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# SOBOLEV'S THEOREM FOR DOUBLE PHASE FUNCTIONALS

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*Abstract.* Our aim in this paper is to establish generalizations of Sobolev's theorem for double phase functionals $\Phi(x,t) = t^p + \{b(x)t(\log(e+t))^{\tau}\}^q$, where $1 < p \leqslant q < \infty$, $\tau > 0$ and $b$ is a nonnegative bounded function satisfying $|b(x) - b(y)| \leqslant C|x-y|^{\theta}(\log(e+|x-y|^{-1}))^{-\tau}$ for $0 \leqslant \theta < 1$.

## 1. Introduction

In harmonic analysis, the maximal operator is a classical tool when studying Sobolev functions and partial differential equations. This also plays a central role in the study of differentiation, singular integrals, smoothness of functions and so on ([5, 21, 25], etc.). It is well known that the maximal operator is bounded on the Lebesgue space $L^p(\mathbf{R}^N)$ if $p > 1$ ([25]).

One of important applications of the boundedness of the maximal operator is Sobolev's inequality; in the classical case,

$$\|I_\alpha * f\|_{p^*} \leqslant C\|f\|_p$$

for $f \in L^p(\mathbf{R}^N)$, $0 < \alpha < N$ and $1 < p < N/\alpha$, where $I_\alpha$ is the Riesz kernel of order $\alpha$ and $1/p^* = 1/p - \alpha/N$ (see, e.g. [2, Theorem 3.1.4]).

There has been a considerable amount of studies on the variable exponent Lebesgue spaces and Sobolev spaces; see [11, 13] for a survey. We refer to [1, 24] for the study of elasticity and fluid mechanics, [7, 19] for the study of image processing, and [9, 10] for double phase variational problems. Capone, Cruz-Uribe and Fiorenza [6] studied a Sobolev type inequality for Riesz potentials in the variable exponent Lebesgue space $L^{p(\cdot)}(\mathbf{R}^N)$. For Sobolev's theorem for Riesz potentials, see also [12], [14], [20] etc..

Recently, regarding regularity theory of differential equations, Baroni, Colombo and Mingione [3, 4, 9, 10] studied a double phase functional $\Phi(x,t) = t^p + a(x)t^q$, $x \in \mathbf{R}^N$, $t \geqslant 0$ where $1 < p < q$, $a(\cdot)$ is non-negative, bounded and Hölder continuous of order $\theta \in (0,1]$. In [9], the minimization problem of the double phase functional was discussed under the assumption $q < (1+\theta/N)p$. Regularity for general functionals

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was studied under the condition $q \leqslant (1+\theta/N)p$ in [4]. In [3], the border-line ($p=q$) double phase functional

$$\Phi(x,t) = t^p + a(x)t^p(\log(e+t)$$

was considered. In [18], Harjulehto, Hästö and Karppinen studied local higher integrability of the gradient of a quasiminimizer of the double phase functional $\Phi(x,t) = t^p + a(x)t^q$. See Colasuonno and Squassina [8] for the eigenvalue problem for the double phase functional $\Phi(x,t) = t^p + a(x)t^q$. See also [16].

In the present paper, for $1 < p \leqslant q < \infty$ and $\tau \geqslant 0$, let us consider the double phase functional

$$\Phi(x,t) = t^p + \{b(x)t(\log(e+t))^\tau\}^q,$$

where $b$ is a nonnegative bounded function satisfying

$$|b(x)-b(y)| \leqslant C|x-y|^\theta(\log(e+|x-y|^{-1}))^{-\tau}$$

for $0 \leqslant \theta < 1$. Hästö [15, Theorem 4.7] showed the boundedness of the maximal operator on $L^\Phi(G)$ when $a(x) = b(x)^q$ is $\theta$-Hölder continuous and $\tau = 0$. See also [17, Proposition 7.2.3].

Our first aim in this paper is to give the boundedness of the maximal operator for the double phase functional $\Phi(x,t)$ (Theorem 1), as an extension of [15, Theorem 4.7]. To show this, we apply [23, Corollary 3.2]. Our strategy is to check all the conditions required in [23, Corollary 3.2] as in the proof of [23, Corollary 5.3]. Next, we give a Sobolev type inequality for $\Phi(x,t)$ (Theorem 2) by applying [23, Theorem 4.9] as in the proof of [23, Corollary 5.9].

For reader's convenience we give direct proofs of Theorems 1 and 2 in the Appendix, by applying the boundedness of the maximal operator on $L^p(\mathbf{R}^N)$ and $L^q(\mathbf{R}^N)$.

Finally, we discuss the continuity of the fractional maximal functions and Riesz potentials for the double phase functional $\Phi(x,t)$ (see Theorems 3-5). The result is new even for the case $\tau = 0$.

Throughout this paper, let $C$ denote various constants independent of the variables in question and $C(a,b,\cdots)$ be a constant that depends on $a,b,\cdots$. The symbol $g \sim h$ means that $C^{-1}h \leqslant g \leqslant Ch$ for some constant $C > 0$.

