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CONTINUOUS FORMS OF GAUSS–PÓLYA TYPE INEQUALITIES INVOLVING DERIVATIVES

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Abstract. The main aim of this paper is to give a continuous form of the Gauss-Pólya type inequalities i.e. to give inequalities involving infinitely many functions. We consider inequalities which involve derivatives and which structure is related to the Hölder inequality. Also, some properties of the corresponding functionals are given.

1. Introduction

Few decades ago an intensive research of relationships among moments of orders $2a$, $2b$ and $a+b$ has been done. The most general result is due to Pečarić and Varošanec and it is given in [7]. This result covers results of Pólya and Szegö from [8], Alzer's result from [2], Volkov's result from [13] etc. Precisely they proved the following theorem.

THEOREM 1. *Let $p_1,\dots,p_n$ be positive real numbers such that $\sum_{i=1}^{n}\frac{1}{p_i}=1$. Let $f,\ x_1,\dots,x_n:[a,b]\to\mathbf{R}$ be non-negative non-decreasing functions such that $x_1,\dots,x_n$, $\prod_{i=1}^{n}x_i^{1/p_i}$ have continuous first derivatives. Then*

$$\int_a^b\left(\prod_{i=1}^{n}(x_i(t))^{1/p_i}\right)' f(t)\,dt \geqslant \prod_{i=1}^{n}\left(\int_a^b x_i'(t)f(t)\,dt\right)^{1/p_i}. \tag{1}$$

If f is a non-increasing non-negative function, x_i satisfy the above assumptions and $x_i(a)=0$ for all $i=1,\dots,n$, then (1) holds with the reversed sign of inequality.

Putting in (1) $x_i(t)=t^{a_ip_i+1}$, where $a_ip_i>-1$ for all $i=1,\dots,n$, then for non-negative non-decreasing f we get

$$\int_a^b t^{a_1+\ldots+a_n}f(t)\,dt \geqslant \frac{\prod_{i=1}^{n}(a_ip_i+1)^{1/p_i}}{1+a_1+\ldots+a_n}\prod_{i=1}^{n}\left(\int_a^b t^{a_ip_i}f(t)\,dt\right)^{1/p_i}. \tag{2}$$

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If $a = 0$ and f is a non-negative non-increasing function, then the reversed inequality in (2) holds. In this case, b can be a real number or even $b = \infty$ (see [12]).

For $n = 2$, $p_1 = p_2 = 2$, $a = 0$, $b = 1$ inequality (2) collapses to an inequality given in [8, Vol. II, p.129], while the reversed inequality with $b = \infty$ is given in [8, Vol. I, p.83]. If $n = 2$, $p_1 = p_2 = 2$, then (2) becomes an inequality which was proved by Alzer under the additional assumptions: $x_1(a) = x_2(a)$, $x_1(b) = x_2(b)$. His method was not suitable for considering the reversed inequality.

It is worth to mention that Volkov's result from [13] (see also [9]) for a non-decreasing function f and functions $x \mapsto x^{a_i}$, $i = 1, \ldots, n$, coincides with the reversed inequality in (2) on the interval $[0, \infty\rangle$.

And finally, the Gauss inequality between the second and the fourth absolute moments, (see [3]),

$$5m_4 \geqslant 9m_2^2, \text{ where } m_r = \int_0^\infty x^r f(x)dx, \quad m_1 = 1,$$

where f is a non-increasing non-negative function, can be recognized as a particular case of the reversed inequality (2). Nowadays, inequality (2) and its reverse are known as Gauss-Pólya type inequalities.

A crucial role in the proof of the above-mentioned Theorem 1 has the well-known Hölder inequality in integral and discrete forms as well as its reverse known as the Popoviciu inequality. Very recently a continuous form (i.e. a form involving infinitely many functions) of the Popoviciu inequality has been obtained in [6]. Here we give results from that paper which will be used in the forthcoming text.

Let (X, μ) and (Y, ν) denote two σ-finite measure spaces. The following inequalities are known.

