Mathematical
Inequalities
& Applications
Volume 22, Number 4 (2019), 1379–1384 doi:10.7153/mia-2019-22-95

NOTE ON GENERALIZATION OF THE JENSEN–MERCER INEQUALITY BY TAYLOR'S POLYNOMIAL

ANITA MATKOVIĆ* AND JOSIP PEČARIĆ

(*Communicated by I. Perić*)

Abstract. We present generalizations of the Jensen-Mercer inequality for the class of n-convex functions. The results are obtained by using Taylor's polynomial and four types of Green's functions.

1. Introduction

In paper [1] the following integral version of the Jensen-Mercer inequality for convex functions was proved.

Theorem A. *Let* $g:[a,b]\to\mathbb{R}$ *be a continuous and monotonic function and* $[\alpha,\beta]$ *be an interval such that the image of* g *is a subset of* $[\alpha,\beta]$. *Let function* $\lambda:[a,b]\to\mathbb{R}$ *be either continuous or of bounded variation satisfying*

$$\lambda(a)\leqslant\lambda(t)\leqslant\lambda(b)\ \ \text{for all } t\in[\alpha,\beta],\quad \lambda(b)-\lambda(a)>0. \tag{1}$$

Then for every continuous convex function $\varphi:[\alpha,\beta]\to\mathbb{R}$ the inequality

$$\varphi\left(\alpha+\beta-\frac{\int_a^b g(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right)\leqslant\varphi(\alpha)+\varphi(\beta)-\frac{\int_a^b \varphi(g(x))\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} \tag{2}$$

holds.

REMARK 1. Inequality (2) is also valid when the condition (1) is replaced with the more strict condition that λ is a nondecreasing function such that $\lambda(a)\neq\lambda(b)$.

Let us recall the definition of n-convex functions.

Mathematics subject classification (2010): 26D15, 26D20.
Keywords and phrases: Jensen-Mercer inequality, n-convex functions.
The work of the second author on this publication was supported by the Ministry of Education and Science of the Russian Federation (the Agreement number No. 02.a03.21.0008).
* Corresponding author.

DEFINITION 1. A function $f:[a,b]\to\mathbb{R}$ is said to be n-convex on $[a,b], n\geqslant 0$, if for all choices of $(n+1)$ distinct points in $[a,b]$, the n-th divided difference of f satisfies $f[x_0,\dots,x_n]\geqslant 0$. If this inequality is reversed, then f is said to be n-concave. If the inequality is strict, then f is said to be a strictly n-convex (n-concave) function.

Thus 0-convex functions are non-negative functions, 1-convex functions are increasing functions and 2-convex functions are simply convex functions. An n-convex function need not be n-times differentiable, however if $\varphi^{(n)}$ exists then φ is n-convex iff $\varphi^{(n)}\geqslant 0$. For more information about n-convex functions see [7] and also [8, pp 14-15].

Consider Green's functions $G_i, i=1,2,3,4$ defined on $[\alpha,\beta]\times[\alpha,\beta]$ by

$$G_1(t,s)=\begin{cases}\alpha-s & \text{for } s\leqslant t,\\ \alpha-t & \text{for } t\leqslant s.\end{cases} \tag{3}$$

$$G_2(t,s)=\begin{cases}t-\beta & \text{for } s\leqslant t,\\ s-\beta & \text{for } t\leqslant s.\end{cases} \tag{4}$$

$$G_3(t,s)=\begin{cases}t-\alpha & \text{for } s\leqslant t,\\ s-\alpha & \text{for } t\leqslant s.\end{cases} \tag{5}$$

$$G_4(t,s)=\begin{cases}\beta-s & \text{for } s\leqslant t,\\ \beta-t & \text{for } t\leqslant s.\end{cases} \tag{6}$$

All four Green's functions are continuous and convex in s and, since they are symmetric, also in t.

