Let $f\colon I \to \mathbb{R}$ be a function, $A \subset I$, $n \in \mathbb{N}$. We put

$$A_{(f,R,+)} = \{\xi \in A\,;\ \forall\ x_0,\dots,x_{n+1} \in I,\ \ \xi = x_0 < \dots < x_{n+1} \quad [x_0,\dots,x_{n+1};f] \geqslant 0\},$$

$$A_{(f,R,-)} = \{\xi \in A\,;\ \forall\ x_0,\dots,x_{n+1} \in I,\ \ \xi = x_0 < \dots < x_{n+1} \quad [x_0,\dots,x_{n+1};f] \leqslant 0\},$$

$$A_{(f,L,+)} = \{\xi \in A\,;\ \forall\ y_{n+1},\dots,y_0 \in I,\ \ y_{n+1} < \dots < y_0 = \xi \quad [y_{n+1},\dots,y_0;f] \geqslant 0\},$$

$$A_{(f,L,-)} = \{\xi \in A\,;\ \forall\ y_{n+1},\dots,y_0 \in I,\ \ y_{n+1} < \dots < y_0 = \xi \quad [y_{n+1},\dots,y_0;f] \leqslant 0\}.$$

LEMMA 4.1. *Let $f\colon I \to \mathbb{R}$ be an $(n+1)$-quasi-affine function, satisfying the condition (ii) in Theorem 4.1 and such that: (a) each of the functions $f|_{I_1\cup I_0}$, $f|_{I_0\cup I_2}$ is neither n-convex nor n-concave, (b) all the sets $I_{0\,(f,R,+)}$, $I_{0\,(f,R,-)}$, $I_{0\,(f,L,+)}$, $I_{0\,(f,L,-)}$ are intervals. Then the interval I_0 is degenerate.*

Proof. Suppose that, contrary to our claim, the interval I_0 is non-degenerate. Then, from the condition (a) it follows that

$$\forall \xi \in int(I_0) \text{ the functions } f|_{I\cap(-\infty,\xi]},\ f|_{I\cap[\xi,\infty)} \text{ are neither } n\text{-convex nor } n\text{-concave.} \tag{4.1}$$

We will prove that

$$I_{0\,(f,R,+)} \neq I_0, \quad I_{0\,(f,R,-)} \neq I_0, \quad I_{0\,(f,L,+)} \neq I_0, \quad I_{0\,(f,L,-)} \neq I_0. \tag{4.2}$$

Suppose that, on the contrary, $I_{0\,(f,R,+)} = I_0$. Then by the definition of $I_{0\,(f,R,+)}$ and taking into account that $f|_{I_2}$ is n-convex, we obtain that $f|_{I_0\cup I_2}$ is n-convex, which contradicts (a). Hence we have $I_{0\,(f,R,+)} \neq I_0$. The other inequalities (4.2) can be proved similarly.

Let $\xi \in I_{0\,(f,R,+)}$. Then there exist $x_0,\ldots,x_{n+1} \in I$, $\xi = x_0 < \ldots < x_{n+1}$ such that $[x_0,\ldots,x_{n+1};f] > 0$. Indeed, if $[x_0,\ldots,x_{n+1};f] = 0$ for all $x_0,\ldots,x_{n+1} \in I$, $\xi = x_0 < \ldots < x_{n+1}$, then two cases may occur. Let $\overline{I_0} = [a,b]$. In the first case $b \in I_0$ and $\xi = b$, hence $f|_{\{\xi\}\cup I_2}$ is n-affine. This implies that $f|_{I_0\cup I_2}$ is n-convex or n-concave, which contradicts (a). In the second case $\xi \in I_0 \setminus \{b\}$. This implies that $f|_{I\cap[\xi,\infty)}$ is n-affine, which contradicts (4.1). Hence there exist $x_0,\ldots,x_{n+1} \in I$, $\xi = x_0 < \ldots < x_{n+1}$ such that $[x_0,\ldots,x_{n+1};f] > 0$ and we can conclude that $\xi \in I_{0\,(f,L,+)}$. Indeed, if $\xi \notin I_{0\,(f,L,+)}$, then there exist $y_{n+1},\ldots,y_0 \in I$, $y_{n+1} < \ldots < y_0 = \xi$ such that $[y_{n+1},\ldots,y_0;f] < 0$. This contradicts assumption that f is quasi-$(n+1)$-affine, and in particular quasi-$(n+1)$-concave. Hence $\xi \in I_{0\,(f,L,+)}$, which means that $I_{0\,(f,R,+)} \subset I_{0\,(f,L,+)}$. In an exactly similar way, we can prove that $I_{0\,(f,L,+)} \subset I_{0\,(f,R,+)}$. Consequently, we have