THEOREM 2. *(Continuous form of the Hölder inequality,* [4, 5]*)*
Let f be positive and measurable on $(X \times Y, \mu \times \nu)$, let u and v be weight functions on the measure spaces (X, μ) and (Y, ν), respectively, with $\int_X u(x)\, d\mu(x) = 1$. Then

$$\begin{aligned} &\int_Y \exp\left(\int_X \log f(x,y) u(x)\, d\mu(x)\right) v(y)\, d\nu(y) \\ &\leqslant \exp\left(\int_X \log\left(\int_Y f(x,y) v(y)\, d\nu(y)\right) u(x)\, d\mu(x)\right). \end{aligned} \tag{3}$$

THEOREM 3. *(Continuous form of the Popoviciu inequality,* [6]*)*
Let u and v be weight functions on the measure spaces (X, μ) and (Y, ν), respectively, and $\int_X u(x)\, d\mu(x) = 1$. Let f be a positive measurable function on $X \times Y$, v_0 is a positive real number and assume that f_0 is a positive function on X such that $v_0 f_0(x) > \int_Y f(x,y) v(y)\, d\nu(y)$ for all $x \in X$. Then the following inequality holds

$$\begin{aligned} &\exp\left(\int_X \log(v_0 f_0(x))\, u(x)\, d\mu(x)\right) - \int_Y \exp\left(\int_X \log f(x,y)\, u(x)\, d\mu(x)\right) v(y)\, d\nu(y) \\ &\geqslant \exp\left[\int_X \log\left(v_0 f_0(x) - \int_Y f(x,y) v(y)\, d\nu(y)\right) u(x)\, d\mu(x)\right]. \end{aligned} \tag{4}$$

In [6] it was open the general question to create a theory concerning such continuous forms of classical inequalities. In this paper we derive such results concerning Pólya type inequalities (see Section 2). Moreover, in Section 3 we show that the corresponding functionals also have more useful properties.

2. Main results

Before our first main theorem let us prove an useful variant of the Popoviciu inequality.

LEMMA 1. *Let $w_1 > 0,\ w_2,\ldots,w_m \geqslant 0$ be reals, $p, a_i, (i = 1,2,\ldots,m)$ be positive functions on X such that $\displaystyle\int_X \frac{d\mu(x)}{p(x)} = 1$ and a_i^p are measurable on X. Then*

$$w_1 \exp\left(\int_X \log a_1(x)\,d\mu(x)\right) - \sum_{i=2}^m w_i \exp\left(\int_X \log a_i(x)\,d\mu(x)\right)$$
$$\geqslant \exp\left\{\int_X \log\left[w_1(a_1(x))^{p(x)} - \sum_{i=2}^m w_i(a_i(x))^{p(x)}\right]\frac{d\mu(x)}{p(x)}\right\}, \tag{5}$$

provided that all integrals exist.

Proof. Without loss of generality we can assume that all w_i are positive. Put in Theorem 3:

$$f_0(x) = w_1(a_1(x))^{p(x)},\ v_0 = 1,\ u(x) = \frac{1}{p(x)},\ v(y) = 1,\ d\nu(y) = dy,$$

$$Y = [1,m],\ Y_i = [i-1,i],\ i = 2,\ldots,m \ \text{ and } f(x,y) = w_i(a_i(x))^{p(x)} \text{ for } y \in Y_i.$$

Then

$$\begin{aligned}
&\exp\left\{\int_X \log\left[w_1(a_1(x))^{p(x)} - \sum_{i=2}^m w_i(a_i(x))^{p(x)}\right]\frac{d\mu(x)}{p(x)}\right\}\\
&= \exp\left[\int_X \log\left(f_0(x) - \int_Y f(x,y)v(y)\,d\nu(y)\right)u(x)\,d\mu(x)\right]\\
&\leqslant \exp\left(\int_X \log f_0(x)u(x)\,d\mu(x)\right)\\
&\quad - \int_Y \exp\left(\int_X \log f(x,y)u(x)\,d\mu(x)\right)v(y)\,d\nu(y)\\
&= w_1 \exp\left(\int_X \log a_1(x)\,d\mu(x)\right) - \sum_{i=2}^m w_i \exp\left(\int_X \log a_i(x)\,d\mu(x)\right). \quad \square
\end{aligned}$$

By applying the same method on the continuous Hölder inequality (3) we obtain the following variant of the Hölder inequality.

