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PÓLYA–SZEGÖ AND CHEBYSHEV TYPES INEQUALITIES VIA AN EXTENDED GENERALIZED MITTAG–LEFFLER FUNCTION

MAJA ANDRIĆ*, GHULAM FARID, SAJID MEHMOOD AND JOSIP PEČARIĆ

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Abstract. In this paper certain Pólya-Szegö type integral inequalities due to Karamata's estimations of the Chebyshev quotient are presented. Those inequalities include an extended generalized Mittag-Leffler function with the corresponding fractional integral operator, and from them, some fractional integral inequalities of Chebyshev type are obtained. Also, several known results are improved.

1. Introduction and preliminaries

The Chebyshev functional $T(f,g)$ for two Lebesgue integrable functions f and g on interval $[a,b]$ is defined by

$$T(f,g)=\frac{1}{b-a}\int_a^b f(x)g(x)dx-\left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right). \tag{1}$$

Majority of problems involving Chebyshev functional are to give a lower bound or an upper bound for T, under various assumptions. For instance, if f and g are monotonic in the same sense (in the opposite sense) then we obtain a well-known Chebyshev inequality ([3])

$$T(f,g)\geqslant 0 \quad (\leqslant 0). \tag{2}$$

Also, if we have constants $m,M,n,N\in\mathbb{R}$ such that for $x\in[a,b]$

$$m\leqslant f(x)\leqslant M, \quad n\leqslant g(x)\leqslant N,$$

then the Grüss inequality ([7]) states

$$|T(f,g)|\leqslant\frac{(M-m)(N-n)}{4}. \tag{3}$$

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* Corresponding author.

For more recent inequalities see [2, 4, 5, 8, 11, 12, 13]. Following inequalities are the subject of our research: the first one was introduced by Pólya and Szegö ([15])

$$\frac{\left(\int_a^b f^2(x)dx\right)\left(\int_a^b g^2(x)dx\right)}{\left(\int_a^b f(x)g(x)dx\right)^2} \leqslant \frac{1}{4}\left(\sqrt{\frac{mn}{MN}}+\sqrt{\frac{MN}{mn}}\right)^2. \tag{4}$$

Next is the inequality by Dragomir and Diamond ([6])

$$|T(f,g)| \leqslant \frac{(M-m)(N-n)}{4\sqrt{mMnN}}\frac{1}{(b-a)^2}\int_a^b f(x)dx\int_a^b g(x)dx. \tag{5}$$

Using Karamata's estimations of the Chebyshev quotient ([9]), Pečarić and Perić give generalized and improved inequality of (5) for positive normalized functional Φ in ([13])

$$\begin{aligned} -\frac{(M-m)(N-n)}{\left(\sqrt{mN}+\sqrt{Mn}\right)^2}\Phi(fg) &\leqslant -\frac{(M-m)(N-n)}{\left(\sqrt{mn}+\sqrt{MN}\right)^2}\Phi(f)\Phi(g) \\ &\leqslant \Phi(fg)-\Phi(f)\Phi(g) \leqslant \frac{(M-m)(N-n)}{\left(\sqrt{mn}+\sqrt{MN}\right)^2}\Phi(fg) \\ &\leqslant (M-m)(N-n)\left(\sqrt{mN}+\sqrt{Mn}\right)^2\Phi(f)\Phi(9). \end{aligned} \tag{6}$$

Motivated by the paper [11], where authors have proved Pólya-Szegö and Chebyshev types fractional integral inequalities for the Riemann-Liouville fractional integral operator, we presents improved and generalized corresponding results using our extended generalized Mittag-Leffler function with its fractional integral operator, both defined in [1]:

DEFINITION 1. Let $\rho,\alpha,\tau,\delta,c \in \mathbb{C}$, $\Re(\rho),\Re(\alpha),\Re(\tau)>0$, $\Re(c)>\Re(\delta)>0$ with $p\geqslant 0$, $r>0$ and $0<q\leqslant r+\Re(\rho)$. Then the extended generalized Mittag-Leffler function $E_{\rho,\alpha,\tau}^{\delta,r,q,c}(z;p)$ is defined by

$$E_{\rho,\alpha,\tau}^{\delta,r,q,c}(z;p)=\sum_{n=0}^{\infty}\frac{\mathrm{B}_p(\delta+nq,c-\delta)}{\mathrm{B}(\delta,c-\delta)}\frac{(c)_{nq}}{\Gamma(\rho n+\alpha)}\frac{z^n}{(\tau)_{nr}}. \tag{7}$$

DEFINITION 2. Let $w,\rho,\alpha,\tau,\delta,c \in \mathbb{C}$, $\Re(\rho),\Re(\alpha),\Re(\tau)>0$, $\Re(c)>\Re(\delta)>0$ with $p\geqslant 0$, $r>0$ and $0<q\leqslant r+\Re(\rho)$. Let $f\in L_1[a,b]$ and $x\in[a,b]$. Then the generalized fractional integral operator $\varepsilon_{a^+,\rho,\alpha,\tau}^{w,\delta,r,q,c}f$ is defined by

$$\left(\varepsilon_{a^+,\rho,\alpha,\tau}^{w,\delta,r,q,c}f\right)(x;p)=\int_a^x (x-t)^{\alpha-1}E_{\rho,\alpha,\tau}^{\delta,r,q,c}(w(x-t)^{\rho};p)f(t)dt. \tag{8}$$

Here $(c)_{nq}$ denotes the generalized Pochhammer symbol

$$(c)_{nq}=\frac{\Gamma(c+nq)}{\Gamma(c)},$$

B_p is an extension of the beta function

$$B_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} e^{-\frac{p}{t(1-t)}}\,dt \quad (\Re(x), \Re(y), \Re(p) > 0)$$

and $L_1[a,b]$ is the space of all Lebesgue measurable functions f for which $|f|$ is Lebesgue integrable on $[a,b]$, i.e. $\|f\|_1 = \int_a^b |f(x)|dx < \infty$.

For different choices of parameters in Definition 1 and Definition 2, we can get corresponding known fractional integral operators (for more details see [1] and references therein). E.g., setting $p = w = 0$ in (8), this operator reduces to the left-sided Riemann-Liouville fractional integral $I_{a^+}^{\alpha} f(x)$ of order α.