## 2. Preliminaries

In this paper, we consider the following double phase functional

$$\Phi(x,t) = t^p + \{b(x)t(\log(e+t))^\tau\}^q,$$

for $1 < p \leqslant q < \infty$ and $\tau \geqslant 0$, where $b$ is a nonnegative bounded function satisfying

$$|b(x)-b(y)| \leqslant C|x-y|^\theta(\log(e+|x-y|^{-1}))^{-\tau} \tag{1}$$

for $0 \leqslant \theta < 1$.

The Musielak-Orlicz space $L^{\Phi}(\mathbf{R}^N)$ is defined by

$$L^{\Phi}(\mathbf{R}^N) = \left\{ f \in L^1_{\mathrm{loc}}(\mathbf{R}^N) : \int_{\mathbf{R}^N} \Phi\left(y, \frac{|f(y)|}{\lambda}\right) dy < \infty \text{ for some } \lambda > 0 \right\}.$$

It is a Banach space with respect to the norm

$$\|f\|_{L^{\Phi}(\mathbf{R}^N)} = \inf\left\{ \lambda > 0 : \int_{\mathbf{R}^N} \Phi\left(y, \frac{|f(y)|}{\lambda}\right) dy \leqslant 1 \right\}.$$

For later use, we prepare the following result.

LEMMA 1. *Let* $1 \leqslant q < \infty$ *and* $\tau \geqslant 0$. *Then there exists a constant* $C > 0$ *such that*

$$\begin{aligned}\frac{1}{|E|}\int_E |f(y)|\,dy \leqslant & C(\log(e+r^{-1}))^{-\tau} \\ & \times \frac{1}{|E|}\int_E |f(y)|\,(\log(e+|f(y)|))^{\tau}\,dy + r^{-N/q}(\log(e+r^{-1}))^{-\tau}\end{aligned}$$

*for all* $r > 0$ *and measurable sets* $E \subset \mathbf{R}^N$ *of positive measure.*

*Proof.* Set $R = r^{-N/q}(\log(e+r^{-1}))^{-\tau}$. We have

$$\begin{aligned}\frac{1}{|E|}\int_E |f(y)|\,dy \leqslant & \frac{1}{|E|}\int_E |f(y)|\left(\frac{\log(e+f(y))}{\log(e+R)}\right)^{\tau} dy + R \\ \leqslant & C(\log(e+r^{-1}))^{-\tau}\frac{1}{|E|}\int_E |f(y)|\,(\log(e+|f(y)|))^{\tau}\,dy \\ & + r^{-N/q}(\log(e+r^{-1}))^{-\tau},\end{aligned}$$

as required. □

COROLLARY 1. *Let* $1 \leqslant q < \infty$ *and* $\tau \geqslant 0$. *Then there exists a constant* $C > 0$ *such that*

$$\begin{aligned}\frac{1}{|E|}\int_E b(y)|f(y)|\,dy \leqslant C\Bigg\{ & r^{-N/q}(\log(e+r^{-1}))^{-\tau} + (\log(e+r^{-1}))^{-\tau} \\ & \times \left(\frac{1}{|E|}\int_E \left(b(y)|f(y)|\,(\log(e+|f(y)|))^{\tau}\right)^q dy\right)^{1/q}\Bigg\}\end{aligned}$$

*for all* $r > 0$ *and measurable sets* $E \subset \mathbf{R}^N$ *of positive measure.*

In fact, Lemma 1, the boundedness of $b$ and Jensen's inequality give

$$\frac{1}{|E|}\int_E b(y)|f(y)|\,dy$$

$$\leqslant C\left\{r^{-N/q}(\log(e+r^{-1}))^{-\tau}+(\log(e+r^{-1}))^{-\tau}\frac{1}{|E|}\int_E b(y)|f(y)|(\log(e+|f(y)|))^{\tau}\,dy\right\}$$

$$\leqslant C\Bigg\{r^{-N/q}(\log(e+r^{-1}))^{-\tau}+(\log(e+r^{-1}))^{-\tau}$$
$$\times\left(\frac{1}{|E|}\int_E\left(b(y)|f(y)|(\log(e+|f(y)|))^{\tau}\right)^q dy\right)^{1/q}\Bigg\}.$$

## 3. Maximal operator

For a locally integrable function $f$ on $\mathbf{R}^N$, the Hardy-Littlewood maximal function $Mf$ is defined by

$$Mf(x)=\sup_{r>0}\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)|\,dy,$$

where $B(x,r)$ is the ball in $\mathbf{R}^N$ with center $x$ and of radius $r>0$ and $|B(x,r)|$ denotes its Lebesgue measure. The mapping $f\mapsto Mf$ is called the maximal operator.

Recall that

$$\Phi(x,t)=t^p+\{b(x)t(\log(e+t))^{\tau}\}^q,$$

where $1<p\leqslant q<\infty$, $\tau\geqslant 0$ and $b$ is a nonnegative bounded function satisfying (1) with $0\leqslant\theta<1$.

