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ON A JENSEN–TYPE INEQUALITY FOR F–CONVEX FUNCTIONS

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Dedicated to Professor Josip Pečarić
on his 70th birthday

(*Communicated by Marjan Praljak*)

Abstract. In this paper we present a counterpart of the Jensen inequality for F-convex functions. We present it in three forms: discrete, integral and operator. As we will see, the generality of the received results will allow to obtain, in particular cases, Jensen-type inequalities for strongly convex functions and superquadratic functions.

Introduction

In the literature the term "Jensen inequality" has many meanings (cf. [5], [8]). In this work we will use this term in the context of the following functional inequality

$$f\left(\sum_{i=1}^{n} t_i x_i\right) \leqslant \sum_{i=1}^{n} t_i f(x_i),$$

where f is a real function defined on an open real interval I, n is an arbitrary natural number, $x_1,\dots,x_n \in I$ and $t_1,\dots,t_n > 0$ with $t_1+\dots+t_n = 1$. It is well known that the above inequality holds true if and only if f is a convex function. Jensen's inequality, but also Jensen-type inequalities, are very significant in convex analysis and we can find their application in different branches of mathematics, economics and engineering sciences (see for example [8], [10]). In this paper we present a counterpart of Jensen's inequality for F-convex functions. The concept of F-convex functions stems from strongly convex functions introduced in [9]. F-convex functions were introduced in [2] and now we shall recall their denotation. Namely, let $F:\mathbb{R}\to\mathbb{R}$ be a fixed function. A function $f: I \to \mathbb{R}$ is called F-convex if

$$f(tx+(1-t)y) \leqslant tf(x)+(1-t)f(y)-t(1-t)F(x-y), \tag{1}$$

for all $x,y \in I$ and $t \in (0,1)$. Observe that if F is the zero function, then we get convex functions and if $F(x) = cx^2$ (where c is a fixed positive number), then F-convex functions become strongly convex functions with modulus $c > 0$. It will turn out that under

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some additional assumptions F-convex functions become superquadratic functions introduced in [1]. Jensen-type inequalities for strongly convex functions can be found in [3] and [4]. For superquadratic functions Jensen-type inequalities are presented in [1]. We will see that the obtained Jensen-type inequalities for F-convex functions are generalization of the results coming from [3], [4] and [1]. In the first section we give tools which will be used in the next sections, but it seems that they are also interesting in and of themselves. In the sections 2-4 we present Jensen type inequalities for F-convex functions, subsequently increasing the generality, with the hope that it will prove beneficial to future readers.

1. Support results

It is well known that convex functions defined on an open interval have an affine support at every point of their domains (see for example [10]). In this section we present a counterpart of that result for F-convex functions. Recall that a function s is a support for a function f at a point x_0, if $s(x_0) = f(x_0)$ and $s(x) \leqslant f(x)$ for every point x from the domain of f.

The first lemma says that starting with an F-convex function, the function F can be replaced by a certain other function.

LEMMA 1. *Let I be a real open interval, $F : \mathbb{R} \to \mathbb{R}$ be a fixed function and put $\hat{F}(x) := \max\{F(x), F(-x)\}$ for every $x \neq 0$ and $\hat{F}(0) := 0$. If a function $f : I \to \mathbb{R}$ is F-convex, then f is $\hat{F}$-convex.*

Proof. Assume that a function f is F-convex i.e.

$$f(tx + (1-t)y) \leqslant tf(x) + (1-t)f(y) - t(1-t)F(x-y),$$

for all $x, y \in I$ and $t \in (0,1)$. Interchanging x with y and t with $1-t$ we get

$$f(tx + (1-t)y) \leqslant tf(x) + (1-t)f(y) - t(1-t)F(y-x).$$

From the above inequalities the following inequality is true

$$f(tx + (1-t)y) \leqslant tf(x) + (1-t)f(y) - t(1-t)\hat{F}(x-y),$$

for all $x, y \in I$ and $t \in (0,1)$. It means that the function f is $\hat{F}$-convex, which ends the proof. □

REMARK 1. If we additionally assume that $F(0) = 0$, then $\hat{F} \geqslant F$ and it means that we have also the reverse implication in the above lemma.