2. Pólya-Szegö type fractional integral inequalities via extended generalized Mittag-Leffler function

In this section we use extended generalized Mittag-Leffler function $E_{\rho,\alpha,\tau}^{\delta,r,q,c}(z;p)$ with the corresponding fractional integral operator $\varepsilon_{a^+,\rho,\alpha,\tau}^{w,\delta,r,q,c} f$ (in real domain) to obtain fractional generalizations of inequalities due to Pólya and Szegö. Following theorems are based on [11] where this was done for the Riemann-Liouville fractional integral operator. The role of the parameter $\alpha > 0$ will be of great significance and for the reader's convenience we will use a simplified notation

$$\begin{aligned}
\boldsymbol{E}_\alpha(z;p) &:= E_{\rho,\alpha,\tau}^{\delta,r,q,c}(z;p) \\
&= \sum_{n=0}^{\infty} \frac{B_p(\delta+nq, c-\delta)}{B(\delta, c-\delta)} \frac{(c)_{nq}}{\Gamma(\rho n+\alpha)} \frac{z^n}{(\tau)_{nr}}, \\
(\boldsymbol{\varepsilon}_\alpha f)(x;p) &:= \left(\varepsilon_{a^+,\rho,\alpha,\tau}^{w,\delta,r,q,c} f\right)(x;p) \\
&= \int_a^x (x-t)^{\alpha-1} \boldsymbol{E}_\alpha(z;p)(w(x-t)^\rho;p) f(t)\,dt.
\end{aligned}$$

THEOREM 1. *Under the various parametric constraints stated already with Definition 1 and Definition 2, in real domain, let $f, g, \varphi_1, \varphi_2, \psi_1$ and ψ_2 be positive integrable functions on $[0,\infty)$ with*

$$0 < \varphi_1(u) \leqslant f(u) \leqslant \varphi_2(u), \quad 0 < \psi_1(u) \leqslant g(u) \leqslant \psi_2(u) \quad (u \in [a,t], t > a). \tag{9}$$

Then the following inequality holds

$$\frac{\left(\boldsymbol{\varepsilon}_\alpha \psi_1 \psi_2 f^2\right)(x;p) \left(\boldsymbol{\varepsilon}_\alpha \varphi_1 \varphi_2 g^2\right)(x;p)}{\left[\left(\boldsymbol{\varepsilon}_\alpha(\varphi_1\psi_1 + \varphi_2\psi_2) fg\right)(x;p)\right]^2} \leqslant \frac{1}{4}. \tag{10}$$

Proof. From the given conditions follows

$$\left(\frac{\varphi_2(u)}{\psi_1(u)} - \frac{f(u)}{g(u)}\right)\left(\frac{f(u)}{g(u)} - \frac{\varphi_1(u)}{\psi_2(u)}\right) \geqslant 0,$$

that is

$$(\varphi_1(u)\psi_1(u)+\varphi_2(u)\psi_2(u))f(u)g(u) \geqslant \psi_1(u)\psi_2(u)f^2(u)+\varphi_1(u)\varphi_2(u)g^2(u).$$

Multiplying above inequality by $(x-u)^{\alpha-1}\boldsymbol{E}_\alpha(w(x-u)^\rho;p)$ and integrating on $[a,x]$ we obtain

$$\begin{aligned}
&\int_a^x (x-u)^{\alpha-1}\boldsymbol{E}_\alpha(w(x-u)^\rho;p)(\varphi_1(u)\psi_1(u)+\varphi_2(u)\psi_2(u))f(u)g(u)du\\
\geqslant &\int_a^x (x-u)^{\alpha-1}\boldsymbol{E}_\alpha(w(x-u)^\rho;p)\psi_1(u)\psi_2(u)f^2(u)du\\
&+\int_a^x (x-u)^{\alpha-1}\boldsymbol{E}_\alpha(w(x-u)^\rho;p)\varphi_1(u)\varphi_2(u)g^2(u)du,
\end{aligned}$$

that is

$$(\boldsymbol{\varepsilon}_\alpha(\varphi_1\psi_1+\varphi_2\psi_2)fg)(x;p) \geqslant \left(\boldsymbol{\varepsilon}_\alpha\psi_1\psi_2 f^2\right)(x;p)+\left(\boldsymbol{\varepsilon}_\alpha\varphi_1\varphi_2 g^2\right)(x;p).$$

Since $a+b \geqslant 2\sqrt{ab}$ for $a,b\in\mathbb{R}$ (the AM-GM inequality), we have

$$(\boldsymbol{\varepsilon}_\alpha(\varphi_1\psi_1+\varphi_2\psi_2)fg)(x;p) \geqslant 2\sqrt{(\boldsymbol{\varepsilon}_\alpha\psi_1\psi_2 f^2)(x;p)\,(\boldsymbol{\varepsilon}_\alpha\varphi_1\varphi_2 g^2)(x;p)},$$

which leads to the inequality (10) as required. □

Fixing the bounds on functions f and g we get the following special case of Theorem 1.

COROLLARY 1. *Under the various parametric constraints stated already with Definition 1 and Definition 2, in real domain, let f and g be two positive integrable functions on $[0,\infty)$ satisfying*

$$0<m\leqslant f(u)\leqslant M<\infty, \quad 0<n\leqslant g(u)\leqslant N<\infty \quad (u\in[a,t], t>a). \tag{11}$$

Then the following inequality holds

$$\frac{\left(\boldsymbol{\varepsilon}_\alpha f^2\right)(x;p)\left(\boldsymbol{\varepsilon}_\alpha g^2\right)(x;p)}{[(\boldsymbol{\varepsilon}_\alpha fg)(x;p)]^2} \leqslant \frac{1}{4}\left(\sqrt{\frac{mn}{MN}}+\sqrt{\frac{MN}{mn}}\right)^2.$$

REMARK 1. Choosing particular values of parameters in Theorem 1, known Pólya-Szegö type inequalities for several fractional integral operators can be deduced (for more details on fractional integral operators see [1] and references therein). For example, setting $w=p=0$ (and $a=0$), we get Pólya-Szegö inequality for the Riemann-Liouville fractional integral operator given in [11, Lemma 3.1].

Now we prove the next Pólya-Szegö type inequality.