LEMMA 2. *Let $w_1,\dots,w_m \geqslant 0$ be reals, $p, a_i, (i=1,2,\dots,m)$ be positive functions on X such that $\int_X \frac{d\mu(x)}{p(x)} = 1$ and a_i^p are measurable on X. Then*

$$\sum_{i=1}^{m} w_i \exp\left(\int_X \log a_i(x)\, d\mu(x)\right) \leqslant \exp\left\{\int_X \log\left[\sum_{i=1}^{m} w_i (a_i(x))^{p(x)}\right] \frac{d\mu(x)}{p(x)}\right\}, \quad (6)$$

provided that all integrals exist.

Now, let us state and prove our first main theorem, i.e. a continuous form of Gauss-Pólya inequality involving derivatives.

THEOREM 4. *(i) Suppose that $f:[a,b]\to\mathbb{R}$ is non-negative and non-decreasing, $g(x,t)$ is a positive measurable function on $X\times[a,b]$ such that the functions $t\mapsto g(x,t)$ (for $x\in X$) are non-decreasing with continuous first derivative.*

If $p(x)>0$, $\int_X \frac{d\mu(x)}{p(x)} = 1$, then

$$\int_a^b \left[\exp\left(\int_X \log g(x,t) \frac{d\mu(x)}{p(x)}\right)\right]' f(t)\, dt \geqslant \exp\left[\int_X \log\left(\int_a^b g_t'(x,t) f(t)\, dt\right) \frac{d\mu(x)}{p(x)}\right], \quad (7)$$

provided that all integrals exist and where $g_t'(x,t) = \frac{d}{dt} g(x,t)$.

(ii) Suppose that $f:[a,b]\to\mathbb{R}$ is non-negative and non-increasing, $g(x,t)$ is a positive measurable function on $X\times[a,b]$ with respect to measure $\mu\times(-f)dx$ such that the functions $t\mapsto g(x,t)$ (for $x\in X$) are non-decreasing with continuous first derivative and $g(x,a)=0$ for all $x\in X$.

If $p(x)>0$, $\int_X \frac{d\mu(x)}{p(x)} = 1$, then the reverse inequality in (7) holds.

Proof.

(i) Without loss of generality we assume that $f(b)>0$. Integrating by parts and then using the continuous Hölder inequality we get:

$$\begin{aligned}
&\int_a^b \left[\exp\left(\int_X \log g(x,t) \frac{d\mu(x)}{p(x)}\right)\right]' f(t)\, dt \\
&= f(b)\exp\left(\int_X \log g(x,b)\frac{d\mu(x)}{p(x)}\right) - f(a)\exp\left(\int_X \log g(x,a)\frac{d\mu(x)}{p(x)}\right) \\
&\quad - \int_a^b \exp\left(\int_X \log g(x,t)\frac{d\mu(x)}{p(x)}\right) df(t) \\
&\geqslant f(b)\exp\left(\int_X \log g(x,b)\frac{d\mu(x)}{p(x)}\right) - f(a)\exp\left(\int_X \log g(x,a)\frac{d\mu(x)}{p(x)}\right) \\
&\quad - \exp\left[\int_X \log\left(\int_a^b g(x,t)\, df(t)\right)\frac{d\mu(x)}{p(x)}\right].
\end{aligned}$$

Putting in Lemma 1

$$m = 3, \;\; w_1 = f(b), \;\; w_2 = f(a), \;\; w_3 = 1,$$

$$a_1(x) = (g(x,b))^{\frac{1}{p(x)}}, \;\; a_2(x) = (g(x,a))^{\frac{1}{p(x)}} \;\text{ and }\; a_3(x) = \left(\int_a^b g(x,t)\,df(t)\right)^{\frac{1}{p(x)}}$$

we get

$$\begin{aligned}
& f(b)\exp\left(\int_X \log g(x,b)\frac{d\mu(x)}{p(x)}\right) - f(a)\exp\left(\int_X \log g(x,a)\frac{d\mu(x)}{p(x)}\right)\\
& - \exp\left[\int_X \log\left(\int_a^b g(x,t)\,df(t)\right)\frac{d\mu(x)}{p(x)}\right]\\
\geqslant\; & \exp\left[\int_X \log\left(f(b)g(x,b) - f(a)g(x,a) - \int_a^b g(x,t)\,df(t)\right)\frac{d\mu(x)}{p(x)}\right]\\
=\; & \exp\left[\int_X \log\left(\int_a^b f(t)g'_t(x,t)\,dt\right)\frac{d\mu(x)}{p(x)}\right].
\end{aligned}$$

Here we integrated by parts once more and the proof of (i) has been established.