In [23], we consider the following conditions for $\Phi(x,t)$. It is easy to check the following conditions on $\Phi$ required in [23]:

($\Phi$1) $\Phi(\cdot,t)$ is measurable on $\mathbf{R}^N$ for each $t\geqslant 0$ and $\Phi(x,\cdot)$ is continuous on $[0,\infty)$ for each $x\in\mathbf{R}^N$;

($\Phi$2) there exists a constant $A_1\geqslant 1$ such that

$$A_1^{-1}\leqslant\Phi(x,1)\leqslant A_1\quad\text{for all } x\in\mathbf{R}^N;$$

($\Phi3;0;p$) $t\mapsto t^{-p}\Phi(x,t)$ is increasing on $(0,1]$ for each $x\in\mathbf{R}^N$;

($\Phi3;\infty;p$) $t\mapsto t^{-p}\Phi(x,t)$ is increasing on $[1,\infty)$ for each $x\in\mathbf{R}^N$.

LEMMA 2. *$\Phi(x,t)$ satisfies*

($\Phi5;\nu$) *for every $\gamma>0$, there exists a constant $B_{\gamma,\nu}\geqslant 1$ such that*

$$\Phi(x,t)\leqslant B_{\gamma,\nu}\Phi(y,t)$$

*whenever $x,y\in\mathbf{R}^N$, $|x-y|\leqslant\gamma t^{-\nu}$ and $t\geqslant 1$*

*for $\nu\geqslant(q-p)/(q\theta)$; and*

($\Phi6;\omega$) *there exist a function* $g$ *on* $\mathbf{R}^N$ *and a constant* $B_\infty \geqslant 1$ *such that* $0 \leqslant g(x) \leqslant 1$ *for all* $x \in \mathbf{R}^N$, $g \in L^\omega(\mathbf{R}^N)$ *and*

$$B_\infty^{-1}\Phi(x,t) \leqslant \Phi(x',t) \leqslant B_\infty\Phi(x,t)$$

*whenever* $x, x' \in \mathbf{R}^N$, $|x'| \geqslant |x|$ *and* $g(x) \leqslant t \leqslant 1$,

*for every* $\omega > 0$.

*Proof.* Let $\nu \geqslant (q-p)/(q\theta)$. If $|x-y| \leqslant \gamma t^{-\nu}$ and $t \geqslant 1$, then

$$\begin{aligned}\Phi(x,t) &= t^p + \{b(x)t(\log(e+t))^\tau\}^q \\ &\leqslant t^p + \left\{b(y)t(\log(e+t))^\tau + C|x-y|^\theta(\log(e+|x-y|^{-1}))^{-\tau}t(\log(e+t))^\tau\right\}^q \\ &\leqslant C\left\{\Phi(y,t) + t^{(1-\theta\nu)q}\right\} \leqslant C\{\Phi(y,t)+t^p\} \leqslant C\Phi(y,t).\end{aligned}$$

Hence $\Phi(x,t)$ satisfies ($\Phi5;\nu$).

Let $g \in L^\omega(\mathbf{R}^N)$ for $\omega > 0$. If $g(x) \leqslant t \leqslant 1$, then

$$\Phi(x,t) \leqslant (1 + C\|b\|^q_{L^\infty(\mathbf{R}^N)})t^p \leqslant (1 + C\|b\|^q_{L^\infty(\mathbf{R}^N)})\Phi(x',t)$$

for every $x$, $x' \in \mathbf{R}^N$ and $|x'| \geqslant |x|$. Therefore $\Phi(x,t)$ satisfies ($\Phi6;\omega$). □

As an extension of [15, Theorem 4.7], we obtain the following result by Lemma 2 and [23, Corollary 3.2] (see also [17, Proposition 7.2.3] and [22]).

THEOREM 1. *Suppose* $1 < p \leqslant q < \infty$, $\tau \geqslant 0$ *and* $1/p - 1/q \leqslant \theta/N$. *Then there is a constant* $C > 0$ *such that*

$$\int_{\mathbf{R}^N} \Phi(x, Mf(x))\,dx < C$$

*for all* $f \in L^\Phi(\mathbf{R}^N)$ *with* $\|f\|_{L^\Phi(\mathbf{R}^N)} \leqslant 1$.

REMARK 1. When $\tau = 0$, $\Phi(x,t) = t^p + a(x)t^q$, $1 < p < q$, $G \subset \mathbf{R}^N$ is bounded, $a \in C^\theta(\bar{G})$ is non-negative and $q \leqslant (1+\theta/N)p$, Hästö showed Theorem 1 in [15, Theorem 4.7].