$$I_{0\,(f,R,+)} = I_{0\,(f,L,+)}. \tag{4.3}$$

Similar arguments to those above show that

$$I_{0\,(f,R,-)} = I_{0\,(f,L,-)}. \tag{4.4}$$

We will prove that

$$I_{0\,(f,R,+)} \cup I_{0\,(f,R,-)} = I_0. \tag{4.5}$$

Contrary to (4.5), suppose that there exists $\xi \in I_0$ such that

$$\xi \notin I_{0\,(f,R,+)} \text{ and } \xi \notin I_{0\,(f,R,-)}. \tag{4.6}$$

In view of (4.3) and (4.4), we have also $\xi \notin I_{0\,(f,L,+)}$ and $\xi \notin I_{0\,(f,L,-)}$. By (4.6), there exist $x_0,\ldots,x_{n+1},x_0',\ldots,x_{n+1}' \in I$, $\xi = x_0 < \ldots < x_{n+1}$, $\xi = x_0' < \ldots < x_{n+1}'$ such that $[x_0,\ldots,x_{n+1};f] > 0$ and $[x_0',\ldots,x_{n+1}';f] < 0$, and there exist $y_{n+1},\ldots,y_0$, $y_{n+1}',\ldots,$ $y_0' \in I$, $y_{n+1} < \ldots < y_0 = \xi$, $y_{n+1}' < \ldots < y_0' = \xi$ such that $[y_{n+1},\ldots,y_0;f] > 0$ and $[y_{n+1}',\ldots,y_0';f] < 0$. Therefore we have $y_{n+1} < \ldots < y_0 = \xi = x_0' < \ldots < x_{n+1}'$, $[y_{n+1},\ldots,y_0;f] > 0$ and $[x_0',\ldots,x_{n+1}';f] < 0$, which contradicts our assumption that f is $(n+1)$-quasi-convex. We have proved (4.5). Then, by (b), (4.5), and taking into account that assumption that I_0 is non-degenerate, it follows that at least one of the intervals $I_{0\,(f,R,+)}$, $I_{0\,(f,R,-)}$ is non-degenerate. Assume, that $I_{0\,(f,R,+)}$ is non-degenerate (when $I_{0\,(f,R,-)}$ is non-degenerate, the proof is analogous). In view of (b), (4.5) and (4.2), one of the two cases may occur $I_{0\,(f,R,-)} \setminus I_{0\,(f,R,+)} < I_{0\,(f,R,+)} \setminus I_{0\,(f,R,-)}$ or $I_{0\,(f,R,+)} \setminus I_{0\,(f,R,-)} < I_{0\,(f,R,-)} \setminus I_{0\,(f,R,+)}$. If the first case is satisfied, then $f|_{I_{0\,(f,R,+)}\cup I_2}$ is n-convex, which contradicts (4.1). Now, let us assume that the second case is satisfied. In view of (4.3) and (4.4), we have $I_{0\,(f,L,+)} \setminus I_{0\,(f,L,-)} < I_{0\,(f,L,-)} \setminus I_{0\,(f,L,+)}$. Then $f|_{I_1\cup I_{0\,(f,L,+)}}$ is n-convex, which contradicts (4.1). The lemma is proved. $\square$

Now, we apply Theorem 4.1 and Lemma 4.1 to obtain a characterization of 2-quasi-affine functions.

THEOREM 4.2. *A function $f\colon I\to\mathbb{R}$ is 2-quasi-affine if, and only if, one of the following conditions holds:*

(i) *f is convex or concave, or*

(ii) *there exist non-degenerate intervals I_1, I_2, a degenerate interval I_0, $I_1<I_0<I_2$, $I_1\cup I_0\cup I_2=I$ such that $f|_{I_1}$, $f|_{I_2}$ are affine, ($[y_2,y_1,y_0;f]<0$ and $[x_0,x_1,x_2;f]<0$) or ($[y_2,y_1,y_0;f]>0$ and $[x_0,x_1,x_2;f]>0$) for all $\xi\in I_0$, $y_2,y_1,y_0,x_0,x_1,x_2\in I$, $y_2<y_1<y_0=\xi=x_0<y_1<x_2$.*