In the further part of this paper we will adopt the notion of $\hat{F}$ as in Lemma 1.

A counterpart of the support theorem for F-convex functions is the following.

THEOREM 1. *Let I be a real open interval and $F:\mathbb{R}\to\mathbb{R}$ be a fixed function. If a function $f:I\to\mathbb{R}$ is F-convex and differentiable, then at every point $x_0\in I$ the function f has a support in the form of*

$$s(x)=f(x_0)+f'(x_0)(x-x_0)+\hat{F}(x-x_0).$$

Proof. Since f is F-convex we know from Lemma 1 it is also $\hat{F}$-convex i.e.

$$f(tx+(1-t)y)\leqslant tf(x)+(1-t)f(y)-t(1-t)\hat{F}(x-y), \tag{2}$$

for all $x,y\in I$ and $t\in(0,1)$. Fix $x,y\in I$ such that $x\leqslant y$ and take arbitrarily $u\in(x,y)$. Writing u as a convex combination $u=tx+(1-t)y$ we derive $t=\frac{y-u}{y-x}$. Now, inequality (2) can be rewritten in equivalent forms

$$\begin{aligned}
&f(u)\leqslant\frac{y-u}{y-x}f(x)+\frac{u-x}{y-x}f(y)-\frac{y-u}{y-x}\cdot\frac{u-x}{y-x}\hat{F}(x-y),\\
&\frac{y-u}{y-x}f(u)-\frac{y-u}{y-x}f(x)\leqslant\frac{u-x}{y-x}f(y)-\frac{u-x}{y-x}f(u)-\frac{y-u}{y-x}\cdot\frac{u-x}{y-x}\hat{F}(x-y),\\
&(y-u)\left(f(u)-f(x)\right)\leqslant(u-x)\left(f(y)-f(u)\right)-\frac{(y-u)(u-x)}{y-x}\hat{F}(x-y),\\
&\frac{f(u)-f(x)}{u-x}\leqslant\frac{f(y)-f(u)}{y-u}-\frac{1}{y-x}\hat{F}(x-y).
\end{aligned}$$

Let u approach to x from the right side. Using the differentiability of the function f we get

$$f'_+(x)\leqslant\frac{f(y)-f(x)}{y-x}-\frac{1}{y-x}\hat{F}(x-y)$$

hence

$$f(y)\geqslant f(x)+f'_+(x)(y-x)+\hat{F}(y-x),\quad y>x. \tag{3}$$

Similarly, if u approach to y form the left side, then, swapping x with y, we have

$$f(y)\geqslant f(x)+f'_-(x)(y-x)+\hat{F}(y-x),\quad y<x. \tag{4}$$

Finally, because of $f'_-(x)=f'_+(x)=f'(x)$, it follows

$$f(y)\geqslant f(x)+f'_-(x)(y-x)+\hat{F}(y-x),\quad x\neq y. \tag{5}$$

Moreover, for $y=x$ the right side of this inequality is equal to $f(x)$, so the function

$$s(x):=f(x_0)+f'(x_0)(x-x_0)+\hat{F}(x-x_0)$$

is a support for a function f at a point x_0. The proof is finished. □

Considering F-convexity for a nonnegative function F, we note that an F-convex function f is also a convex function. Thus, in particular, f has the left-hand derivative f'_- and the right-hand derivative f'_+ at every point of the domain of f. Moreover, the inequality $f'_{-} \leqslant f'_{+}$ holds true. Now, observe that instead of inequalities (3) and (4) we have the following inequality

$$f(x) \geqslant f(x_0) + a(x - x_0) + \hat{F}(x - x_0), \quad x \in I,$$

where $f'_{-}(x_0) \leqslant a \leqslant f'_{+}(x_0)$, which implies the following corollary.