## 4. Sobolev's inequality

For $0 < \alpha < N$, we define the Riesz potential of order $\alpha$ of a function $f \in L^1_{\mathrm{loc}}(\mathbf{R}^N)$ by

$$I_\alpha f(x) = \int_{\mathbf{R}^N} |x-y|^{\alpha-N} f(y)\,dy.$$

In this section, let $g(x) = (1+|x|)^{-(N+1)}$. Then

$$g^*(x) = \min\{g(x), Mg(x)\} \leqslant C(1+|x|)^{-N}.$$

Further set

$$\Phi_\infty(t) = t^p$$

as in [23]. Then it satisfies the following conditions:

($\Phi_\infty 0$) $\Phi_\infty(t)$ is continuous, $\Phi_\infty(t) > 0$ for $t > 0$ and $\Phi_\infty(t)/t$ is increasing on $[0,\infty)$;

($\Phi_\infty 1$) there exists a constant $\tilde{B}_\infty \geqslant 1$ such that

$$\tilde{B}_\infty^{-1}\Phi(x,t) \leqslant \Phi_\infty(t) \leqslant \tilde{B}_\infty\Phi(x,t) \quad \text{whenever } g(x) \leqslant t \leqslant 1$$

for $g(x)$ in condition $(\Phi 6;\omega)$;

($\Phi_\infty 2$) there exists a constant $c_\infty \geqslant 1$ such that

$$\Phi_\infty(g^*(x)) \leqslant c_\infty(1+|x|)^{-N}$$

for all $x \in \mathbf{R}^N$;

($\Phi_\infty N$) $r \mapsto r^\gamma \Phi_\infty^{-1}(r^{-N})$ is increasing on $(1,\infty)$ for some $0 < \gamma < N$.

LEMMA 3. (1) *If $q < N/\alpha$, then*

($\Phi N\alpha$) $r \mapsto r^{\alpha+\varepsilon}\Phi^{-1}(x,r^{-N})$ *is uniformly almost decreasing on $(0,\infty)$ for some $\varepsilon > 0$, that is, there exists a constant $C > 0$ such that*

$$r^{\alpha+\varepsilon}\Phi^{-1}(x,r^{-N}) \leqslant C s^{\alpha+\varepsilon}\Phi^{-1}(x,s^{-N})$$

*for all $0 < s < r$ and $x \in \mathbf{R}^N$.*

(2) *If $p < N/\alpha$, then*

($\Phi_\infty N\alpha$) $r \mapsto r^{\alpha+\varepsilon}\Phi_\infty^{-1}(r^{-N})$ *is increasing on $(1,\infty)$ for some $\varepsilon > 0$.*

*Proof.*

(1) Since

$$\Phi^{-1}(x,s) \sim \min\left\{s^{1/p}, \left(b(x)^{-q}s\right)^{1/q}\left(\log(e+b(x)^{-q}s)\right)^{-\tau}\right\}$$

for $x \in \mathbf{R}^N$, we have

$$r^{\alpha+\varepsilon}\Phi^{-1}(x,r^{-N})$$
$$\sim \min\left\{r^{\alpha-N/p+\varepsilon}, b(x)^{-(\alpha+\varepsilon)q/N}\left(b(x)^q r^N\right)^{(\alpha-N/q+\varepsilon)/N}\left(\log(e+b(x)^{-q}r^{-N})\right)^{-\tau}\right\}.$$

Choose $\varepsilon > 0$ such that $N/q - \alpha > \varepsilon$. Then we obtain that $r \mapsto r^{\alpha+\varepsilon}\Phi^{-1}(x,r^{-N})$ is uniformly almost decreasing.

(2) Since $\Phi_\infty^{-1}(r^{-N}) = r^{-N/p}$ for $r \geqslant 1$, we find $r^\alpha\Phi_\infty^{-1}(r^{-N}) = r^{\alpha-N/p}$. Thus, ($\Phi_\infty N\alpha$) holds if $p < N/\alpha$. □

Now, consider the double phase functional

$$\Psi(x,t) = t^{p^*} + \{b(x)t(\log(e+t))^\tau\}^{q^*}$$

and

$$\Psi_1(x,t) = t^{p^*} + \left\{b(x)t\left(\log\left(e+b(x)^{\alpha q/N}t\right)\right)^\tau\right\}^{q^*},$$

where $1/p^* = 1/p - \alpha/N > 0$ and $1/q^* = 1/q - \alpha/N > 0$. Then there is $C > 1$ such that

$$C^{-1}\Psi(x,t) \leqslant \Psi_1(x,t) \leqslant C\Psi(x,t) \tag{2}$$

for all $x \in \mathbf{R}^N$ and $t > 0$. In fact, the right inequality is clear since $b$ is bounded. To show the left inequality for $t > 1$, take $\varepsilon$ such that $0 < \varepsilon < p^*/q^*$. If $b(x)t \leqslant t^\varepsilon$ and $t > 1$, then

$$\{b(x)t(\log(e+t))^\tau\}^{q^*} \leqslant Ct^{p^*}$$

and if $b(x)t \geqslant t^\varepsilon > 1$, then $b(x)^{\alpha q/N}t \geqslant t^{1-(\alpha q/N)(1-\varepsilon)}$, so that

$$\Psi_1(x,t) \geqslant t^{p^*} + \left\{b(x)t\left(\log\left(e+t^{1-(\alpha q/N)(1-\varepsilon)}\right)\right)^\tau\right\}^{q^*} \geqslant C\Psi(x,t)$$

since $\alpha q/N < 1$, which proves (2).