THEOREM 2. *Suppose that the assumptions of Theorem 1 hold with* $\beta > 0$. *Then*

$$\frac{\left(\boldsymbol{\varepsilon}_{\alpha}\varphi_1\varphi_2\right)(x;p)\left(\boldsymbol{\varepsilon}_{\beta}\psi_1\psi_2\right)(x;p)\left(\boldsymbol{\varepsilon}_{\alpha}f^2\right)(x;p)\left(\boldsymbol{\varepsilon}_{\beta}g^2\right)(x;p)}{\left[\left(\boldsymbol{\varepsilon}_{\alpha}\varphi_1 f\right)(x;p)\left(\boldsymbol{\varepsilon}_{\beta}\psi_1 g\right)(x;p)+\left(\boldsymbol{\varepsilon}_{\alpha}\varphi_2 f\right)(x;p)\left(\boldsymbol{\varepsilon}_{\beta}\psi_2 g\right)(x;p)\right]^2} \leqslant \frac{1}{4}. \tag{12}$$

Proof. Under given conditions on f,g and $\varphi_i,\psi_i (i=1,2)$ in (9), for $u,v \in [a,t]$ we have

$$\frac{\varphi_2(u)}{\psi_1(v)} - \frac{f(u)}{g(v)} \geqslant 0 \quad \text{and} \quad \frac{f(u)}{g(v)} - \frac{\varphi_1(u)}{\psi_2(v)} \geqslant 0,$$

which imply

$$\left(\frac{\varphi_1(u)}{\psi_2(v)} + \frac{\varphi_2(u)}{\psi_1(v)}\right)\frac{f(u)}{g(v)} \geqslant \frac{f^2(u)}{g^2(v)} + \frac{\varphi_1(u)\varphi_2(u)}{\psi_1(v)\psi_2(v)}$$

that is

$$\varphi_1(u)f(u)\psi_1(v)g(v) + \varphi_2(u)f(u)\psi_2(v)g(v) \geqslant \psi_1(v)\psi_2(v)f^2(u) + \varphi_1(u)\varphi_2(u)g^2(v).$$

Multiplying above inequality by

$$(x-u)^{\alpha-1}(x-v)^{\beta-1}\boldsymbol{E}_{\alpha}(w(x-u)^{\rho};p)\boldsymbol{E}_{\beta}(w(x-v)^{\rho};p)$$

and integrating, we obtain

$$\begin{aligned}
&\int_a^x\int_a^x (x-u)^{\alpha-1}(x-v)^{\beta-1}\boldsymbol{E}_{\alpha}(w(x-u)^{\rho};p)\\
&\quad\times\boldsymbol{E}_{\beta}(w(x-v)^{\rho};p)\varphi_1(u)f(u)\psi_1(v)g(v)dudv\\
&\quad+\int_a^x\int_a^x (x-u)^{\alpha-1}(x-v)^{\beta-1}\boldsymbol{E}_{\alpha}(w(x-u)^{\rho};p)\\
&\quad\times\boldsymbol{E}_{\beta}(w(x-v)^{\rho};p)\varphi_2(u)f(u)\psi_2(v)g(v)dudv\\
&\geqslant \int_a^x\int_a^x (x-u)^{\alpha-1}(x-v)^{\beta-1}\boldsymbol{E}_{\alpha}(w(x-u)^{\rho};p)\\
&\quad\times\boldsymbol{E}_{\beta}(w(x-v)^{\rho};p)\psi_1(v)\psi_2(v)f^2(u)dudv\\
&\quad+\int_a^x\int_a^x (x-u)^{\alpha-1}(x-v)^{\beta-1}\boldsymbol{E}_{\alpha}(w(x-u)^{\rho};p)\\
&\quad\times\boldsymbol{E}_{\beta}(w(x-v)^{\rho};p)\varphi_1(u)\varphi_2(u)g^2(v)dudv,
\end{aligned}$$

that is

$$\begin{aligned}
&\left(\boldsymbol{\varepsilon}_{\alpha}\varphi_1 f\right)(x;p)\left(\boldsymbol{\varepsilon}_{\beta}\psi_1 g\right)(x;p)+\left(\boldsymbol{\varepsilon}_{\alpha}\varphi_2 f\right)(x;p)\left(\boldsymbol{\varepsilon}_{\beta}\psi_2 g(x;p)\right)\\
&\geqslant \left(\boldsymbol{\varepsilon}_{\alpha}f^2\right)(x;p)\left(\boldsymbol{\varepsilon}_{\beta}\psi_1\psi_2\right)(x;p)+\left(\boldsymbol{\varepsilon}_{\alpha}\varphi_1\varphi_2\right)(x;p)\left(\boldsymbol{\varepsilon}_{\beta}g^2\right)(x;p).
\end{aligned}$$

Now if we apply the AM-GM inequality we get

$$\begin{aligned}
&\left(\boldsymbol{\varepsilon}_{\alpha}\varphi_1 f\right)(x;p)\left(\boldsymbol{\varepsilon}_{\beta}\psi_1 g\right)(x;p)+\left(\boldsymbol{\varepsilon}_{\alpha}\varphi_2 f\right)(x;p)\left(\boldsymbol{\varepsilon}_{\beta}\psi_2 g\right)(x;p)\\
&\geqslant 2\sqrt{\left(\boldsymbol{\varepsilon}_{\alpha}f^2\right)(x;p)\left(\boldsymbol{\varepsilon}_{\beta}\psi_1\psi_2\right)(x;p)\left(\boldsymbol{\varepsilon}_{\alpha}\varphi_1\varphi_2\right)(x;p)\left(\boldsymbol{\varepsilon}_{\beta}g^2\right)(x;p)}
\end{aligned}$$

which leads to the inequality (12). □

In the results that follow, we need next equality:

$$\begin{aligned}
(\boldsymbol{\varepsilon}_\alpha 1)(x;p) &= \int_a^x (x-t)^{\alpha-1}\boldsymbol{E}_\alpha(w(x-t)^\rho;p)dt \\
&= \int_a^x (x-t)^{\alpha-1}\sum_{n=0}^{\infty}\frac{\mathrm{B}_p(\delta+nq,c-\delta)}{\mathrm{B}(\delta,c-\delta)}\frac{(c)_{nq}}{\Gamma(\rho n+\alpha)}\frac{w^n(x-t)^{\rho n}}{(\tau)_{n\rho}}dt \\
&= \sum_{n=0}^{\infty}\frac{\mathrm{B}_p(\delta+nq,c-\delta)}{\mathrm{B}(\delta,c-\delta)}\frac{(c)_{nq}}{\Gamma(\rho n+\alpha)}\frac{w^n}{(\tau)_{n\rho}}\int_a^x (x-t)^{\rho n+\alpha-1}dt \\
&= (x-a)^\alpha\sum_{n=0}^{\infty}\frac{\mathrm{B}_p(\delta+nq,c-\delta)}{\mathrm{B}(\delta,c-\delta)}\frac{(c)_{nq}}{\Gamma(\rho n+\alpha)}\frac{w^n}{(\tau)_{n\rho}}\frac{(x-a)^{\rho n}}{\rho n+\alpha}.
\end{aligned}$$

Hence,

$$(\boldsymbol{\varepsilon}_\alpha 1)(x;p) = (x-a)^\alpha \boldsymbol{E}_{\alpha+1}(w(x-a)^\rho;p). \tag{13}$$

More details on the properties of the fractional integral operator $\boldsymbol{\varepsilon}_\alpha f$ can be found in [1, Section 2].