Proof. Let $n=1$. From Theorem 4.1 it follows that if f satisfies the above conditions, then it is 2-quasi-affine. Now let us assume that f is 2-quasi-affine. Then by Theorem 4.1 the conditions (i) and (ii) that appears in Theorem 4.1 are satisfied. To prove the theorem remains to consider the condition (ii) in Theorem 4.1, with a non-degenerate interval I_0. Without loss of generality we may assume, that the condition (a) in Lemma 4.1 is satisfied. We will prove that the condition (b) is also fulfilled, then the theorem will be an immediate consequence of Lemma 4.1. Now we will prove that $I_{0\,(f,R,+)}$ is an interval. Since $f|_{I_0}$ is affine, there exist $a,b\in\mathbb{R}$ such that $f(x)=ax+b$ for all $x\in I_0$. Let $\xi_1,\xi_2\in I_{0\,(f,R,+)}$, $\xi_1<\xi_2$. Consider $\xi_1<\xi_3<\xi_2$. Let $x_1,x_2\in I$ be such that $\xi_3<x_1<x_2$. To prove that $\xi_3\in I_{0\,(f,R,+)}$, we need to check that $[\xi_3,x_1,x_2;f]\geqslant 0$. There are two possible cases. In the first case $x_1\in I_0$. We obtain

$$[\xi_3,x_1,x_2;f]=\frac{[x_1,x_2;f]-a}{x_2-\xi_3},\quad [\xi_1,x_1,x_2;f]=\frac{[x_1,x_2;f]-a}{x_2-\xi_1}. \tag{4.7}$$

Since $\xi_1\in I_{0\,(f,R,+)}$, we have $[\xi_1,x_1,x_2;f]\geqslant 0$. By (4.7), taking into account that $\xi_1<\xi_3<x_1<x_2$, we obtain that $[\xi_3,x_1,x_2;f]\geqslant 0$. In the second case $x_1\notin I_0$. Because $\xi_3<x_1$ and $\xi_3\in I_0$, we have that $x_1\in I_2$. Taking into account that $\xi_2\in I_0$, we obtain $\xi_2<x_1$. Since $\xi_1,\xi_2\in I_{0\,(f,R,+)}$, we have $[\xi_1,x_1,x_2;f]\geqslant 0$ and $[\xi_2,x_1,x_2;f]\geqslant 0$, which can be written in the form

$$\frac{[x_1,x_2;f]-[\xi_1,x_1;f]}{x_2-\xi_1}\geqslant 0 \quad\text{and}\quad \frac{[x_1,x_2;f]-[\xi_2,x_1;f]}{x_2-\xi_2}\geqslant 0. \tag{4.8}$$

Taking into account that $\xi_1<\xi_2<x_1<x_2$, inequalities (4.8) are equivalent to

$$[\xi_1,x_1;f]\leqslant[x_1,x_2;f] \quad\text{and}\quad [\xi_2,x_1;f]\leqslant[x_1,x_2;f]. \tag{4.9}$$

We will prove

$$[\xi_3,x_1;f]\leqslant[x_1,x_2;f], \tag{4.10}$$

which is equivalent to $[\xi_3,x_1,x_2;f]\geqslant 0$. Fixing x_1,x_2, we put the function

$$\gamma(x)=[x,x_1;f]=\frac{f(x_1)-f(x)}{x_1-x}\quad\text{for }\ \xi_1\leqslant x\leqslant\xi_2.$$

Since $\xi_1,\xi_2\in I_0$, we have $\gamma(x)=\frac{f(x_1)-ax-b}{x_1-x}$ for $\xi_1\leqslant x\leqslant\xi_2$, which implies

$$\gamma'(x)=\frac{f(x_1)-ax_1-b}{(x_1-x)^2}\quad\text{for }\ \xi_1\leqslant x\leqslant\xi_2. \tag{4.11}$$

By (4.11), we conclude, that $\gamma'(x) > 0$ for all $\xi_1 \leqslant x \leqslant \xi_2$ or $\gamma'(x) < 0$ for all $\xi_1 \leqslant x \leqslant \xi_2$, which implies that $\gamma(x)$ is increasing on $[\xi_1, \xi_2]$ or $\gamma(x)$ is decreasing on $[\xi_1, \xi_2]$. Taking into account, that by (4.9), $\gamma(\xi_1) \leqslant [x_1, x_2; f]$ and $\gamma(\xi_2) \leqslant [x_1, x_2; f]$, we conclude that $\gamma(x) \leqslant [x_1, x_2; f]$ for all $\xi_1 \leqslant x \leqslant \xi_2$, and in particular $\gamma(\xi_3) \leqslant [x_1, x_2; f]$, which means that (4.10) is satisfied. We have proved that $I_{0\,(f,R,+)}$ is an interval.