(ii) In the proof of (ii) we use the same method as in the previous proof only instead of Lemma 1 we use Lemma 2 with

$$m = 2, \quad w_1 = f(b), \; w_2 = 1,$$

$$a_1(x) = (g(x,b))^{\frac{1}{p(x)}}, \quad a_2(x) = \left(\int_a^b g(x,t)\,d(-f(t))\right)^{\frac{1}{p(x)}}. \quad \square$$

REMARK 1. If $X = [0,n]$, $p(x) = p_i$, for $x \in [i-1,i]$ and $g(x,t) = x_i(t)$, for $x \in [i-1,i]$, $i = 1,\dots,n$, then Theorem 4 coincides with Theorem 1.

If in Theorem 4 we put $g(x,t) = (g(t))^{a(x)p(x)+1}$, then we get the following result. It is a continuous form of the Gauss-Polya inequality proved in [12].

COROLLARY 1. *Let $g : [a,b] \to R$ be a non-negative increasing differentiable function and let $f : [a,b] \to R$ be a non-negative function such that the quotient f/g' is non-decreasing. Let p be a positive function such that $\int_X \frac{dx}{p(x)} = 1$. If $a(x)$ is such that $a(x) > -1/p(x)$, then*

$$\begin{aligned}
& \int_a^b (g(t))^{\int_X a(x)\,dx} f(t)\,dt\\
\geqslant\; & \frac{\exp\left(\int_X \log[a(x)p(x)+1]\frac{dx}{p(x)}\right)}{1 + \int_X a(x)\,dx}\exp\left\{\int_X \left[\log\left(\int_a^b (g(t))^{a(x)p(x)} f(t)\,dt\right)\right]\frac{dx}{p(x)}\right\},
\end{aligned}$$

provided that all integrals exist.

The following theorem is also one generalization of the Gauss-Pólya inequalities. Integral form of that result is given in [1] and here we give a continuous form of it.

THEOREM 5. *Let* $w, w_x, (x \in X)$ *be non-negative and integrable functions on* $[a,b]$ *such that* $\int_a^b w_x(s)\,ds \neq 0$, $\int_a^b w(s)\,ds \neq 0$ *and let* W *and* W_x *be defined by*

$$W(t) = \frac{\int_a^t w(s)\,ds}{\int_a^b w(s)\,ds} \text{ and } W_x(t) = \frac{\int_a^t w_x(s)\,ds}{\int_a^b w_x(s)\,ds}, \text{ respectively.}$$

Let $p(x) > 0$ *and* $\int_X \frac{dx}{p(x)} = 1$.

a) *If* f *is a non-negative non-increasing function on* $[a,b]$ *and if*

$$\exp \int_X \log W_x(t) \frac{dx}{p(x)} \geqslant W(t) \tag{8}$$

for all $t \in [a,b]$, *then*

$$\frac{\int_a^b w(t)f(t)\,dt}{\int_a^b w(t)\,dt} \leqslant \exp\left[\int_X \log\left(\frac{\int_a^b w_x(t)f(t)\,dt}{\int_a^b w_x(t)\,dt}\right)\frac{dx}{p(x)}\right]. \tag{9}$$

b) *If* f *is a non-negative non-decreasing function on* $[a,b]$ *and the reverse inequality in (8) is valid, then the inequality (9) is reversed.*

Proof.

a) Let us denote the right-hand side in (9) by A. Using integration by parts we get

$$\begin{aligned} A &= \exp\left[\int_X \log\left(\int_a^b \frac{d}{dt}W_x(t)f(t)\,dt\right)\frac{dx}{p(x)}\right] \\ &= \exp\left[\int_X \log\left(f(b) + \int_a^b W_x(t)d(-f(t))\,dt\right)\frac{dx}{p(x)}\right]. \end{aligned}$$

Putting in Lemma 2 the following:

$$m=2, \quad w_1 = w_2 = 1,\ a_1(x) = (f(b))^{\frac{1}{p(x)}} \text{ and } a_2(x) = \left(\int_a^b W_x(t)\,d(-f(t))\right)^{\frac{1}{p(x)}}$$

we get that

$$A \geqslant f(b) + \exp\left[\int_X \log\left(\int_a^b W_x(t)\,d(-f(t))\right)\frac{dx}{p(x)}\right].$$