COROLLARY 1. *Let I be a real open interval and $F : \mathbb{R} \to \mathbb{R}$ be a fixed nonnegative function. If a function $f : I \to \mathbb{R}$ is F-convex, then at every point $x_0 \in I$ the function f has a support in the form of*

$$s(x) = f(x_0) + a(x - x_0) + \hat{F}(x - x_0),$$

where $f'_{-}(x_0) \leqslant a \leqslant f'_{+}(x_0)$.

In view of the definition of the function $\hat{F}$, notice that if F is an even function with $F(0) = 0$, then $\hat{F} = F$. Thus, we have the following corollary.

COROLLARY 2. *Let I be a real open interval and $F : \mathbb{R} \to \mathbb{R}$ be a fixed even function with $F(0) = 0$. If a function $f : I \to \mathbb{R}$ is F-convex and differentiable, then at every point $x_0 \in I$ the function f has a support in the form of*

$$s(x) = f(x_0) + f'(x_0)(x - x_0) + F(x - x_0).$$

If F is a nonnegative function, then automatically $F(0) = 0$ (it is enough to take $x = y$ in inequality (1)). And we have the next corollary.

COROLLARY 3. *Let I be a real open interval and $F : \mathbb{R} \to \mathbb{R}$ be a fixed even and nonnegative function. If a function $f : I \to \mathbb{R}$ is F-convex, then at every point $x_0 \in I$ the function f has a support in the form of*

$$s(x) = f(x_0) + a(x - x_0) + F(x - x_0),$$

where $f'_{-}(x_0) \leqslant a \leqslant f'_{+}(x_0)$.

Notice that in the light of Corollary 3, if we take $F(x) = 0$, then we get the classical result for convex functions; if we take $F(x) = cx^2$, then we get the result for strongly convex functions, which follows from the representation theorem of strongly convex functions proved in [7] (see also [9]); and for $F(x) = f(|x|)$ we obtain the superquadratic functions introduced in [1].

2. Discreet Jensen-type inequalities

A counterpart of Jensen's inequality for F-convex functions is the following.

THEOREM 2. *Let I be a real open interval and $F:\mathbb{R}\to\mathbb{R}$ be a fixed function. If a function $f:I\to\mathbb{R}$ is F-convex and differentiable, then inequality*

$$f\left(\sum_{i=1}^{n} t_i x_i\right) \leqslant \sum_{i=1}^{n} t_i f(x_i) - \sum_{i=1}^{n} t_i \hat{F}\left(x_i - \sum_{j=1}^{n} t_j x_j\right),$$

holds for all $x_1,\dots,x_n \in I$ and $t_1,\dots,t_n > 0$ with $t_1+\cdots+t_n = 1$.

Proof. In the proof of this theorem we will use Theorem 1 and standard methods for Jensen type inequalities. Fix $x_1,\dots,x_n \in I$ and $t_1,\dots,t_n > 0$ such that $t_1+\cdots+t_n = 1$ and denote $x_0 := t_1x_1+\ldots+t_nx_n$. From Theorem 1, a function $s:\mathbb{R}\to\mathbb{R}$ of the form $s(x) = f(x_0)+f'(x_0)(x-x_0)+\hat{F}(x-x_0)$ is a support of f at the point x_0. Thus, we have

$$f(x_i) \geqslant f(x_0)+f'(x_0)(x-x_0)+\hat{F}(x_i-x_0),$$

for all $i=1,\dots,n$. Multiplying both sides by t_i and summing side by side, we obtain

$$\sum_{i=1}^{n} t_i f(x_i) \geqslant f(x_0) + f'(x_0)\sum_{i=1}^{n} t_i(x_i-x_0) + \sum_{i=1}^{n} t_i \hat{F}(x_i-x_0).$$

Finally, because $\sum_{i=1}^{n} t_i(x_i-x_0) = 0$ and $x_0 = \sum_{i=1}^{n} t_i x_i$, we get

$$f\left(\sum_{i=1}^{n} t_i x_i\right) \leqslant \sum_{i=1}^{n} t_i f(x_i) - \sum_{i=1}^{n} t_i \hat{F}\left(x_i - \sum_{j=1}^{n} t_j x_j\right),$$

which ends the proof. □

From Theorem 2 and the previous section we have the following corollaries.