We continue with a special case of Theorem 2.

COROLLARY 2. *Suppose that the assumptions of Corollary 1 hold with $\beta > 0$. Then*

$$\frac{(\boldsymbol{\varepsilon}_\alpha 1)(x;p)\left(\boldsymbol{\varepsilon}_\beta 1\right)(x;p)\left(\boldsymbol{\varepsilon}_\alpha f^2\right)(x;p)\left(\boldsymbol{\varepsilon}_\beta g^2\right)(x;p)}{\left[(\boldsymbol{\varepsilon}_\alpha f)(x;p)\left(\boldsymbol{\varepsilon}_\beta g\right)(x;p)\right]^2} \leqslant \frac{1}{4}\left(\sqrt{\frac{mn}{MN}}+\sqrt{\frac{MN}{mn}}\right)^2.$$

THEOREM 3. *Suppose that the assumptions of Theorem 1 hold with $\beta > 0$. Then*

$$\left(\boldsymbol{\varepsilon}_\alpha f^2\right)(x;p)\left(\boldsymbol{\varepsilon}_\beta g^2\right)(x;p) \leqslant \left(\boldsymbol{\varepsilon}_\alpha\left(\varphi_2 fg/\psi_1\right)\right)(x;p)\left(\boldsymbol{\varepsilon}_\beta\left(\psi_2 fg/\varphi_1\right)\right)(x;p). \tag{14}$$

Proof. Under given conditions on f,g and φ_i,ψ_i $(i=1,2)$ in (9), for $u,v\in[a,t]$ we have

$$\frac{\varphi_2(u)f(u)g(u)}{\psi_1(u)} - f^2(u) \geqslant 0 \quad \text{and} \quad \frac{\psi_2(v)f(v)g(v)}{\varphi_1(v)} - g^2(v) \geqslant 0,$$

hence

$$\begin{aligned}
&\int_a^x (x-u)^{\alpha-1}\boldsymbol{E}_\alpha(w(x-u)^\rho;p)f^2(u)du \\
&\leqslant \int_a^x (x-u)^{\alpha-1}\boldsymbol{E}_\alpha(w(x-u)^\rho;p)\frac{\varphi_2(u)}{\psi_1(u)}f(u)g(u)du,
\end{aligned}$$

and

$$\begin{aligned}
&\int_a^x (x-v)^{\beta-1}\boldsymbol{E}_\beta(w(x-v)^\rho;p)g^2(v)dv \\
&\leqslant \int_a^x (x-v)^{\beta-1}\boldsymbol{E}_\beta(w(x-v)^\rho;p)\frac{\psi_2(v)}{\varphi_1(v)}f(v)g(v)dv,
\end{aligned}$$

which imply

$$\left(\boldsymbol{\varepsilon}_\alpha f^2\right)(x;p) \leqslant \left(\boldsymbol{\varepsilon}_\alpha(\varphi_2 fg/\psi_1)\right)(x;p),$$

$$\left(\boldsymbol{\varepsilon}_\beta g^2\right)(x;p) \leqslant \left(\boldsymbol{\varepsilon}_\beta(\psi_2 fg/\varphi_1)\right)(x;p).$$

Multiplying above inequalities we obtain (14). □

COROLLARY 3. *Suppose that the assumptions of Corollary 1 hold with* $\beta > 0$. *Then*

$$\frac{\left(\boldsymbol{\varepsilon}_\alpha f^2\right)(x;p)\left(\boldsymbol{\varepsilon}_\beta g^2\right)(x;p)}{\left(\boldsymbol{\varepsilon}_\alpha fg\right)(x;p)\left(\boldsymbol{\varepsilon}_\beta fg\right)(x;p)} \leqslant \frac{MN}{mn}.$$

REMARK 2. As before, choosing particular values of parameters in Theorem 2 and Theorem 3, known Pólya-Szegö type inequalities for several fractional integral operators can be deduced, such as inequalities for the Riemann-Liouville fractional integral operator in [11, Lemma 3.3, Lemma 3.4] if we set $w = p = 0$.

3. Chebyshev type fractional integral inequalities

Using Pólya-Szegö type inequality in Theorem 1, we obtain following Chebyshev inequalities based on [13] and [11].

THEOREM 4. *Under the various parametric constraints stated already with Definition 1 and Definition 2, in real domain, let* $f, g, \varphi_1, \varphi_2, \psi_1$ *and* ψ_2 *be positive integrable functions on* $[0,\infty)$ *with*

$$0 < \varphi_1(u) \leqslant f(u) \leqslant \varphi_2(u), \quad 0 < \psi_1(u) \leqslant g(u) \leqslant \psi_2(u) \quad (u \in [a,t], t > a).$$

Suppose also $\beta > 0$. *Then*

$$\begin{aligned}
&\Big|\left(\boldsymbol{\varepsilon}_\alpha 1\right)(x;p)\left(\boldsymbol{\varepsilon}_\beta fg\right)(x;p) + \left(\boldsymbol{\varepsilon}_\beta 1\right)(x;p)\left(\boldsymbol{\varepsilon}_\alpha fg\right)(x;p) \\
&\quad - \left(\boldsymbol{\varepsilon}_\alpha f\right)(x;p)\left(\boldsymbol{\varepsilon}_\beta g\right)(x;p) - \left(\boldsymbol{\varepsilon}_\alpha g\right)(x;p)\left(\boldsymbol{\varepsilon}_\beta f\right)(x;p)\Big| \\
&\leqslant \left|G_{\alpha,\beta}(f,\varphi_1,\varphi_2)(x;p) + G_{\beta,\alpha}(f,\varphi_1,\varphi_2)(x;p)\right|^{\frac{1}{2}} \\
&\quad \times \left|G_{\alpha,\beta}(g,\psi_1,\psi_2)(x;p) + G_{\beta,\alpha}(g,\psi_1,\psi_2)(x;p)\right|^{\frac{1}{2}},
\end{aligned} \tag{15}$$

where

$$\begin{aligned}
G_{\alpha,\beta}(u,v,w)(x;p) &= \frac{\left(\boldsymbol{\varepsilon}_\beta 1\right)(x;p)\left[\left(\boldsymbol{\varepsilon}_\alpha(v+w)u\right)(x;p)\right]^2}{4\left(\boldsymbol{\varepsilon}_\alpha vw\right)(x;p)} \\
&\quad - \left(\boldsymbol{\varepsilon}_\alpha u\right)(x;p)\left(\boldsymbol{\varepsilon}_\beta u\right)(x;p).
\end{aligned} \tag{16}$$