Once again we use the Hölder inequality and get that

$$\begin{aligned} A &\geqslant f(b) + \int_a^b \exp\left[\int_X \log\left(W_x(t)\right)\frac{dx}{p(x)}\right]d(-f(t)) \geqslant f(b) + \int_a^b W(t)\,d(-f(t)) \\ &= \int_a^b W'(t)f(t)\,dt = \frac{\int_a^b w(t)f(t)\,dt}{\int_a^b w(t)\,dt}. \end{aligned}$$

b) The proof is similar, only instead of the discrete Hölder inequality from Lemma 2 we use the discrete Popoviciu inequality from Lemma 1. □

REMARK 2. Putting $w_x(t) = g_t'(x,t), w(t) = \left[\exp\left(\int_X \log g(x,t)\frac{dx}{p(x)}\right)\right]'$, then we get

$$W_x(t) = \frac{g(x,t)}{g(x,b)} \text{ and } W(t) = \frac{\exp\left(\int_X \log g(x,t)\frac{dx}{p(x)}\right)}{\exp\left(\int_X \log g(x,b)\frac{dx}{p(x)}\right)}$$

and, hence,

$$\exp\int_X \log W_x(t)\frac{dx}{p(x)} = W(t).$$

So we can apply Theorem 5 in both cases a) and b) and we get the result of Theorem 4.

3. Some further results for the corresponding functionals

There are several functionals which are connected with inequalities from the previous chapter. These functionals have properties which lead to refinements and improvements of the original inequalities.

Firstly, let us define the functional G:

$$G(f) = \exp\left[\int_X \log\left(\int_a^b g_t'(x,t)f(t)\,dt\right)\frac{d\mu(x)}{p(x)}\right] - \int_a^b \left[\exp\left(\int_X \log g(x,t)\frac{d\mu(x)}{p(x)}\right)\right]' f(t)\,dt.$$

Under the assumptions of Theorem 4 (i) G is non-positive, while under the assumptions given in Theorem 4 (ii) G is non-negative. Also, note that $G(f)$ is positive homogeneous. But, G has more useful properties which are described in the following theorem.

THEOREM 6. *Let the functions p and g satisfy the assumptions from Theorem 4. Let f_1, f_2 be non-negative functions monotone in the same sense and such that $G(f_1)$ and $G(f_2)$ are well-defined. Then*

$$G(f_1 + f_2) \geqslant G(f_1) + G(f_2).$$

Moreover, if $g(x,a) = 0$ for all $x \in X$ and if f_1 and f_2 are non-negative non-increasing functions such that $f_2 \geqslant f_1$, $f_2 - f_1$ is non-increasing, $G(f_1)$, $G(f_2)$ and $G(f_2 - f_1)$ are well-defined, then

$$G(f_2) \geqslant G(f_1).$$

Proof.

(a) Let us estimate the difference $G(f_1+f_2)-G(f_1)-G(f_2)$.

$$\begin{aligned}
&G(f_1+f_2)-G(f_1)-G(f_2)\\
&=\exp\left[\int_X \log\left(\int_a^b g_t'(x,t)[f_1(t)+f_2(t)]\,dt\right)\frac{d\mu(x)}{p(x)}\right]\\
&\quad-\int_a^b\left[\exp\left(\int_X \log g(x,t)\frac{d\mu(x)}{p(x)}\right)\right]'[f_1(t)+f_2(t)]\,dt\\
&\quad-\exp\left[\int_X \log\left(\int_a^b g_t'(x,t)f_1(t)\,dt\right)\frac{d\mu(x)}{p(x)}\right]\\
&\quad-\exp\left[\int_X \log\left(\int_a^b g_t'(x,t)f_2(t)\,dt\right)\frac{d\mu(x)}{p(x)}\right]\\
&\quad+\int_a^b\left[\exp\left(\int_X \log g(x,t)\frac{d\mu(x)}{p(x)}\right)\right]' f_1(t)\,dt\\
&\quad+\int_a^b\left[\exp\left(\int_X \log g(x,t)\frac{d\mu(x)}{p(x)}\right)\right]' f_2(t)\,dt\\
&=\exp\left[\int_X \log\left(\int_a^b g_t'(x,t)[f_1(t)+f_2(t)]\,dt\right)\frac{d\mu(x)}{p(x)}\right]\\
&\quad-\exp\left[\int_X \log\left(\int_a^b g_t'(x,t)f_1(t)\,dt\right)\frac{d\mu(x)}{p(x)}\right]\\
&\quad-\exp\left[\int_X \log\left(\int_a^b g_t'(x,t)f_2(t)\,dt\right)\frac{d\mu(x)}{p(x)}\right]\geqslant 0,
\end{aligned}$$