It can be easily shown by integrating by parts that every function $\varphi:[\alpha,\beta]\to\mathbb{R}$, $\varphi\in C^2([\alpha,\beta])$ can be represented in the following four forms:

$$\varphi(x)=\varphi(\alpha)+(x-\alpha)\varphi'(\beta)+\int_\alpha^\beta G_1(x,s)\varphi''(s)\,\mathrm{d}s. \tag{7}$$

$$\varphi(x)=\varphi(\beta)+(x-\beta)\varphi'(\alpha)+\int_\alpha^\beta G_2(x,s)\varphi''(s)\,\mathrm{d}s. \tag{8}$$

$$\varphi(x)=\varphi(\beta)-(\beta-\alpha)\varphi'(\beta)+(x-\alpha)\varphi'(\alpha)+\int_\alpha^\beta G_3(x,s)\varphi''(s)\,\mathrm{d}s. \tag{9}$$

$$\varphi(x)=\varphi(\alpha)+(\beta-\alpha)\varphi'(\alpha)-(\beta-x)\varphi'(\beta)+\int_\alpha^\beta G_4(x,s)\varphi''(s)\,\mathrm{d}s. \tag{10}$$

2. Main results

LEMMA 1. *Let $g:[a,b]\to\mathbb{R}$ be a continuous function, and $[\alpha,\beta]$ an interval such that the image of g is a subset of $[\alpha,\beta]$. Let function $\lambda:[a,b]\to\mathbb{R}$ be either continuous or of bounded variation, and such that $\lambda(a)\neq\lambda(b)$ and* $\dfrac{\int_a^b g(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\in$

$[\alpha,\beta]$. *Let the Green's functions* $G_i, i=1,2,3,4$ *be defined by* $(3)-(6)$. *Then for every function* $\varphi\in C^2([\alpha,\beta])$ *the identities*

$$\begin{aligned}
&\varphi\left(\alpha+\beta-\frac{\int_a^b g(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right)-\left(\varphi(\alpha)+\varphi(\beta)-\frac{\int_a^b \varphi(g(x))\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right)\\
&=\int_\alpha^\beta\left[G_i\left(\alpha+\beta-\frac{\int_a^b g(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)},s\right)\right.\\
&\left.\quad-\left(G_i(\alpha,s)+G_i(\beta,s)-\frac{\int_a^b G_i(g(x),s)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right)\right]\varphi''(s)\,\mathrm{d}s
\end{aligned}\tag{11}$$

hold for $i=1,2,3,4$.

Proof. Using (7) we obtain

$$\begin{aligned}
&\varphi\left(\alpha+\beta-\frac{\int_a^b g(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right)-\left(\varphi(\alpha)+\varphi(\beta)-\frac{\int_a^b \varphi(g(x))\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right)\\
&=\varphi(\alpha)+\left(\beta-\frac{\int_a^b g(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right)\varphi'(\beta)+\int_\alpha^\beta G_1\left(\alpha+\beta-\frac{\int_a^b g(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)},s\right)\varphi''(s)\,\mathrm{d}s\\
&\quad-\left(\varphi(\alpha)+\varphi(\alpha)+(\beta-\alpha)\varphi'(\beta)+\int_\alpha^\beta G_1(\beta,s)\varphi''(s)\,\mathrm{d}s\right)\\
&\quad+\frac{\int_a^b\left[\varphi(\alpha)+(g(x)-\alpha)\varphi'(\beta)+\int_\alpha^\beta G_1(g(x),s)\varphi''(s)\,\mathrm{d}s\right]\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\\
&=\int_\alpha^\beta\left[G_1\left(\alpha+\beta-\frac{\int_a^b g(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)},s\right)\right.\\
&\left.\quad-\left(G_1(\alpha,s)+G_1(\beta,s)-\frac{\int_a^b G_1(g(x),s)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right)\right]\varphi''(s)\,\mathrm{d}s,
\end{aligned}$$