Similar arguments to those above show that $I_{0\,(f,R,-)}$, $I_{0\,(f,L,+)}$, $I_{0\,(f,L,-)}$ are intervals. Now the theorem follows immediately from Lemma 4.1. $\square$

5. Strongly quasi-convex, strongly quasi-concave and strongly quasi-affine functions of higher order

The definition of quasi-convex functions of higher order and Proposition 2.2 on strongly quasi-convex functions motivate the introduction of the following strongly quasi-convex functions of higher order.

DEFINITION 5.1. Let $c > 0$. We say that a function $f\colon I \to \mathbb{R}$ is

(i) *strongly n-quasi-convex* if $[y_n, \dots, y_0; f] \leqslant -c$ or $[x_0, \dots, x_n; f] \geqslant c$,

(ii) *strongly n-quasi-concave* if $[y_n, \dots, y_0; f] \geqslant c$ or $[x_0, \dots, x_n; f] \leqslant -c$

for all $\xi \in I$ and $y_n, \dots, y_0, x_0, \dots, x_n \in I$, $y_n < \dots < y_0 = \xi = x_0 < \dots < x_n$. f is *strongly n-quasi-affine* if f is strongly n-quasi-convex and strongly n-quasi-concave.

REMARK 5.1. It is easy to see that the definition of strongly n-quasi-convex functions with $c = 0$ gives the concept of n-quasi-convex functions. Obviously, if the function f is strongly n-quasi-convex with $c > 0$, then f is n-quasi-convex.

PROPOSITION 5.1. ([16]) *Let $n \in \mathbb{N}$ and $c > 0$. Let $f\colon I \to \mathbb{R}$ be an n-convex function. Then f is strongly n-convex with modulus c if, and only if, $f^{(n+1)}(x) \geqslant c(n+1)!$ for $x \in I$ λ a.e (λ denotes the Lebesgue measure).*

PROPOSITION 5.2. *Let I_0 be a non-degenerate subinterval of I. If a function $f\colon I \to \mathbb{R}$ is $(n+1)$-quasi-convex (strongly $(n+1)$-quasi-convex with modulus c), then $f|_{I_0}$ is $(n+1)$-quasi-convex (strongly $(n+1)$-quasi-convex with modulus c).*

PROPOSITION 5.3. *(i) An n-convex function $f\colon I \to \mathbb{R}$ is strongly $(n+1)$-quasi-convex with modulus c if, and only if, f is strongly n-convex with modulus c.*

(ii) An n-concave function $f\colon I \to \mathbb{R}$ is strongly $(n+1)$-quasi-convex with modulus c if, and only if, f is strongly n-concave with modulus c.

The following theorem gives a characterization of strongly quasi-convex functions of higher order, which generalizes that given in Theorem 2.2 for strongly quasi-convex functions.

THEOREM 5.1. *Let $c>0$. A function $f\colon I\to\mathbb{R}$ is strongly $(n+1)$-quasi-convex with modulus c if, and only if, there exist (possible degenerate) intervals I_1, I_2, $I_1<I_2$, $I_1\cup I_2=I$, such that (i) $f|_{I_1}$ is strongly n-concave with modulus c, (ii) $f|_{I_2}$ is strongly n-convex with modulus c.*