where we use Lemma 2 with $m=2$, $w_i=1$ and $a_i(x)=[\int_a^b g_t'(x,t)f_i(t)dt]^{\frac{1}{p(x)}}$ for $i=1,2$.

(b) Since $G(f_2-f_1)\geqslant 0$ by Theorem 4 we have

$$G(f_2)=G(f_1+(f_2-f_1))\geqslant G(f_1)+G(f_2-f_1)\geqslant G(f_1). \qquad \square$$

Having in mind Corollary 1 we define the functional GP as

$$\begin{aligned}
GP(f)=\exp&\left(\int_X \log[a(x)p(x)+1]\frac{dx}{p(x)}\right)\exp\left\{\int_X\left[\log\left(\int_a^b g(t)^{a(x)p(x)}f(t)\,dt\right)\right]\frac{dx}{p(x)}\right\}\\
&-\left(1+\int_X a(x)\,dx\right)\int_a^b g(t)^{\int_X a(x)\,dx}f(t)\,dt.
\end{aligned}$$

Using the same method of proving we have results about positivity, superadditivity and monotonicity which generalizes Theorem 2.1 from [10].

Moreover, we can define the new functional as follows

$$G_W(f)=\exp\left[\int_X \log\left(\frac{\int_a^b w_x(t)f(t)\,dt}{\int_a^b w_x(t)\,dt}\right)\frac{dx}{p(x)}\right]-\frac{\int_a^b w(t)f(t)\,dt}{\int_a^b w(t)\,dt}$$

and under the conditions like in Theorem 5 we have that the functional G_W is superadditive and monotone.

Let w be a positive function on X and let denote $W = \int_X w(x)d\mu(x)$. The following functionals are regarded as the functions of weight w.

$$K(w) = \left\{ \int_a^b \left[\exp\left(\int_X \log g(x,t) \frac{w(x)}{W} d\mu(x) \right) \right]' f(t)\, dt \right\}^W,$$

$$H(w) = \frac{\exp\left[\int_X \log\left(\int_a^b g_t'(x,t) f(t)\, dt \right) w(x) d\mu(x) \right]}{K(w)}.$$

THEOREM 7. *Let the functions f and g satisfy the assumptions from Theorem 4 (ii). Let v and w be positive functions such that K and H are well-defined for them. Then*

$$K(v+w) \leqslant K(v) \cdot K(w), \quad H(v+w) \geqslant H(v) \cdot H(w).$$

If, additionally, $v \geqslant w$, such that $H(v-w)$ is well-defined, then

$$H(v) \geqslant H(w).$$

Proof. For the weights v and w denote $V = \int_X v(x)d\mu(x)$ and $W = \int_X w(x)d\mu(x)$. Transforming $K(v+w)$ we get

$$\begin{aligned}
&K(v+w) \\
&= \left\{ \int_a^b \left[\exp\left(\int_X \log g(x,t) \left(\frac{v(x)}{V} \cdot \frac{V}{V+W} + \frac{w(x)}{W} \cdot \frac{W}{V+W} \right) d\mu(x) \right) \right]' f(t)\, dt \right\}^{V+W} \\
&= \left\{ \int_a^b \left[\exp\left(\int_X \log g(x,t) \cdot \frac{v(x)}{V} \cdot \frac{V}{V+W} d\mu(x) \right) \right.\right. \\
&\qquad \left.\left. \cdot \exp\left(\int_X \log g(x,t) \cdot \frac{w(x)}{W} \cdot \frac{W}{V+W} d\mu(x) \right) \right]' f(t)\, dt \right\}^{V+W} \\
&= \left\{ \int_a^b \left[\left(\exp\left(\int_X \log g(x,t) \cdot \frac{v(x)}{V} d\mu(x) \right) \right)^{\frac{V}{V+W}} \right.\right. \\
&\qquad \left.\left. \cdot \left(\exp\left(\int_X \log g(x,t) \cdot \frac{w(x)}{W} d\mu(x) \right) \right)^{\frac{W}{V+W}} \right]' f(t)\, dt \right\}^{V+W} \\
&\leqslant \left\{ \int_a^b \left[\exp\left(\int_X \log g(x,t) \cdot \frac{v(x)}{V} d\mu(x) \right) \right]' f(t)\, dt \right\}^{V}
\end{aligned}$$