Proof. Let f and g be two positive integrable function on $[0,\infty)$. For $u, v \in [a,t]$ we define $A(u,v)$ as

$$A(u,v) = (f(u) - f(v))(g(u) - g(v)),$$

that is

$$A(u,v) = f(u)g(u) + f(v)g(v) - f(u)g(v) - f(v)g(u).$$

Multiplying above equality by

$$(x-u)^{\alpha-1}(x-v)^{\beta-1}\boldsymbol{E}_{\alpha}(w(x-u)^{\rho};p)\boldsymbol{E}_{\beta}(w(x-v)^{\rho};p)$$

and integrating, we obtain

$$\begin{aligned}
&\int_a^x\int_a^x (x-u)^{\alpha-1}(x-v)^{\beta-1}\boldsymbol{E}_{\alpha}(w(x-u)^{\rho};p)\boldsymbol{E}_{\beta}\left(w(x-v)^{\rho};p\right)A(u,v)dudv \\
&= \left(\boldsymbol{\varepsilon}_{\beta}1\right)(x;p)\left(\boldsymbol{\varepsilon}_{\alpha}fg\right)(x;p) + \left(\boldsymbol{\varepsilon}_{\alpha}1\right)(x;p)\left(\boldsymbol{\varepsilon}_{\beta}fg\right)(x;p) \\
&\quad - \left(\boldsymbol{\varepsilon}_{\alpha}f\right)(x;p)\left(\boldsymbol{\varepsilon}_{\beta}g\right)(x;p) - \left(\boldsymbol{\varepsilon}_{\beta}f\right)(x;p)\left(\boldsymbol{\varepsilon}_{\alpha}g\right)(x;p).
\end{aligned} \tag{17}$$

By the Cauchy-Schwartz inequality for double integrals we have

$$\begin{aligned}
&\left|\int_a^x\int_a^x (x-u)^{\alpha-1}(x-v)^{\beta-1}\boldsymbol{E}_{\alpha}(w(x-u)^{\rho};p)\boldsymbol{E}_{\beta}(w(x-v)^{\rho};p)A(u,v)dudv\right| \\
&\leqslant \left[\int_a^x\int_a^x (x-u)^{\alpha-1}(x-v)^{\beta-1}\boldsymbol{E}_{\alpha}(w(x-u)^{\rho};p)\boldsymbol{E}_{\beta}(w(x-v)^{\rho};p)f^2(u)dudv\right. \\
&\quad + \int_a^x\int_a^x (x-u)^{\alpha-1}(x-v)^{\beta-1}\boldsymbol{E}_{\alpha}(w(x-u)^{\rho};p)\boldsymbol{E}_{\beta}(w(x-v)^{\rho};p)f^2(v)dudv \\
&\quad \left.-2\int_a^x\int_a^x (x-u)^{\alpha-1}(x-v)^{\beta-1}\boldsymbol{E}_{\alpha}(w(x-u)^{\rho};p)\boldsymbol{E}_{\beta}(w(x-v)^{\rho};p)f(u)f(v)dudv\right]^{\frac{1}{2}} \\
&\quad \times\left[\int_a^x\int_a^x (x-u)^{\alpha-1}(x-v)^{\beta-1}\boldsymbol{E}_{\alpha}(w(x-u)^{\rho};p)\boldsymbol{E}_{\beta}(w(x-v)^{\rho};p)g^2(u)dudv\right. \\
&\quad + \int_a^x\int_a^x (x-u)^{\alpha-1}(x-v)^{\beta-1}\boldsymbol{E}_{\alpha}(w(x-u)^{\rho};p)\boldsymbol{E}_{\beta}(w(x-v)^{\rho};p)g^2(v)dudv \\
&\quad \left.-2\int_a^x\int_a^x (x-u)^{\alpha-1}(x-v)^{\beta-1}\boldsymbol{E}_{\alpha}(w(x-u)^{\rho};p)\boldsymbol{E}_{\beta}(w(x-v)^{\rho};p)g(u)g(v)dudv\right]^{\frac{1}{2}} \\
&= \left[\left(\boldsymbol{\varepsilon}_{\beta}1\right)(x;p)\left(\boldsymbol{\varepsilon}_{\alpha}f^2\right)(x;p) + \left(\boldsymbol{\varepsilon}_{\alpha}1\right)(x;p)\left(\boldsymbol{\varepsilon}_{\beta}f^2\right)(x;p) - 2\left(\boldsymbol{\varepsilon}_{\alpha}f\right)(x;p)\left(\boldsymbol{\varepsilon}_{\beta}f\right)(x;p)\right]^{\frac{1}{2}} \\
&\quad \times\left[\left(\boldsymbol{\varepsilon}_{\beta}1\right)(x;p)\left(\boldsymbol{\varepsilon}_{\alpha}g^2\right)(x;p) + \left(\boldsymbol{\varepsilon}_{\alpha}1\right)(x;p)\left(\boldsymbol{\varepsilon}_{\beta}g^2\right)(x;p) - 2\left(\boldsymbol{\varepsilon}_{\alpha}g\right)(x;p)\left(\boldsymbol{\varepsilon}_{\beta}g\right)(x;p)\right]^{\frac{1}{2}}
\end{aligned}$$

For $\psi_1(t) = \psi_2(t) = g(t) = 1$ by Theorem 1 follows

$$\left(\boldsymbol{\varepsilon}_{\alpha}f^2\right)(x;p) \leqslant \frac{\left[\left(\boldsymbol{\varepsilon}_{\alpha}(\varphi_1+\varphi_2)f\right)(x;p)\right]^2}{4\left(\boldsymbol{\varepsilon}_{\alpha}\varphi_1\varphi_2\right)(x;p)}.$$