Further we see that both $\Psi$ and $\Psi_1$ satisfy

($\Psi$1) $\Psi(\cdot,t)$ is measurable on $\mathbf{R}^N$ for each $t \geqslant 0$ and $\Psi(x,\cdot)$ is continuous on $[0,\infty)$ for each $x \in \mathbf{R}^N$;

($\Psi$2) there exists a constant $\widehat{Q}_1 \geqslant 1$ such that

$$\widehat{Q}_1^{-1} \leqslant \Psi(x,1) \leqslant \widehat{Q}_1 \quad \text{for all } x \in \mathbf{R}^N;$$

($\Psi$3) $t \mapsto \Psi(x,t)/t$ is increasing on $(0,\infty)$ for all $x \in \mathbf{R}^N$.

LEMMA 4. *Both $\Psi$ and $\Psi_1$ satisfy*

($\Psi$4) *there exists a constant $\widehat{Q}_3 \geqslant 1$ such that*

$$\Psi\left(x, t\Phi(x,t)^{-\alpha/N}\right) \leqslant \widehat{Q}_3\Phi(x,t)$$

*for all $x \in \mathbf{R}^N$ and $t > 0$.*

*Proof.* Since $\Phi(x,t) \geqslant \max\{t^p, \{b(x)t(\log(e+t))^\tau\}^q\}$, we see that

$$t\Phi(x,t)^{-\alpha/N} \leqslant \min\left\{t^{p/p^*}, b(x)^{-\alpha q/N}t^{q/q^*}(\log(e+t))^{-\alpha q\tau/N}\right\}.$$

Hence

$$\Psi_1\left(x, t\Phi(x,t)^{-\alpha/N}\right) \leqslant C\left[\left\{t^{p/p^*}\right\}^{p^*} + \left\{b(x)^{q/q^*}t^{q/q^*}(\log(e+t))^{-\alpha q\tau/N}\right.\right.$$

$$\times \left( \log \left( e + t^{q/q^*} (\log(e+t))^{-\alpha q \tau/N} \right) \right)^{\tau} \Big\}^{q^*} \Big]$$
$$\leqslant C \left[ t^p + \{ b(x) t (\log(e+t))^{\tau} \}^q \right] = C\Phi(x,t),$$

as required. □

Consequently we apply [23, Theorem 4.9] to obtain the following result.

THEOREM 2. *Suppose* $1 < p \leqslant q < N/\alpha$, $\tau \geqslant 0$ *and* $1/p - 1/q \leqslant \theta/N$. *Then there is a constant* $C > 0$ *such that*

$$\int_{\mathbf{R}^N} \Psi(x, |I_\alpha f(x)|)\, dx < C$$

*for all* $f \in L^{\Phi}(\mathbf{R}^N)$ *with* $\|f\|_{L^{\Phi}(\mathbf{R}^N)} \leqslant 1$.

REMARK 2. If $1/p - 1/q \leqslant \theta/N$, then take $\theta_1$ such that

$$1/p - 1/q = \theta_1/N.$$

Then $b$ is $\theta_1$-Hölder continuous when it is $\theta$-Hölder continuous and bounded, and thus we may assume from the beginning that $1/p - 1/q = \theta/N$.

## 5. Continuity

For $0 < \sigma < N$ and a function $f \in L^1_{\mathrm{loc}}(\mathbf{R}^N)$ we define the fractional maximal function by

$$M_\sigma f(x) = \sup_{r>0} \frac{r^\sigma}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy.$$

THEOREM 3. *Suppose* $1 < p \leqslant q < \infty$, $\tau \geqslant 0$, $\sigma - N/p \leqslant 0$, $0 \leqslant \sigma + \theta - N/p < 1/p'$ *and* $1/p - 1/q = \theta/N \geqslant 0$. *Then there is a constant* $C > 0$ *such that*

$$|b(x) M_\sigma f(x) - b(z) M_\sigma f(z)| \leqslant C|x-z|^{\sigma+\theta-N/p} (\log(e + |x-z|^{-1}))^{-\tau} \tag{3}$$

*for all* $x, z \in \mathbf{R}^N$ *with* $0 < |x-z| < 1/2$ *and* $f \in L^{\Phi}(\mathbf{R}^N)$ *with* $\|f\|_{L^{\Phi}(\mathbf{R}^N)} \leqslant 1$.

*Proof.* Let $f$ be a measurable function on $\mathbf{R}^N$ with $\|f\|_{L^{\Phi}(\mathbf{R}^N)} \leqslant 1$. For $x \in \mathbf{R}^N$ and $r > 0$ set

$$I(x,r) = b(x) \frac{r^\sigma}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy.$$

First we consider the case $0 < r < 2|x-z| < 1$. Then we have

$$I(x,r) = \frac{r^\sigma}{|B(x,r)|} \int_{B(x,r)} (b(x) - b(y)) |f(y)|\, dy + \frac{r^\sigma}{|B(x,r)|} \int_{B(x,r)} b(y) |f(y)|\, dy$$

$$\leqslant Cr^{\sigma+\theta}(\log(e+r^{-1}))^{-\tau}\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)|\,dy+\frac{r^{\sigma}}{|B(x,r)|}\int_{B(x,r)}b(y)|f(y)|\,dy$$
$$= I_1+I_2.$$