COROLLARY 4. *Let I be a real open interval and $F:\mathbb{R}\to\mathbb{R}$ be a fixed nonnegative function. If a function $f:I\to\mathbb{R}$ is F-convex, then*

$$f\left(\sum_{i=1}^{n} t_i x_i\right) \leqslant \sum_{i=1}^{n} t_i f(x_i) - \sum_{i=1}^{n} t_i \hat{F}\left(x_i - \sum_{j=1}^{n} t_j x_j\right),$$

for all $x_1,\dots,x_n \in I$ and $t_1,\dots,t_n > 0$ with $t_1+\cdots+t_n = 1$.

COROLLARY 5. *Let I be a real open interval and $F:\mathbb{R}\to\mathbb{R}$ be a fixed even function with $F(0)=0$. If a function $f:I\to\mathbb{R}$ is F-convex and differentiable, then*

$$f\left(\sum_{i=1}^{n} t_i x_i\right) \leqslant \sum_{i=1}^{n} t_i f(x_i) - \sum_{i=1}^{n} t_i F\left(x_i - \sum_{j=1}^{n} t_j x_j\right),$$

for all $x_1,\dots,x_n \in I$ and $t_1,\dots,t_n > 0$ with $t_1+\cdots+t_n = 1$.

COROLLARY 6. *Let I be a real open interval and $F:\mathbb{R}\to\mathbb{R}$ be a fixed nonnegative even function. If a function $f:I\to\mathbb{R}$ is F-convex, then*

$$f\left(\sum_{i=1}^{n} t_i x_i\right) \leqslant \sum_{i=1}^{n} t_i f(x_i) - \sum_{i=1}^{n} t_i F\left(x_i - \sum_{j=1}^{n} t_j x_j\right),$$

for all $x_1,\dots,x_n\in I$ and $t_1,\dots,t_n>0$ with $t_1+\cdots+t_n=1$.

Of course, if F is the zero function in the last corollary, then we have the classical Jensen's inequality for convex functions, but from the same corollary we can also derive two known results for strongly convex functions and superquadratic functions. We will present them in the next two remarks.

REMARK 2. Taking function $F(x)=cx^2$ we get the inequality

$$f\left(\sum_{i=1}^{n} t_i x_i\right) \leqslant \sum_{i=1}^{n} t_i f(x_i) - c\sum_{i=1}^{n} t_i \left(x_i - \sum_{j=1}^{n} t_j x_j\right)^2,$$

proved for strongly convex functions in [4] and [6].

REMARK 3. Taking a nonnegative function f and $F(x)=f(|x|)$, we get the inequality

$$f\left(\sum_{i=1}^{n} t_i x_i\right) \leqslant \sum_{i=1}^{n} t_i f(x_i) - \sum_{i=1}^{n} t_i f\left(\left|x_i - \sum_{j=1}^{n} t_j x_j\right|\right),$$

proved for superquadratic functions in [1].

3. Integral Jensen-type inequalities

THEOREM 3. *Let (Ω,Σ,μ) be a probability measure space, I be a real open interval, $F:\mathbb{R}\to\mathbb{R}$ be a fixed function and $\varphi:\Omega\to I$ be a μ-integrable function such that also the function $\hat{F}(\varphi-\int_\Omega\varphi(t)d\mu)$ is μ-integrable. If a function $f:I\to\mathbb{R}$ is F-convex and differentiable, then*

$$f\left(\int_\Omega \varphi(t)d\mu\right) \leqslant \int_\Omega f(\varphi(t))\,d\mu - \int_\Omega \hat{F}\left(\varphi(t)-\int_\Omega\varphi(t)d\mu\right)d\mu.$$