since $G_1(\alpha,s)=0$ and

$$\begin{aligned}
&\varphi(\alpha)+\left(\beta-\frac{\int_a^b g(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right)\varphi'(\beta)-\left(\varphi(\alpha)+\varphi(\alpha)+(\beta-\alpha)\varphi'(\beta)\right)\\
&\quad+\frac{\int_a^b[\varphi(\alpha)+(g(x)-\alpha)\varphi'(\beta)]\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}=\beta\varphi'(\beta)-\frac{\int_a^b g(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\varphi'(\beta)\\
&\quad-\varphi(\alpha)-\beta\varphi'(\beta)+\alpha\varphi'(\beta)+\varphi(\alpha)+\frac{\int_a^b g(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\varphi'(\beta)-\alpha\varphi'(\beta)=0.
\end{aligned}$$

Analogously, we can prove the identities for another three Green's functions.

In the rest of the paper, for the sake of simplicity, let us denote

$$\mathscr{G}_i(s) = G_i\left(\alpha+\beta-\frac{\int_a^b g(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}, s\right) - \left(G_i(\alpha,s)+G_i(\beta,s)-\frac{\int_a^b G_i(g(x),s)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right), \tag{12}$$

for $i=1,2,3,4$.

In the same way as in [6] we generalize inequality (2) for n-convex functions using the following Taylor's formula with the integral remainder.

Let n be a positive integer, function $\varphi:[\alpha,\beta]\to\mathbb{R}$ such that $\varphi^{(n-1)}$ is absolutely continuous, and $c\in[\alpha,\beta]$. Then for all $x\in[\alpha,\beta]$

$$\varphi(x) = T_{n-1}(\varphi;c,x)+R_{n-1}(\varphi;c,x) \tag{13}$$

holds, where

$$T_{n-1}(\varphi;c,x) = \sum_{k=0}^{n-1}\frac{\varphi^{(k)}(c)}{k!}(x-c)^k \tag{14}$$

is Taylor's polynomial of degree $n-1$, and the remainder is given by

$$R_{n-1}(\varphi;c,x) = \frac{1}{(n-1)!}\int_c^x \varphi^{(n)}(t)(x-t)^{n-1}\,\mathrm{d}t. \tag{15}$$

Applying Taylor's formula (13) at the points α and β, respectively, and replacing n by $n-2$ $(n\geqslant 3)$ we have

$$\varphi''(s) = \sum_{k=0}^{n-3}\frac{\varphi^{(k+2)}(\alpha)}{k!}(s-\alpha)^k + \frac{1}{(n-3)!}\int_\alpha^s \varphi^{(n)}(t)(s-t)^{n-3}\,\mathrm{d}t, \tag{16}$$

and

$$\varphi''(s) = \sum_{k=0}^{n-3}\frac{\varphi^{(k+2)}(\beta)}{k!}(-1)^k(\beta-s)^k - \frac{1}{(n-3)!}\int_s^\beta \varphi^{(n)}(t)(s-t)^{n-3}\,\mathrm{d}t. \tag{17}$$

Proofs of the following theorems utilize the main ideas from the proofs of the similar theorems in [6], so we omit them here.

LEMMA 2. *Let $g:[a,b]\to\mathbb{R}$, $\lambda:[a,b]\to\mathbb{R}$ be as in Lemma 1, and $\mathscr{G}_i:[\alpha,\beta]\to\mathbb{R}$, $i=1,2,3,4$ defined by (12). Then for every function $\varphi:[\alpha,\beta]\to\mathbb{R}$ such that $\varphi^{(n-1)}$ is absolutely continuous for some $n\geqslant 3$, the identities*