$$\cdot\left\{\int_a^b\left[\exp\left(\int_X \log g(x,t)\cdot\frac{w(x)}{W}d\mu(x)\right)\right]' f(t)\,dt\right\}^W = K(v)\cdot K(w),$$

where the above inequality follows from Theorem 1 for

$$n=2,\quad p_1=\frac{V+W}{V},\quad p_2=\frac{V+W}{W},$$

$x_1(t)=\exp\left(\int_X \log g(x,t)\cdot\frac{v(x)}{V}d\mu(x)\right)$ and $x_2(t)=\exp\left(\int_X \log g(x,t)\cdot\frac{w(x)}{W}d\mu(x)\right)$.

The above inequality is used in the proof of the property for H. Namely, we get

$$\begin{aligned}
H(v+w) &= \frac{\exp\left[\int_X \log\left(\int_a^b g_t'(x,t)f(t)\,dt\right)(v(x)+w(x))d\mu(x)\right]}{K(v+w)}\\
&\geqslant \frac{1}{K(v)K(w)}\cdot\exp\left[\int_X \log\left(\int_a^b g_t'(x,t)f(t)\,dt\right)v(x)d\mu(x)\right]\\
&\qquad\cdot\exp\left[\int_X \log\left(\int_a^b g_t'(x,t)f(t)\,dt\right)w(x)d\mu(x)\right]\\
&= H(v)\cdot H(w).
\end{aligned}$$

The last inequality follows since $H(v-w)\geqslant 1$ by Theorem 4 (ii), so that

$$H(v)=H(v-w+w)\geqslant H(v-w)\cdot H(w)\geqslant H(w). \quad \square$$

If m, M are positive real numbers such that $mv\leqslant w\leqslant Mv$ and if the assumptions of the previous theorem hold, then

$$H^m(v)\leqslant H(w)\leqslant H^M(v).$$

For the particular choice of v and w we have the following interesting refinement and improvement of the continuous form of the Gauss-Pólya inequality.

COROLLARY 2. *Let f and g satisfy the assumptions from Theorem 4 (ii) and $\mu(X)=\int_X d\mu(x)\neq 0$ and finite. Let w be a positive function such that K and H are well-defined for w and for the constant function. Then*

$$\begin{aligned}
&\left\{\frac{\exp\left[\int_X \log\left(\int_a^b g_t'(x,t)f(t)\,dt\right)d\mu(x)\right]}{\left\{\int_a^b\left[\exp\left(\int_X \log g(x,t)\frac{d\mu(x)}{\mu(X)}\right)\right]' f(t)\,dt\right\}^{\mu(X)}}\right\}^{w_{\min}}\\
&\leqslant \frac{\exp\left[\int_X \log\left(\int_a^b g_t'(x,t)f(t)\,dt\right)w(x)d\mu(x)\right]}{\left\{\int_a^b\left[\exp\left(\int_X \log g(x,t)\frac{w(x)}{W}d\mu(x)\right)\right]' f(t)\,dt\right\}^{W}}
\end{aligned}$$

$$\leqslant \left\{ \frac{\exp\left[\int_X \log\left(\int_a^b g_t'(x,t) f(t)\, dt\right) d\mu(x)\right]}{\left\{\int_a^b \left[\exp\left(\int_X \log g(x,t) \frac{d\mu(x)}{\mu(X)}\right)\right]' f(t)\, dt\right\}^{\mu(X)}} \right\}^{w_{\max}},$$

where $w_{\min} = \min\{w(x) : x \in X\}$ *and* $w_{\max} = \max\{w(x) : x \in X\}$.

Proof. It is a consequence of Theorem 7 for the weights $w_{\min}$, w and $w_{\max}$. □

REMARK 3. The results of Theorem 7 and Corollary 2 are the continuous forms of some results given in [11].

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