Proof. Since f is F-convex and differentiable we know from Theorem 1 that it satisfies the inequality

$$f(x)\geqslant f(x_0)+f'(x_0)(x-x_0)+\hat{F}(x-x_0),\quad x_0,x\in I.$$

Fix a Lebesgue integrable function $\varphi:\Omega\to I$. Taking $x_0=\int_\Omega\varphi(t)d\mu$ and $x=\varphi(t)$ we get the inequality

$$\begin{aligned} f(\varphi(t)) \geqslant & f\left(\int_\Omega\varphi(t)d\mu\right)+f'\left(\int_\Omega\varphi(t)d\mu\right)\left(\varphi(t)-\int_\Omega\varphi(t)d\mu\right)\\ &+\hat{F}\left(\varphi(t)-\int_\Omega\varphi(t)d\mu\right), \end{aligned}$$

for all $t \in \Omega$, which can be written in the form of

$$f\left(\int_\Omega \varphi(t)d\mu\right) \leqslant f(\varphi(t)) - \hat{F}\left(\varphi(t) - \int_\Omega \varphi(t)d\mu\right) + f'\left(\int_\Omega \varphi(t)d\mu\right) \cdot \left(\varphi(t) - \int_\Omega \varphi(t)d\mu\right).$$

Now, integrating both sides of the last inequality, we obtain

$$f\left(\int_\Omega \varphi(t)d\mu\right) \leqslant \int_\Omega f(\varphi(t))\,d\mu - \int_\Omega \hat{F}\left(\varphi(t) - \int_\Omega \varphi(t)d\mu\right)d\mu.$$

The proof is ended. □

The counterparts of the corollaries from the previous section are the following.

COROLLARY 7. *Let* (Ω,Σ,μ) *be a probability measure space,* I *be a real open interval,* $F:\mathbb{R}\to\mathbb{R}$ *be a nonnegative fixed function and* $\varphi:\Omega\to I$ *be a* μ*-integrable function such that also the function* $\hat{F}(\varphi - \int_\Omega \varphi(t)d\mu)$ *is* μ*-integrable. If a function* $f:I\to\mathbb{R}$ *is* F*-convex, then*

$$f\left(\int_\Omega \varphi(t)d\mu\right) \leqslant \int_\Omega f(\varphi(t))\,d\mu - \int_\Omega \hat{F}\left(\varphi(t) - \int_\Omega \varphi(t)d\mu\right)d\mu.$$

COROLLARY 8. *Let* (Ω,Σ,μ) *be a probability measure space,* I *be a real open interval,* $F:\mathbb{R}\to\mathbb{R}$ *be an even fixed function with* $F(0)=0$ *and* $\varphi:\Omega\to I$ *be a* μ*-integrable function such that also the function* $F(\varphi - \int_\Omega \varphi(t)d\mu)$ *is* μ*-integrable. If a function* $f:I\to\mathbb{R}$ *is* F*-convex and differentiable, then*

$$f\left(\int_\Omega \varphi(t)d\mu\right) \leqslant \int_\Omega f(\varphi(t))\,d\mu - \int_\Omega F\left(\varphi(t) - \int_\Omega \varphi(t)d\mu\right)d\mu.$$

COROLLARY 9. *Let* (Ω,Σ,μ) *be a probability measure space,* I *be a real open interval,* $F:\mathbb{R}\to\mathbb{R}$ *be a nonnegative, even fixed function and* $\varphi:\Omega\to I$ *be a* μ*-integrable function such that also the function* $F(\varphi - \int_\Omega \varphi(t)d\mu)$ *is* μ*-integrable. If a function* $f:I\to\mathbb{R}$ *is* F*-convex, then*

$$f\left(\int_\Omega \varphi(t)d\mu\right) \leqslant \int_\Omega f(\varphi(t))\,d\mu - \int_\Omega F\left(\varphi(t) - \int_\Omega \varphi(t)d\mu\right)d\mu.$$

Analogously as in Section 2, but using Corollary 9, we obtain integral versions of Jensen type inequalities for strongly convex functions and superquadratic functions. The counterparts of Remark 2 and Remark 3 are as follows.