$$\varphi\left(\alpha+\beta-\frac{\int_a^b g(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right) - \left(\varphi(\alpha)+\varphi(\beta)-\frac{\int_a^b \varphi(g(x))\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right)$$
$$= \sum_{k=0}^{n-3}\frac{\varphi^{(k+2)}(\alpha)}{k!}\int_\alpha^\beta \mathscr{G}_i(s)(s-\alpha)^k\,\mathrm{d}s + \frac{1}{(n-3)!}\int_\alpha^\beta\left(\int_t^\beta \mathscr{G}_i(s)(s-t)^{n-3}\,\mathrm{d}s\right)\varphi^{(n)}(t)\,\mathrm{d}t, \tag{18}$$

and

$$\varphi\left(\alpha+\beta-\frac{\int_a^b g(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right)-\left(\varphi(\alpha)+\varphi(\beta)-\frac{\int_a^b \varphi(g(x))\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right)$$
$$=\sum_{k=0}^{n-3}\frac{\varphi^{(k+2)}(\beta)}{k!}(-1)^k\int_\alpha^\beta \mathscr{G}_i(s)(\beta-s)^k\mathrm{d}s-\frac{1}{(n-3)!}\int_\alpha^\beta\left(\int_\alpha^t \mathscr{G}_i(s)(s-t)^{n-3}\mathrm{d}s\right)\varphi^{(n)}(t)\mathrm{d}t \tag{19}$$

hold for $i=1,2,3,4$.

Using Lemma 2 we can get the following generalizations of the Jensen-Mercer inequality for n-convex functions.

THEOREM 1. *Let* $g:[a,b]\to\mathbb{R}$ *be a continuous function, and* $[\alpha,\beta]$ *be an interval such that the image of* g *is a subset of* $[\alpha,\beta]$. *Let* $\lambda:[a,b]\to\mathbb{R}$ *be either continuous or of bounded variation, and such that* $\lambda(a)\neq\lambda(b)$ *and* $\frac{\int_a^b g(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\in[\alpha,\beta]$. *Let* $\mathscr{G}_i:[\alpha,\beta]\to\mathbb{R}, i=1,2,3,4$ *be defined by* (12). *Let* $\varphi:[\alpha,\beta]\to\mathbb{R}$ *be a* n*-convex function such that* $\varphi^{(n-1)}$ *is absolutely continuous for some* $n\geqslant 3$.

(*i*) *Then the inequality*

$$\varphi\left(\alpha+\beta-\frac{\int_a^b g(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right)-\left(\varphi(\alpha)+\varphi(\beta)-\frac{\int_a^b \varphi(g(x))\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right)$$
$$\leqslant\sum_{k=0}^{n-3}\frac{\varphi^{(k+2)}(\alpha)}{k!}\int_\alpha^\beta \mathscr{G}_i(s)(s-\alpha)^k\,\mathrm{d}s \tag{20}$$

holds for $i=1,2,3,4$. *Moreover, if* $\varphi^{(k)}(\alpha)\geqslant 0$ *for* $k=2,3,\ldots,n-1$, *then the right hand side of* (20) *is negative or equals zero and* (2) *holds.*

(*ii*) *If* n *is even then the inequality*

$$\varphi\left(\alpha+\beta-\frac{\int_a^b g(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right)-\left(\varphi(\alpha)+\varphi(\beta)-\frac{\int_a^b \varphi(g(x))\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right)$$
$$\leqslant\sum_{k=0}^{n-3}\frac{\varphi^{(k+2)}(\beta)}{k!}(-1)^k\int_\alpha^\beta \mathscr{G}_i(s)(\beta-s)^k\,\mathrm{d}s \tag{21}$$

holds for $i=1,2,3,4$. *Moreover, if* $\varphi^{(k)}(\beta)\geqslant 0$ *for* $k=2,4,\ldots,n-2$ *and* $\varphi^{(k)}(\beta)\leqslant 0$ *for* $k=3,5,\ldots,n-1$, *then the right hand side of* (21) *is negative or equals zero and* (2) *holds.*