By Hölder's inequality, we have

$$\begin{aligned} I_1 &\leqslant Cr^{\sigma+\theta}(\log(e+r^{-1}))^{-\tau}\left(\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)|^p\,dy\right)^{1/p} \\ &\leqslant Cr^{\sigma+\theta-N/p}(\log(e+r^{-1}))^{-\tau}. \end{aligned}$$

By Corollary 1 with $E=B(x,r)$, we have

$$I_2 \leqslant Cr^{\sigma-N/q}(\log(e+r^{-1}))^{-\tau}.$$

Therefore

$$\begin{aligned} I(x,r) &\leqslant C\left\{r^{\sigma+\theta-N/p}(\log(e+r^{-1}))^{-\tau}+r^{\sigma-N/q}(\log(e+r^{-1}))^{-\tau}\right\} \\ &\leqslant Cr^{\sigma+\theta-N/p}(\log(e+r^{-1}))^{-\tau}\leqslant C|x-z|^{\sigma+\theta-N/p}(\log(e+|x-z|^{-1}))^{-\tau}, \end{aligned}$$

since $\sigma+\theta-N/p\geqslant 0$.

Next we consider the case $0<2|x-z|<r<1$. We have

$$\begin{aligned} I(x,r)-I(z,r) =& b(x)\frac{r^{\sigma}}{|B(x,r)|}\int_{B(x,r)}|f(y)|\,dy-b(z)\frac{r^{\sigma}}{|B(z,r)|}\int_{B(z,r)}|f(y)|\,dy \\ &\leqslant |b(x)-b(z)|\frac{r^{\sigma}}{|B(x,r)|}\int_{B(x,r)}|f(y)|\,dy \\ &\quad +b(z)\frac{r^{\sigma}}{|B(z,r)|}\int_{B(x,r)\Delta B(z,r)}|f(y)|\,dy \\ &\leqslant C|x-z|^{\theta}(\log(e+|x-z|^{-1}))^{-\tau}\frac{r^{\sigma}}{|B(z,r)|}\int_{B(z,r)}|f(y)|\,dy \\ &\quad +\frac{r^{\sigma}}{|B(x,r)|}\int_{B(x,r)\Delta B(z,r)}|b(z)-b(y)|\,|f(y)|\,dy \\ &\quad +\frac{r^{\sigma}}{|B(z,r)|}\int_{B(x,r)\Delta B(z,r)}b(y)|f(y)|\,dy \\ &\leqslant C\left\{|x-z|^{\theta}(\log(e+|x-z|^{-1}))^{-\tau}\frac{r^{\sigma}}{|B(x,r)|}\int_{B(x,r)}|f(y)|\,dy\right. \\ &\quad +r^{\theta}(\log(e+|x-z|^{-1}))^{-\tau}\frac{r^{\sigma}}{|B(z,r)|}\int_{B(x,r)\Delta B(z,r)}|f(y)|\,dy \\ &\quad \left.+\frac{r^{\sigma}}{|B(z,r)|}\int_{B(x,r)\Delta B(z,r)}b(y)|f(y)|\,dy\right\}. \end{aligned}$$

By Hölder's inequality and Corollary 1 with $E=B(x,r)\Delta B(z,r)$, we establish

$$I(x,r)-I(z,r) \leqslant C\left\{|x-z|^{\theta}(\log(e+|x-z|^{-1}))^{-\tau}r^{\sigma-N/p}\right.$$

$$
\begin{aligned}
&\quad + |B(x,r)\Delta B(z,r)|^{1/p'} r^{\sigma+\theta-N} (\log(e+r^{-1}))^{-\tau} \\
&\quad + |B(x,r)\Delta B(z,r)| r^{\sigma-N-N/q} (\log(e+r^{-1}))^{-\tau} \\
&\quad + |B(x,r)\Delta B(z,r)|^{1/q'} r^{\sigma-N} (\log(e+r^{-1}))^{-\tau} \Big\} \\
&\leqslant C\Big\{ |x-z|^{\theta} (\log(e+|x-z|^{-1}))^{-\tau} r^{\sigma-N/p} \\
&\quad + (r^{N-1}|x-z|)^{1/p'} r^{\sigma+\theta-N} (\log(e+r^{-1}))^{-\tau} \\
&\quad + r^{N-1}|x-z| r^{\sigma-N-N/q} (\log(e+r^{-1}))^{-\tau} \\
&\quad + (r^{N-1}|x-z|)^{1/q'} r^{\sigma-N} (\log(e+r^{-1}))^{-\tau} \Big\} \\
&\leqslant C\Big\{ |x-z|^{\theta} (\log(e+|x-z|^{-1}))^{-\tau} r^{\sigma-N/p} \\
&\quad + |x-z|^{1/p'} r^{\sigma+\theta-N/p-1/p'} (\log(e+r^{-1}))^{-\tau} \\
&\quad + |x-z| r^{\sigma-1-N/q} (\log(e+r^{-1}))^{-\tau} \\
&\quad + |x-z|^{1/q'} r^{\sigma-N/q-1/q'} (\log(e+r^{-1}))^{-\tau} \Big\} \\
&\leqslant C|x-z|^{\sigma+\theta-N/p} (\log(e+|x-z|^{-1}))^{-\tau},
\end{aligned}
$$

since $\sigma - N/p \leqslant 0$ and $\sigma - N/q = \sigma+\theta-N/p < 1/p' \leqslant 1/q' < 1$.