REMARK 4. Taking function $F(x)=cx^2$, we get the inequality

$$f\left(\int_\Omega \varphi(t)d\mu\right) \leqslant \int_\Omega f(\varphi(t))\,d\mu - c\int_\Omega \left(\varphi(t) - \int_\Omega \varphi(t)d\mu\right)^2 d\mu,$$

proved for strongly convex functions in [4] and [6].

REMARK 5. Taking a nonnegative function f and $F(x) = f(|x|)$ we get the inequality

$$f\left(\int_\Omega \varphi(t)d\mu\right) \leqslant \int_\Omega f(\varphi(t))\,d\mu - \int_\Omega f\left(\left|\varphi(t) - \int_\Omega \varphi(t)d\mu\right|\right)d\mu,$$

proved for superquadratic functions in [1].

Moreover, notice that the results presented in Section 2 are a special case of the results from this section. Let us see the following remark.

REMARK 6. For $x_1,\dots,x_n \in I$, $t_1,\dots,t_n > 0$ such that $t_1+\cdots+t_n = 1$ and $\Omega = \{x_1,\dots,x_n\}$, $\Sigma = 2^\Omega$, $\mu(x_i) = t_i$ and function $\varphi(x) = x$, results from Section 3 become results from Section 2, respectively.

4. Operator Jensen-type inequalities

In this section, we present a counterpart of Jensen's inequality for F-convex functions involving the concept of a linear positive normalized functional.

Let us recall from [4] the concept of a linear positive normalized functional. Let T be a nonempty set and $\mathscr{L}$ be a linear class of functions $\varphi : T \to \mathbb{R}$. Functional $A : \mathscr{L} \to \mathbb{R}$ is a linear positive normalized functional if it satisfies the following conditions:

(A1) $A(a\varphi + b\psi) = aA(\varphi) + bA(\psi)$, for all $\varphi,\psi \in \mathscr{L}$ and $a,b \in \mathbb{R}$;

(A2) $A(\varphi) \geqslant 0$, for every non-negative function $\varphi \in \mathscr{L}$;

(A3) $\mathbf{1} \in \mathscr{L}$ ($\mathbf{1}$ is the unit function i.e. $\mathbf{1}(t) = 1$, $t \in T$);

(A4) $A(\mathbf{1}) = 1$.

THEOREM 4. *Let T be a nonempty set and $\mathscr{L}$ be a linear class of functions $\varphi : T \to \mathbb{R}$, $A : \mathscr{L} \to \mathbb{R}$ be a linear positive normalized functional and $F : \mathbb{R} \to \mathbb{R}$ be a fixed function. If a function $f : I \to \mathbb{R}$ defined on a real open interval I is F-convex and differentiable, then for every $\varphi \in \mathscr{L}$ such that $\hat{F}(\varphi - A(\varphi)) \in \mathscr{L}$ and $f \circ \varphi \in \mathscr{L}$ we have*

$$f(A(\varphi)) \leqslant A(f \circ \varphi) - A\left(\hat{F}(\varphi - A(\varphi))\right).$$

Proof. Take a function $\varphi : T \to \mathbb{R}$ such that $\varphi, \hat{F}(\varphi - A(\varphi)), f \circ \varphi \in \mathscr{L}$. Using Theorem 1, we conclude that f satisfies the following inequality

$$f(x) \geqslant f(A(\varphi)) + f'(A(\varphi))(x - (A(\varphi))) + \hat{F}(x - (A(\varphi)), \quad x \in I,$$

and consequently

$$f(\varphi(t)) \geqslant f(A(\varphi)) + f'(A(\varphi))(\varphi(t) - (A(\varphi))) + \hat{F}(\varphi(t) - (A(\varphi)), \quad t \in T.$$