(*iii*) *If* n *is odd then the reversed inequality* (21) *holds. Moreover, if* $\varphi^{(k)}(\beta)\leqslant 0$ *for* $k=2,4,\ldots,n-1$ *and* $\varphi^{(k)}(\beta)\geqslant 0$ *for* $k=3,5,\ldots,n-2$, *then the right hand side of the reversed inequality* (21) *is nonnegative and reverse inequality in* (2) *holds.*

REMARK 2. The right hand side of (20) can be written in the form

$$\int_{\alpha}^{\beta} \mathscr{G}_i(s) \left(\sum_{k=0}^{n-3} \frac{\varphi^{(k+2)}(\alpha)}{k!} (s-\alpha)^k \right) \mathrm{d}s.$$

Hence, in case $T_{n-3}(\varphi;\alpha,s) = \sum_{k=0}^{n-3} \frac{\varphi^{(k+2)}(\alpha)}{k!} (s-\alpha)^k \geqslant 0$ it is negative or equals zero and inequality (2) holds. Similarly, the right hand side of (21) can be written in the form

$$\int_{\alpha}^{\beta} \mathscr{G}_i(s) \left(\sum_{k=0}^{n-3} \frac{\varphi^{(k+2)}(\beta)}{k!} (s-\beta)^k \right) \mathrm{d}s.$$

Therefore, in case n is even and $T_{n-3}(\varphi;\beta,s) \geqslant 0$ inequality (2) holds, while in case n is odd and $T_{n-3}(\varphi;\beta,s) \leqslant 0$ reverse inequality in (2) holds.

REMARK 3. By use of inequalities (20) and (21) we can obtain similar results to those from [6] related to Čebyšev functionals and Ostrowski type inequalities. We can also obtain other related results, analogous to those in [2] and [5], using the main ideas from [3] and [4]. In particular, we can produce new families of n-exponentially convex and exponentially convex functions, applying functionals, constructed as differences of the right hand side and left hand side of some of the inequalities derived earlier, on some given families with the same property.

REFERENCES

[1] J. BARIĆ, A. MATKOVIĆ, *Bounds for the normalized Jensen-Mercer functional*, J. Math. Inequal. 3 (4) (2009), 529–541.

[2] S. I. BUTT, K. A. KHAN, J. PEČARIĆ, *Popovicu type inequalities via Green function and generalized Montgomery identity*, Math. Inequal. Appl. 18(4) (2015), 1519–1538.

[3] J. JAKŠETIĆ, J. PEČARIĆ, *Exponential convexity method*, J. Convex Anal. 20 (1) (2013), 181–197.

[4] J. JAKŠETIĆ, J. PEČARIĆ, A. PERUŠIĆ, *Steffensen inequality, higher order convexity and exponential convexity*, Rend. Circ. Mat. Palermo. 63(1) (2014), 109–127.

[5] M. ADIL KHAN, N. LATIF AND J. PEČARIĆ, *Generalization of majorization theorem*, J. Math. Inequal. 9 (3) (2015), 847–872.

[6] A. MATKOVIĆ, *Generalization of the Jensen-Mercer inequality by Taylor's polynomial*, Math. Inequal. Appl. 19(4) (2016), 1387–1398.

[7] T. POPOVICIU, *Les Fonctions Convexes*, Hermann, Paris, 1944.

[8] J. E. PEČARIĆ, F. PROSCHAN AND Y. L. TONG, *Convex Functions, Partial Orderings, and Statistical Applications*, Academic Press, Inc. 1992.

(Received October 30, 2018)

Anita Matković
Faculty of Electrical Engineering
Mechanical Engineering and Naval Architecture, University of Split
Ruđera Boškovića 32, 21000 Split, Croatia
e-mail: anita.matkovic@fesb.hr

Josip Pečarić
RUDN University
Miklukho-Maklaya str.6, 117198 Moscow, Russia
e-mail: pecaric@element.hr