This implies

$$\begin{aligned}
&\left(\boldsymbol{\varepsilon}_{\beta}1\right)(x;p)\left(\boldsymbol{\varepsilon}_{\alpha}f^2\right)(x;p) - \left(\boldsymbol{\varepsilon}_{\alpha}f\right)(x;p)\left(\boldsymbol{\varepsilon}_{\beta}f\right)(x;p) \\
&\leqslant \frac{\left(\boldsymbol{\varepsilon}_{\beta}1\right)(x;p)\left[\left(\boldsymbol{\varepsilon}_{\alpha}(\varphi_1+\varphi_2)f\right)(x;p)\right]^2}{4\left(\boldsymbol{\varepsilon}_{\alpha}\varphi_1\varphi_2\right)(x;p)} - \left(\boldsymbol{\varepsilon}_{\alpha}f\right)(x;p)\left(\boldsymbol{\varepsilon}_{\beta}f\right)(x;p) \\
&= G_{\alpha,\beta}(f,\varphi_1,\varphi_2)(x;p)
\end{aligned} \tag{18}$$

and

$$\begin{aligned}&(\boldsymbol{\varepsilon}_{\alpha}1)(x;p)\left(\boldsymbol{\varepsilon}_{\beta}f^{2}\right)(x;p)-(\boldsymbol{\varepsilon}_{\alpha}f)(x;p)\left(\boldsymbol{\varepsilon}_{\beta}f\right)(x;p)\\&\leqslant\frac{(\boldsymbol{\varepsilon}_{\alpha}1)(x;p)\left[\left(\boldsymbol{\varepsilon}_{\beta}(\varphi_{1}+\varphi_{2})f\right)(x;p)\right]^{2}}{4\left(\boldsymbol{\varepsilon}_{\beta}\varphi_{1}\varphi_{2}\right)(x;p)}-(\boldsymbol{\varepsilon}_{\alpha}f)(x;p)\left(\boldsymbol{\varepsilon}_{\beta}f\right)(x;p)\\&=G_{\beta,\alpha}(f,\varphi_{1},\varphi_{2})(x;p)\end{aligned}\tag{19}$$

Applying the same procedure for $\varphi_1(t)=\varphi_2(t)=f(t)=1$, we get

$$\begin{aligned}&\left(\boldsymbol{\varepsilon}_{\beta}1\right)(x;p)\left(\boldsymbol{\varepsilon}_{\alpha}g^{2}\right)(x;p)-(\boldsymbol{\varepsilon}_{\alpha}g)(x;p)\left(\boldsymbol{\varepsilon}_{\beta}g\right)(x;p)\\&\qquad\leqslant G_{\alpha,\beta}(g,\psi_{1},\psi_{2})(x;p)\end{aligned}\tag{20}$$

$$\begin{aligned}&(\boldsymbol{\varepsilon}_{\alpha}1)(x;p)\left(\boldsymbol{\varepsilon}_{\beta}g^{2}\right)(x;p)-(\boldsymbol{\varepsilon}_{\alpha}g)(x;p)\left(\boldsymbol{\varepsilon}_{\beta}g\right)(x;p)\\&\qquad\leqslant G_{\beta,\alpha}(g,\psi_{1},\psi_{2})(x;p)\end{aligned}\tag{21}$$

Finally, considering (17) to (21), we arrive at the desired result in (15). This completes the proof. □

Setting $\alpha=\beta$ in Theorem 4, next inequality follows.

COROLLARY 4. *Suppose that the assumptions of Theorem 4 hold. Then*

$$\begin{aligned}&|(\boldsymbol{\varepsilon}_{\alpha}1)(x;p)(\boldsymbol{\varepsilon}_{\alpha}fg)(x;p)-(\boldsymbol{\varepsilon}_{\alpha}f)(x;p)(\boldsymbol{\varepsilon}_{\alpha}g)(x;p)|\\&\leqslant|G_{\alpha,\alpha}(f,\varphi_{1},\varphi_{2})(x;p)\,G_{\alpha,\alpha}(g,\psi_{1},\psi_{2})(x;p)|^{\frac{1}{2}},\end{aligned}$$

where $G_{\alpha,\alpha}$ is given by (16).

Setting $\varphi_1=m$, $\varphi_2=M$, $\psi_1=n$ and $\psi_2=N$, we obtain

$$G_{\alpha,\alpha}(f,m,M)(x;p)=\frac{(M-m)^{2}}{4mM}\left[(\boldsymbol{\varepsilon}_{\alpha}f)(x;p)\right]^{2},$$

$$G_{\alpha,\alpha}(g,n,N)(x;p)=\frac{(N-n)^{2}}{4nN}\left[(\boldsymbol{\varepsilon}_{\alpha}g)(x;p)\right]^{2}.$$

COROLLARY 5. *Under the various parametric constraints stated already with Definition 1 and Definition 2, in real domain, let f and g be two positive integrable functions on $[0,\infty)$ satisfying*

$$0<m\leqslant f(u)\leqslant M<\infty,\quad 0<n\leqslant g(u)\leqslant N<\infty\quad(u\in[a,t],t>a).$$

Then

$$\begin{aligned}&|(\boldsymbol{\varepsilon}_{\alpha}1)(x;p)(\boldsymbol{\varepsilon}_{\alpha}fg)(x;p)-(\boldsymbol{\varepsilon}_{\alpha}f)(x;p)(\boldsymbol{\varepsilon}_{\alpha}g)(x;p)|\\&\leqslant\frac{(M-m)(N-n)}{4\sqrt{mMnN}}(\boldsymbol{\varepsilon}_{\alpha}f)(x;p)(\boldsymbol{\varepsilon}_{\alpha}g)(x;p).\end{aligned}$$

REMARK 3. Setting $w = p = 0$ (and $a = 0$) in Theorem 4, Corollary 4 and Corollary 5 we get Pólya-Szegö type inequalities for the Riemann-Liouville fractional integral operator given in [11, Theorem 3.6, Theorem 3.7, Corollary 3.4].

Recently, in [10] Nikolova and Varošanec generalized results from [11] for any two linear isotonic functionals. Here, we will give another approach. In the next theorem we will use Karamata's estimations of the Chebyshev quotient ([9]),

$$\frac{1}{K^2} \leqslant \frac{\left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right)}{\frac{1}{b-a}\int_a^b f(x)g(x)dx} \leqslant K^2, \tag{22}$$

where

$$K = \frac{\sqrt{mn}+\sqrt{MN}}{\sqrt{mN}+\sqrt{Mn}}, \tag{23}$$

and the result (6) by Pečarić and Perić ([13]). In this way we will obtain even better upper and lower estimations than those in Corollary 5.