Proof. Assume that f is strongly $(n+1)$-quasi-convex on I with modulus $c>0$. By Remark 5.1, f is $(n+1)$-quasi-convex. By Theorem 3.2, there exist (possibly degenerate) intervals I_1, I_0, I_2, $I_1<I_0<I_2$ such that $I_1\cup I_0\cup I_2=I$, $f|_{I_1}$ is n-concave, $f|_{I_2}$ is n-convex, $f|_{I_0}$ is n-affine and satisfies the conditions that appear in Theorem3.2. By Proposition 5.3, $f|_{I_1}$ is strongly n-concave, $f|_{I_2}$ is strongly n-convex and $f|_{I_0}$ is strongly n-concave and strongly n-convex with modulus $c>0$. Suppose that I_0 is non-degenerate. Taking into account Proposition 5.1, we obtain $f|_{I_0}^{(n+1)}(x)\geqslant c(n+1)!$ and $f|_{I_0}^{(n+1)}(x)\leqslant -c(n+1)!$ for $x\in I_0$ λ a.e., which is a contradiction. Now consider the case when the interval I_0 is degenerate. If $I_0=\emptyset$, then our statement holds true. Assume that I_0 is a one-point set, say $I_0=\{\xi\}$. There are three possible cases. In the first case, there exist $x_0,x_1,\dots,x_{n+1}\in I$, $\xi=x_0<x_1<\dots<x_{n+1}$ such that

$$[x_0,\dots,x_{n+1};f]<c. \tag{5.1}$$

Since f is strongly $(n+1)$-quasi-convex on I with modulus $c>0$, it follows that $[y_{n+1},\dots,y_0;f]\leqslant -c$ or $[x_0,\dots,x_{n+1};f]\geqslant c$ for all $y_{n+1},\dots,y_0\in I$ such that $y_{n+1}<\dots<y_0=\xi=x_0$. By (5.1), we conclude that $[y_{n+1},\dots,y_0;f]\leqslant -c$ for all $y_{n+1},\dots,y_0$ such that $a<y_{n+1}<\dots<y_0=\xi=x_0$. Taking into account that f is strongly n-concave on I_1, we obtain that f is strongly n-concave on $I_1\cup I_0$. Since f is strongly n-convex on I_2, the statement holds true in this case. In the second case, there exist $y_{n+1},\dots,y_0\in I$ such that $y_{n+1}<\dots<y_0=\xi$ and $[y_{n+1},\dots,y_0;f]>-c$. Then, by the arguments similar to those above, we conclude that f is strongly n-convex on $I_2\cup I_0$, f is strongly n-concave on I_1 and f is not strongly n-concave on $I_1\cup I_0$. So in this case the statement holds true. In the third case $[y_{n+1},\dots,y_0;f]\leqslant -c$ and $[x_0,\dots,x_{n+1};f]\geqslant c$ for all $y_{n+1},\dots,y_0,x_0,x_1,\dots,x_{n+1}\in I$ such that $y_{n+1}<\dots<y_0=\xi=x_0<x_1<\dots<x_{n+1}$. Taking into account that f is strongly n-convex on I_2 and it is strongly n-concave on I_1, this implies that f is strongly n-convex on $I_2\cup I_0$ and it is strongly n-concave on $I_1\cup I_0$. Our statement holds true in this case.

Since the converse is obvious, the theorem is proved. □

THEOREM 5.2. *Let $c>0$. A function $f\colon I\to\mathbb{R}$ is strongly $(n+1)$-quasi-concave with modulus c if, and only if, there exist (possible degenerate) intervals J_1, J_2, $J_1<J_2$, $J_1\cup J_2=I$, such that $f|_{J_1}$ is strongly n-convex and $f|_{J_2}$ is strongly n-concave with modulus c.*

As a corollary, we obtain the following characterization of strongly $(n+1)$-quasi-affine functions (the proof is analogous to the proof of Theorem 2.3 and can be omitted).

THEOREM 5.3. *Let $c>0$. A function $f\colon I\to\mathbb{R}$ is strongly $(n+1)$-quasi-affine with modulus c if, and only if, f is strongly n-concave with modulus c or f is strongly n-convex with modulus c.*

6. Other quasi-convex functions of higher order

E. Popoviciu [13] gave the following definition of quasi-convex functions of order n, $n \geqslant 0$. The function $f\colon E \to \mathbb{R}$ $(E \subset \mathbb{R})$ is quasi-convex of order n on E, if for every system of points $x_1 < \ldots < x_{n+3}$ of E

$$[x_2,\ldots,x_{n+2};f] \leqslant \max\{[x_1,\ldots,x_{n+1};f],\ [x_3,\ldots,x_{n+3};f]\}. \tag{6.1}$$

If $n = 0$, the inequality in (6.1) becomes the definition of the usual quasi-convex function.