Therefore

$$
I(x,r) \leqslant I(z,r) + C|x-z|^{\sigma+\theta-N/p} (\log(e+|x-z|^{-1}))^{-\tau},
$$

which gives the theorem. □

THEOREM 4. *Suppose* $1<p<q<\infty$, $\tau > 1/q'$, $\alpha+\theta = N/p$ *and* $1/p-1/q = \theta/N > 0$. *Then there is a constant* $C>0$ *such that*

$$
|b(x)I_\alpha f(x) - b(z)I_\alpha f(z)| \leqslant C(\log(e+|x-z|^{-1}))^{-\tau+1/q'}
$$

*for all* $x,z \in \mathbf{R}^N$ *with* $0<|x-z|<1/2$ *and* $f \in L^{\Phi}(\mathbf{R}^N)$ *with* $\|f\|_{L^{\Phi}(\mathbf{R}^N)} \leqslant 1$.

*Proof.* Let $f$ be a measurable function on $\mathbf{R}^N$ with $\|f\|_{L^{\Phi}(\mathbf{R}^N)} \leqslant 1$. For $x,z \in \mathbf{R}^N$, write

$$
\begin{aligned}
&b(x)I_\alpha f(x) - b(z)I_\alpha f(z) \\
&= \int_{\mathbf{R}^N} |x-y|^{\alpha-N}(b(x)-b(y))f(y)\,dy + \int_{\mathbf{R}^N} |x-y|^{\alpha-N} b(y)f(y)\,dy \\
&\quad - \int_{\mathbf{R}^N} |z-y|^{\alpha-N}(b(z)-b(y))f(y)\,dy - \int_{\mathbf{R}^N} |z-y|^{\alpha-N} b(y)f(y)\,dy \\
&= J_1(x) + J_2(x) - J_1(z) - J_2(z).
\end{aligned}
$$

For $\delta = 2|x-z| < 1$, we have by Hölder's inequality

$$
J_{11}(x) = \int_{B(x,\delta)} |x-y|^{\alpha-N}(b(x)-b(y))f(y)\,dy
$$

$$\leqslant C\int_{B(x,\delta)}|x-y|^{\alpha-N+\theta}\left(\log(e+|x-y|^{-1})\right)^{-\tau}|f(y)|\,dy$$
$$\leqslant C\left(\int_{B(x,\delta)}\left(|x-y|^{\alpha-N+\theta}\left(\log(e+|x-y|^{-1})\right)^{-\tau}\right)^{p'}dy\right)^{1/p'}\left(\int_{B(x,\delta)}|f(y)|^{p}dy\right)^{1/p}$$
$$\leqslant C\left(\int_0^{\delta}t^N\left(t^{\alpha-N+\theta}\left(\log(e+t^{-1})\right)^{-\tau}\right)^{p'}\frac{dt}{t}\right)^{1/p'}\leqslant C\left(\log(e+\delta^{-1})\right)^{-\tau+1/p'},$$

since $\alpha+\theta-N/p=0$ and $\tau>1/p'$. Further, we obtain by Hölder's inequality for $0<\beta<\alpha$

$$\begin{aligned}J_{21}(x)=&\int_{B(x,\delta)}|x-y|^{\alpha-N}b(y)f(y)\,dy\\ \leqslant& C\int_{B(x,\delta)}|x-y|^{\alpha-N}\left(\frac{\log(e+|f(y)|)}{\log(e+|x-y|^{-\beta})}\right)^{\tau}b(y)|f(y)|\,dy\\ &+\int_{B(x,\delta)}|x-y|^{\alpha-N}b(y)|x-y|^{-\beta}\,dy\\ \leqslant& C\Bigg\{\left(\int_{B(x,\delta)}\left(|x-y|^{\alpha-N}\left(\log(e+|x-y|^{-1})\right)^{-\tau}\right)^{q'}dy\right)^{1/q'}\\ &\times\left(\int_{B(x,\delta)}(b(y)|f(y)|\,(\log(e+|f(y)|))^{\tau})^{q}\,dy\right)^{1/q}+\delta^{\alpha-\beta}\Bigg\}\\ \leqslant& C\left\{\left(\int_0^{\delta}t^N\left(t^{\alpha-N}\left(\log(e+t^{-1})\right)^{-\tau}\right)^{q'}\frac{dt}{t}\right)^{1/q'}+\delta^{\alpha-\beta}\right\}\\ \leqslant& C\left(\log(e+\delta^{-1})\right)^{-\tau+1/q'},\end{aligned}$$

since $\alpha-N/q=0$ and $\tau>1/q'$. Similarly,

$$J_{11}(z)=\int_{B(x,\delta)}|z-y|^{\alpha-N}(b(z)-b(y))f(y)\,dy\leqslant C\left(\log(e+\delta^{-1})\right)^{-\tau+1/p'}$$

and

$$J_{21}(z)=\int_{B(x,\delta)}|z-y|^{\alpha-N}b(y)f(y)\,dy\leqslant C\left(\log(e+\delta^{-1})\right)^{-\tau+1/q'}.$$