Taking on both sides of the last inequality the linear positive normalized functional A, and using its properties, we get

$$\begin{aligned}A(f\circ\varphi) &\geqslant A\left(f(A(\varphi))+f'(A(\varphi))(\varphi(t)-(A(\varphi)))+\hat{F}(\varphi(t)-(A(\varphi)))\right)\\ &= f(A(\varphi))+A\left(\hat{F}(\varphi-A(\varphi))\right),\end{aligned}$$

which is equivalent to the inequality from the thesis. This ends the proof. □

The counterparts of the corollaries from the previous sections are the following.

COROLLARY 10. *Let T be a nonempty set and $\mathscr{L}$ be a linear class of functions $\varphi: T\to\mathbb{R}$, $A:\mathscr{L}\to\mathbb{R}$ be a linear positive normalized functional, and $F:\mathbb{R}\to\mathbb{R}$ be a fixed nonnegative function. If a function $f: I\to\mathbb{R}$ defined on a real open interval I is F-convex, then for every $\varphi\in\mathscr{L}$ such that $\hat{F}(\varphi-A(\varphi))\in\mathscr{L}$ and $f\circ\varphi\in\mathscr{L}$ we have*

$$f(A(\varphi))\leqslant A(f\circ\varphi)-A\left(\hat{F}(\varphi-A(\varphi))\right).$$

COROLLARY 11. *Let T be a nonempty set and $\mathscr{L}$ be a linear class of functions $\varphi: T\to\mathbb{R}$, $A:\mathscr{L}\to\mathbb{R}$ be a linear positive normalized functional and $F:\mathbb{R}\to\mathbb{R}$ be a fixed even function. If a function $f: I\to\mathbb{R}$ defined on a real open interval I is F-convex and differentiable, then for every $\varphi\in\mathscr{L}$ such that $F(\varphi-A(\varphi))\in\mathscr{L}$ and $f\circ\varphi\in\mathscr{L}$ we have*

$$f(A(\varphi))\leqslant A(f\circ\varphi)-A(F(\varphi-A(\varphi))).$$

COROLLARY 12. *Let T be a nonempty set and $\mathscr{L}$ be a linear class of functions $\varphi: T\to\mathbb{R}$, $A:\mathscr{L}\to\mathbb{R}$ be a linear positive normalized functional and $F:\mathbb{R}\to\mathbb{R}$ be a fixed nonnegative even function. If a function $f: I\to\mathbb{R}$ defined on a real open interval I is F-convex, then for every $\varphi\in\mathscr{L}$ such that $F(\varphi-A(\varphi))\in\mathscr{L}$ and $f\circ\varphi\in\mathscr{L}$ we have*

$$f(A(\varphi))\leqslant A(f\circ\varphi)-A(F(\varphi-A(\varphi))).$$

Similarly to the previous sections, but using the last corollary, we can write operator versions of a Jensen-type inequality for strongly convex functions and superquadratic functions. We present them for the sake of completeness in the next two remarks.

REMARK 7. Taking function $F(x)=cx^2$ we get the inequality

$$f(A(\varphi))\leqslant A(f\circ\varphi)-cA\left((\varphi-A(\varphi))^2\right).$$

proved for strongly convex functions in [4].

REMARK 8. Taking a nonnegative function f and $F(x)=f(|x|)$, we get the inequality

$$f(A(\varphi))\leqslant A(f\circ\varphi)-A(f(|\varphi-A(\varphi)|)),$$

as a counterpart of the inequality presented in Remark 5.

The last remark shows that the results obtained in Section 3 are a special case of the results presented in this section.

REMARK 9. Let (Ω, Σ, μ) be a probability measure space. Taking $T = \Omega$, a family $\mathscr{L}$ as the class of μ-integrable function $\varphi : T \to \mathbb{R}$ and a linear positive normalized functional $A(\varphi) = \int_T \varphi(t) d\mu$, results from Section 4 become results from Section 3, respectively.

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