PROPOSITION 6.1. ([10]) *If a function $f\colon E \to \mathbb{R}$ is quasi-convex of order n in the sense of E. Popoviciu* [13]*, then there exist (possibly one of them being the empty set) sets E_1, E_2, $E_1 < E_2$ such that $E_1 \cup E_2 = E$, $f|_{E_1}$ is n-concave and $f|_{E_2}$ is n-convex.*

REMARK 6.1. ([10]) According to Proposition 2.1, the converse of the implication in Proposition 6.1 holds too, in the case $k = 0$. But in the general case $k \geqslant 1$ that is not true without supplementary conditions. For example, the function $f(x) = (-x^3)\,\chi_{(-\infty,0]}(x) + (x^3+5x)\,\chi_{(0,\infty)}(x)$ $(x \in \mathbb{R})$ is continuous, it is 2-concave on $(-\infty,0]$ and 2-convex on $(0,\infty)$, but it is not quasi-convex of order 2 in the sense of E. Popoviciu [13], because $[-2,-1,0,1;f] = 1/6$ and $[-1,0,1,2;f] = -1/6$.

Motivated by higher order quasi-convex functions defined by E. Popoviciu [13], we introduce (n,k)-quasi-convex functions. Note that the condition (6.1) is equivalent to $[x_1,\ldots,x_{n+2};f] \leqslant 0 \quad \text{or} \quad [x_2,\ldots,x_{n+3};f] \geqslant 0$.

DEFINITION 6.1. Let $n,k \geqslant 1$. We say that a function $f\colon I \to \mathbb{R}$ is

(i) *(n,k)-quasi-convex* if $[x_0,\ldots,x_n;f] \leqslant 0 \quad \text{or} \quad [x_k,\ldots,x_{n+k};f] \geqslant 0$,

(ii) *(n,k)-quasi-concave* if $[x_0,\ldots,x_n;f] \geqslant 0 \quad \text{or} \quad [x_k,\ldots,x_{n+k};f] \leqslant 0$,

for all $x_0 < \ldots < x_{n+k}$. The function f is *(n,k)-quasi-affine* if it is simultaneously (n,k)-quasi-convex and (n,k)-quasi-concave.

DEFINITION 6.2. Let $n,k \geqslant 1$ and $c > 0$. We say that a function $f\colon I \to \mathbb{R}$ is

(i) *strongly (n,k)-quasi-convex* if $[x_0,\ldots,x_n;f] \leqslant -c \quad \text{or} \quad [x_k,\ldots,x_{n+k};f] \geqslant c$

(ii) *strongly (n,k)-quasi-concave* if $[x_0,\ldots,x_n;f] \geqslant c \quad \text{or} \quad [x_k,\ldots,x_{n+k};f] \leqslant -c$

for all $x_0 < \ldots < x_{n+k}$. The function f is *strongly (n,k)-quasi-affine* if it is simultaneously strongly (n,k)-quasi-convex and strongly (n,k)-quasi-concave.

REMARK 6.2. Note that

(i) $(n+1,1)$-quasi-convex functions are quasi-convex functions of order n in the sense of E. Popoviciu [13],

(ii) (n,n)-quasi-convex functions are n-quasi-convex functions,

(iii) for every $k = 1,2,\dots$, the class of $(n,n+1+k)$-quasi-convex functions coincides with the class of $(n,n+1)$-quasi-convex functions.

REMARK 6.3. A function $f\colon I \to \mathbb{R}$ is

(i) *(n,n+1)-quasi-convex* if, and only if, $[y_n,\dots,y_0;f] \leqslant 0$ or $[x_0,\dots,x_n;f] \geqslant 0$,

(ii) *(n,n+1)-quasi-concave* if, and only if, $[y_n,\dots,y_0;f] \geqslant 0$ or $[x_0,\dots,x_n;f] \leqslant 0$

for all $y_n,\dots,y_0,x_0,\dots,x_n \in I$, $y_n < \dots < y_0 < x_0 < \dots < x_n$.

THEOREM 6.1. *Let $n \in \mathbb{N}$. Let $f\colon (a,b) \to \mathbb{R}$ $(a<b)$ be a function. Then f is $((n+1),(n+2))$-quasi-convex on (a,b) if, and only if, one of the following conditions holds: (a) f is n-convex, (b) f is n-concave, (c) there exists $x_0 \in (a,b)$, such that f is n-concave on (a,x_0) and n-convex on (x_0,b).*

Proof. The proof is similar to the proof of Theorem 3.1. Aiming for a contradiction, we suppose that $\neg[(a)\vee(b)\vee(c)]$, which is equivalent to $\neg(a)\wedge\neg(b)\wedge\neg(c)$.