THEOREM 5. *Suppose that the assumptions of Corollary 5 hold. Then*

$$\frac{1}{K^2} \leqslant \frac{(\boldsymbol{\varepsilon}_\alpha 1)(x;p)(\boldsymbol{\varepsilon}_\alpha fg)(x;p)}{(\boldsymbol{\varepsilon}_\alpha f)(x;p)(\boldsymbol{\varepsilon}_\alpha g)(x;p)} \leqslant K^2 \tag{24}$$

where K is given by (23).

Proof. Without loss of generality we can assume

$$1 \leqslant f(u) \leqslant \mu_1, \quad 1 \leqslant g(u) \leqslant \mu_2$$

for every $u \in [a,t]$ and some $\mu_1, \mu_2 \geqslant 1$. From the obvious inequality

$$[\mu_1 - f(u)][f(v)-1][\mu_2 g(v) - g(u)] \geqslant 0$$

we obtain

$$\begin{aligned}\mu_1\mu_2 f(v)g(v) - \mu_1\mu_2 g(v) - \mu_1 f(v)g(u) + \mu_1 g(u)\\ -\mu_2 f(u)f(v)g(v) + f(u)f(v)g(u) + \mu_2 f(u)g(v) - f(u)g(u) \geqslant 0.\end{aligned}$$

Multiplying above inequality by

$$(x-u)^{\alpha-1}(x-v)^{\alpha-1}\boldsymbol{E}_\alpha(w(x-u)^\rho;p)\boldsymbol{E}_\alpha(w(x-v)^\rho;p)$$

and then integrating, we have

$$\begin{aligned}\mu_1\mu_2(\boldsymbol{\varepsilon}_\alpha 1)(x;p)(\boldsymbol{\varepsilon}_\alpha fg)(x;p) - \mu_1\mu_2(\boldsymbol{\varepsilon}_\alpha 1)(x;p)(\boldsymbol{\varepsilon}_\alpha g)(x;p)\\ -\mu_1(\boldsymbol{\varepsilon}_\alpha f)(x;p)(\boldsymbol{\varepsilon}_\alpha g)(x;p) + \mu_1(\boldsymbol{\varepsilon}_\alpha 1)(x;p)(\boldsymbol{\varepsilon}_\alpha g)(x;p)\\ -\mu_2(\boldsymbol{\varepsilon}_\alpha f)(x;p)(\boldsymbol{\varepsilon}_\alpha fg)(x;p) + (\boldsymbol{\varepsilon}_\alpha f)(x;p)(\boldsymbol{\varepsilon}_\alpha fg)(x;p)\\ +\mu_2(\boldsymbol{\varepsilon}_\alpha f)(x;p)(\boldsymbol{\varepsilon}_\alpha g)(x;p) - (\boldsymbol{\varepsilon}_\alpha 1)(x;p)(\boldsymbol{\varepsilon}_\alpha fg)(x;p)) \geqslant 0\end{aligned}$$

from which follows

$$\frac{\mu_2\left[\mu_1\left(\boldsymbol{\varepsilon}_\alpha 1\right)(x;p)-\left(\boldsymbol{\varepsilon}_\alpha f\right)(x;p)\right]+\mu_1\left[\left(\boldsymbol{\varepsilon}_\alpha f\right)(x;p)-\left(\boldsymbol{\varepsilon}_\alpha 1\right)(x;p)\right]}{\mu_2\left[\mu_1\left(\boldsymbol{\varepsilon}_\alpha 1\right)(x;p)-\left(\boldsymbol{\varepsilon}_\alpha f\right)(x;p)\right]+\left(\boldsymbol{\varepsilon}_\alpha f\right)(x;p)-\left(\boldsymbol{\varepsilon}_\alpha 1\right)(x;p)} \leqslant \frac{\left(\boldsymbol{\varepsilon}_\alpha fg\right)(x;p)}{\left(\boldsymbol{\varepsilon}_\alpha g\right)(x;p)}. \tag{25}$$

Similarly, from

$$[\mu_1 - f(u)]\,[f(v)-1]\,[\mu_2 g(u)-g(v)] \geqslant 0$$

follows

$$\frac{\left(\boldsymbol{\varepsilon}_\alpha fg\right)(x;p)}{\left(\boldsymbol{\varepsilon}_\alpha g\right)(x;p)} \leqslant \frac{\mu_1\left(\boldsymbol{\varepsilon}_\alpha 1\right)(x;p)-\left(\boldsymbol{\varepsilon}_\alpha f\right)(x;p)+\mu_1\mu_2\left[\left(\boldsymbol{\varepsilon}_\alpha f\right)(x;p)-\left(\boldsymbol{\varepsilon}_\alpha 1\right)(x;p)\right]}{\mu_1\left(\boldsymbol{\varepsilon}_\alpha 1\right)(x;p)-\left(\boldsymbol{\varepsilon}_\alpha f\right)(x;p)+\mu_2\left[\left(\boldsymbol{\varepsilon}_\alpha f\right)(x;p)-\left(\boldsymbol{\varepsilon}_\alpha 1\right)(x;p)\right]}. \tag{26}$$

Hence, from (25) and (26) we have

$$\frac{\mu_2\left[\mu_1-\frac{(\boldsymbol{\varepsilon}_\alpha f)(x;p)}{(\boldsymbol{\varepsilon}_\alpha 1)(x;p)}\right]+\mu_1\left[\frac{(\boldsymbol{\varepsilon}_\alpha f)(x;p)}{(\boldsymbol{\varepsilon}_\alpha 1)(x;p)}-1\right]}{\mu_2\left[\mu_1-\frac{(\boldsymbol{\varepsilon}_\alpha f)(x;p)}{(\boldsymbol{\varepsilon}_\alpha 1)(x;p)}\right]+\frac{(\boldsymbol{\varepsilon}_\alpha f)(x;p)}{(\boldsymbol{\varepsilon}_\alpha 1)(x;p)}-1} \leqslant \frac{\left(\boldsymbol{\varepsilon}_\alpha fg\right)(x;p)}{\left(\boldsymbol{\varepsilon}_\alpha g\right)(x;p)} \leqslant \frac{\mu_1-\frac{(\boldsymbol{\varepsilon}_\alpha f)(x;p)}{(\boldsymbol{\varepsilon}_\alpha 1)(x;p)}+\mu_1\mu_2\left[\frac{(\boldsymbol{\varepsilon}_\alpha f)(x;p)}{(\boldsymbol{\varepsilon}_\alpha 1)(x;p)}-1\right]}{\mu_1-\frac{(\boldsymbol{\varepsilon}_\alpha f)(x;p)}{(\boldsymbol{\varepsilon}_\alpha 1)(x;p)}+\mu_2\left[\frac{(\boldsymbol{\varepsilon}_\alpha f)(x;p)}{(\boldsymbol{\varepsilon}_\alpha 1)(x;p)}-1\right]}. \tag{27}$$