The condition $\neg(c)$ (the proof in the cases $\neg(a)$ and $\neg(b)$ is analogous) is equivalent to the condition $\forall\xi \in (a,b)\ \neg\,(f\ \ is\ \ n-concave\ \ on\ \ (a,\xi)\,)\ \vee\ \neg\,(f\ \ is\ \ n-convex\ \ on\ \ (\xi,b)\,)$, which implies, that for any $\xi \in (a,b)$, one of the following two conditions is satisfied: $\exists\ a < y_{n+1} < \dots < y_0 < \xi \quad [y_{n+1},\dots,y_0;f] > 0,$, $\exists\ \xi < x_0 < \dots < x_{n+1} < b \quad [x_0,\dots,x_{n+1};f] < 0$. Putting

$$L_f = \{\xi : \exists\ a < y_{n+1} < \dots < y_0 < \xi \quad [y_{n+1},\dots,y_0;f] > 0\}, \tag{6.2}$$

$$R_f = \{\xi : \exists\ \xi < x_0 < \dots < x_{n+1} < b \quad [x_0,\dots,x_{n+1};f] < 0\}. \tag{6.3}$$

we have $L_f \cup R_f = (a,b)$. There are four possible cases: (A) $L_f = (a.b) \wedge R_f = \emptyset$, (B) $L_f = \emptyset \wedge R_f = (a.b)$, (C) $L_f \neq \emptyset \wedge R_f \neq \emptyset \wedge L_f \cap R_f = \emptyset$, (D) $L_f \cap R_f \neq \emptyset$.

We consider the case (A). Then we have: $\xi \in (a,b) \implies \{\ \xi \in L_f \wedge \xi \notin R_f\ \} \implies \{\ \exists\ a < y_{n+1} < \dots < y_0 < \xi\ [y_{n+1},\dots,y_0;f] > 0\ \wedge\ \forall\,\xi < x_0 < \dots < x_{n+1} < b$ $[x_0,\dots,x_{n+1};f] \geqslant 0\ \} \implies \{\ f\ is\ n-convex\ on\ (\xi,b)\ \}$. Consequently, we obtain, that for all $\xi \in (a,b)$, f is n-convex on (ξ,b), which implies that f is n-convex on (a,b). This contradicts the assumption $\neg$(a). Analysis similar to that in the case (A) shows, that in the case (B), f is n-concave on (a,b), contrary to the assumption $\neg$(b).

Now consider the case (C), i.e. $L_f \neq \emptyset$, $R_f \neq \emptyset$ and $L_f \cap R_f = \emptyset$. By the definitions of L_f and R_f ((6.2), (6.3)), it follows that, if $\xi_1 \in L_f$, then $\xi_1' \in L_f$ for any $\xi_1' \geqslant \xi_1$, and if $\xi_2 \in R_f$, then $\xi_2' \in R_f$ for any $\xi_2' \leqslant \xi_2$. This implies

$$\xi_1 \in L_f \implies [\xi_1,b) \subset L_f, \quad \xi_2 \in R_f \implies (a,\xi_2] \subset R_f. \tag{6.4}$$

Since $L_f \cap R_f = \emptyset$, it follows that if $\xi \in L_f$, then $\xi \notin R_f$. This implies that, if $\xi \in L_f$, then for any $\xi < x_0 < \dots < x_{n+1} < b \quad [x_0,\dots,x_{n+1};f] \geqslant 0$, i.e. f is n-convex on (ξ,b). Similarly, if $\xi \in R_f$, then $\xi \notin L_f$, which implies that, if $\xi \in R_f$, then for any

$a < y_{n+1} < \ldots < y_0 < \xi \quad [y_{n+1},\ldots,y_0;f] \leqslant 0$, i.e. f is n-concave on (a,ξ). Consequently, we have: $\forall \xi_1 \in (a,b)\ \xi_1 \in L_f \Longrightarrow$ f is n-convex on (ξ_1,b), $\forall \xi_2 \in (a,b)\ \xi_2 \in R_f \Longrightarrow$ f is n-concave on (a,ξ_2). Because $L_f \neq \emptyset$ and $R_f \neq \emptyset$, there exist $\xi_1 \in L_f$ and $\xi_2 \in R_f$. Putting $\alpha = \inf\{\ \xi :\ \ \xi \in L_f\ \}$, $\beta = \sup\{\ \xi :\ \ \xi \in R_f\ \}$, we have that

$$f \text{ is } n\text{-concave on } (a,\beta) \quad \text{and} \quad f \text{ is } n\text{-convex on } (\alpha,b). \tag{6.5}$$