Next, we define functions $h, H : [1,\mu_1] \to (0,\infty)$ by

$$H(t) = \frac{1}{t}\,\frac{\mu_1 - t + \mu_1\mu_2(t-1)}{\mu_1 - t + \mu_2(t-1)}, \quad h(t) = \frac{1}{H(\mu_1/t)}.$$

For $t_1 = \frac{\sqrt{\mu_1}(\sqrt{\mu_1}+\sqrt{\mu_2})}{\sqrt{\mu_1\mu_2}+1}$ it is straightforward to check that

$$\max_{t\in[1,\mu_1]} H(t) = H(t_1) = \left(\frac{1+\sqrt{\mu_1\mu_2}}{\sqrt{\mu_1}+\sqrt{\mu_2}}\right)^2 = K^2$$

and

$$\min_{t\in[1,\mu_1]} h(t) = h(\mu_1/t_1) = 1/H(t_1) = 1/K^2.$$

Using (27) we obtain

$$h\left(\frac{\left(\boldsymbol{\varepsilon}_\alpha f\right)(x;p)}{\left(\boldsymbol{\varepsilon}_\alpha 1\right)(x;p)}\right) \leqslant \frac{\left(\boldsymbol{\varepsilon}_\alpha 1\right)(x;p)\left(\boldsymbol{\varepsilon}_\alpha fg\right)(x;p)}{\left(\boldsymbol{\varepsilon}_\alpha f\right)(x;p)\left(\boldsymbol{\varepsilon}_\alpha g\right)(x;p)} \leqslant H\left(\frac{\left(\boldsymbol{\varepsilon}_\alpha f\right)(x;p)}{\left(\boldsymbol{\varepsilon}_\alpha 1\right)(x;p)}\right)$$

from which follows (24). □

COROLLARY 6. *If the assumptions of Theorem 5 hold, then*

$$\begin{aligned}
&-\frac{(M-m)(N-n)}{\left(\sqrt{mN}+\sqrt{Mn}\right)^2}\left(\boldsymbol{\varepsilon}_\alpha 1\right)(x;p)\left(\boldsymbol{\varepsilon}_\alpha fg\right)(x;p)\\
&\leqslant -\frac{(M-m)(N-n)}{\left(\sqrt{mn}+\sqrt{MN}\right)^2}\left(\boldsymbol{\varepsilon}_\alpha f\right)(x;p)\left(\boldsymbol{\varepsilon}_\alpha g\right)(x;p)\\
&\leqslant \left(\boldsymbol{\varepsilon}_\alpha 1\right)(x;p)\left(\boldsymbol{\varepsilon}_\alpha fg\right)(x;p)-\left(\boldsymbol{\varepsilon}_\alpha f\right)(x;p)\left(\boldsymbol{\varepsilon}_\alpha g\right)(x;p)\\
&\leqslant \frac{(M-m)(N-n)}{\left(\sqrt{mn}+\sqrt{MN}\right)^2}\left(\boldsymbol{\varepsilon}_\alpha 1\right)(x;p)\left(\boldsymbol{\varepsilon}_\alpha fg\right)(x;p)\\
&\leqslant \frac{(M-m)(N-n)}{\left(\sqrt{mN}+\sqrt{Mn}\right)^2}\left(\boldsymbol{\varepsilon}_\alpha f\right)(x;p)\left(\boldsymbol{\varepsilon}_\alpha g\right)(x;p).
\end{aligned} \tag{28}$$

Proof. As in [13, Corrolary 1], we see that direct consequences of (24) are the first and the last inequality in (28). From the lower bound in (24) we have

$$\begin{aligned}
&\left[\frac{1}{K^2}-1\right]\left(\boldsymbol{\varepsilon}_\alpha 1\right)(x;p)\left(\boldsymbol{\varepsilon}_\alpha fg\right)(x;p)\\
&\leqslant \left(\boldsymbol{\varepsilon}_\alpha 1\right)(x;p)\left(\boldsymbol{\varepsilon}_\alpha fg\right)(x;p)-\left(\boldsymbol{\varepsilon}_\alpha f\right)(x;p)\left(\boldsymbol{\varepsilon}_\alpha g\right)(x;p)
\end{aligned}$$

from which follows the second inequality in (28). Analogously, from the upper bound in (24) follows the third inequality in (28). □

REMARK 4. If we observe results from Corollary 5 and Corollary 6 we can see that the upper estimate from Corollary 6 is better, i.e. inequality

$$\frac{(M-m)(N-n)}{\left(\sqrt{mN}+\sqrt{Mn}\right)^2}\leqslant\frac{(M-m)(N-n)}{4\sqrt{mMnN}}$$

is equivalent with inequality

$$0\leqslant\left(\sqrt{Mn}-\sqrt{mN}\right)^2.$$

The upper estimates are equal if and only if $M/m=N/n$.

The lower estimate from Corollary 6 is also better, i.e. inequality

$$-\frac{(M-m)(N-n)}{4\sqrt{mMnN}}\leqslant-\frac{(M-m)(N-n)}{\left(\sqrt{mn}+\sqrt{MN}\right)^2}$$

is equivalent with inequality

$$0\leqslant\left(\sqrt{mn}-\sqrt{MN}\right)^2.$$

The lower estimates are equal if and only if $m=M$ and $n=N$.

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Maja Andrić
Faculty of Civil Engineering, Architecture and Geodesy
University of Split
Matice hrvatske 15, 21000 Split, Croatia
e-mail: maja.andric@gradst.hr

Ghulam Farid
Department of Mathematics
COMSATS University Islamabad
Attock Campus, Pakistan
e-mail: faridphdsms@hotmail.com;
ghlmfarid@ciit-attock.edu.pk

Sajid Mehmood
GBPS Sherani
Hazro Attock 43440, Pakistan
e-mail: smjg227@gmail.com

Josip Pečarić
RUDN University
Miklukho-Maklaya str. 6, 117198 Moscow, Russia
e-mail: pecaric@element.hr