By (6.4), we obtain $(a,\beta) \subset R_f$ and $(\alpha,b) \subset L_f$. Moreover, we have $\alpha = \beta$. Indeed, suppose that $\alpha \neq \beta$. If $\alpha < \beta$, then $(\alpha,\beta) \subset L_f \cap R_f$, contrary to $L_f \cap R_f = \emptyset$. If $\alpha > \beta$, then by (6.4), we obtain that $(\beta,\alpha) \subset (a,b) \setminus [L_f \cup R_f]$, which contradicts the assumption $L_f \cup R_f = (a,b)$. Consequently, we have that $\alpha = \beta$. In view of (6.5), we obtain f is n-concave on (a,α) and f is n-convex on (α,b), which contradicts the assumption $\neg$(c). It remains to consider the case (D), i.e. $L_f \cap R_f \neq \emptyset$. It follows that $\exists\ \ \xi_1 \in (a,b)\ \exists\ \ a < y_{n+1} < \ldots < y_0 < \xi_1 < x_0 < \ldots < x_{n+1} < b \quad [y_{n+1},\ldots,y_0;f] > 0\ \wedge\ [x_0,\ldots,x_{n+1};f] < 0$, which contradicts the assumption that f is $(n+1,n+2)$-quasi-convex on (a,b). Since the converse is obvious, the theorem is proved. □

REMARK 6.4. By Theorem 3.1, the function $f(x) = (x+1)\,\chi_{(-\infty,-1)}(x) + (-x)\,\chi_{[-1,1]}(x) + (x-1)\,\chi_{(1,\infty)}(x)$ $(x \in \mathbb{R})$ is $(2,2)$-quasi-convex. Since $[\frac{-9}{4},\frac{-5}{4},\frac{-3}{4},f] = \frac{2}{3}$ and $[\frac{3}{4},\frac{5}{4},\frac{9}{4},f] = \frac{-2}{3}$, f is not $(2,3)$-quasi-convex.

REMARK 6.5. In general, it is not true that the class of $(n,n+1)$-quasi-convex functions is contained in the class of (n,n)-quasi-convex functions. For example, by Theorem 6.1, the function $f(x) = (-x^2-100)\,\chi_{(-\infty,0)}(x) + (x^2+100)\,\chi_{(0,\infty)}(x)$ $(x \in \mathbb{R})$ is $(2,3)$-quasi-convex. Since $[-2,-1,0;f] = 49$ and $[0,1,2;f] = -49$, f is not $(2,2)$-quasi-convex. Moreover, however $f|_{(-\infty,0)}$ is concave and $f|_{(0,\infty)}$ is convex, there are no intervals I_1, I_2, $I_1 < I_2$, $I_1 \cup I_2 = \mathbb{R}$ such that $f|_{I_1}$ is concave and $f|_{I_2}$ is convex.

It is not difficult to prove the following theorem on strongly $((n+1),(n+2))$-quasi-convex functions. We omit the proof.

THEOREM 6.2. *Let $n \in \mathbb{N}$ and $c > 0$. Let $f\colon (a,b) \to \mathbb{R}$ $(a < b)$ be a function. Then f is strongly $((n+1),(n+2))$-quasi-convex on (a,b) if, and only if, one of the following conditions holds: (a) f is strongly n-convex, (b) f is strongly n-concave, (c) there exists $x_0 \in (a,b)$, such that f is strongly n-concave on (a,x_0) and strongly n-convex on (x_0,b),*

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Jacek Mrowiec
University of Bielsko-Biala
Department of Mathematics
ul. Willowa 2, 43–309 Bielsko-Biala, Poland
e-mail: jmrowiec@ath.bielsko.pl

Teresa Rajba
University of Bielsko-Biala
Department of Mathematics
ul. Willowa 2, 43–309 Bielsko-Biala, Poland
e-mail: trajba@ath.bielsko.pl