Noting that

$$\left||x-y|^{\alpha-N}-|z-y|^{\alpha-N}\right|\leqslant C|x-z||x-y|^{\alpha-N-1}$$

when $|x-y|>2|x-z|$, we have by Hölder's inequality

$$\begin{aligned}J_{31}=&\int_{\mathbf{R}^N\setminus B(x,\delta)}\left(|x-y|^{\alpha-N}-|z-y|^{\alpha-N}\right)(b(x)-b(y))f(y)\,dy\\ \leqslant& C|x-z|\int_{\mathbf{R}^N\setminus B(x,\delta)}|x-y|^{\alpha-N-1+\theta}\left(\log(e+|x-y|^{-1})\right)^{-\tau}|f(y)|\,dy\end{aligned}$$

$$\leqslant C|x-z|\left(\int_{\mathbf{R}^N\setminus B(x,\delta)}\left(|x-y|^{\alpha-N-1+\theta}\left(\log(e+|x-y|^{-1})\right)^{-\tau}\right)^{p'}dy\right)^{1/p'}$$
$$\times\left(\int_{\mathbf{R}^N\setminus B(x,\delta)}|f(y)|^p dy\right)^{1/p}$$
$$\leqslant C|x-z|\left(\int_\delta^\infty t^N\left(t^{\alpha-N-1+\theta}\left(\log(e+t^{-1})\right)^{-\tau}\right)^{p'}\frac{dt}{t}\right)^{1/p'}\leqslant C\left(\log(e+\delta^{-1})\right)^{-\tau}$$

and

$$J_{32}=\int_{\mathbf{R}^N\setminus B(x,\delta)}|z-y|^{\alpha-N}(b(x)-b(z))f(y)\,dy$$
$$\leqslant C|x-z|^\theta\left(\log(e+|x-z|^{-1})\right)^{-\tau}\int_{\mathbf{R}^N\setminus B(x,\delta)}|z-y|^{\alpha-N}f(y)\,dy$$
$$\leqslant C|x-z|^\theta\left(\log(e+|x-z|^{-1})\right)^{-\tau}\left(\int_{\mathbf{R}^N\setminus B(x,\delta)}|z-y|^{(\alpha-N)p'}dy\right)^{1/p'}$$
$$\times\left(\int_{\mathbf{R}^N\setminus B(x,\delta)}|f(y)|^p\,dy\right)^{1/p}$$
$$\leqslant C|x-z|^\theta\left(\log(e+|x-z|^{-1})\right)^{-\tau}\delta^{\alpha-N/p}\leqslant C\left(\log(e+\delta^{-1})\right)^{-\tau}.$$

Therefore

$$J_{31}+J_{32}\leqslant C\left(\log(e+\delta^{-1})\right)^{-\tau}.$$

Similarly, we have for $\max\{0,\alpha-1\}<\beta<\alpha$

$$J_{33}=\int_{\mathbf{R}^N\setminus B(x,\delta)}\left(|x-y|^{\alpha-N}-|z-y|^{\alpha-N}\right)b(y)f(y)\,dy$$
$$\leqslant C|x-z|\int_{\mathbf{R}^N\setminus B(x,\delta)}|x-y|^{\alpha-N-1}b(y)|f(y)|\,dy$$
$$\leqslant C|x-z|\left\{\int_{\mathbf{R}^N\setminus B(x,\delta)}|x-y|^{\alpha-N-1}\left(\frac{\log(e+|f(y)|)}{\log(e+|x-y|^{-\beta})}\right)^\tau b(y)|f(y)|\,dy\right.$$
$$\left.+\int_{\mathbf{R}^N\setminus B(x,\delta)}|x-y|^{\alpha-N-1}b(y)|x-y|^{-\beta}\,dy\right\}$$
$$\leqslant C|x-z|\left\{\left(\int_{\mathbf{R}^N\setminus B(x,\delta)}\left(|x-y|^{\alpha-N-1}\left(\log(e+|x-y|^{-1})\right)^{-\tau}\right)^{q'}dy\right)^{1/q'}\right.$$
$$\left.\times\left(\int_{\mathbf{R}^N\setminus B(x,\delta)}(b(y)|f(y)|\,(\log(e+|f(y)|))^\tau)^q\,dy\right)^{1/q}+\delta^{\alpha-\beta-1}\right\}$$
$$\leqslant C\left\{|x-z|\left(\int_\delta^\infty t^N\left(t^{\alpha-N-1}\left(\log(e+t^{-1})\right)^{-\tau}\right)^{q'}\frac{dt}{t}\right)^{1/q'}+\delta^{\alpha-\beta}\right\}$$
$$\leqslant C\left(\log(e+\delta^{-1})\right)^{-\tau}.$$

Now we establish

$$\begin{aligned} J(x) - J(z) &= J_1(x) + J_2(x) - J_1(z) - J_2(z) \\ &= J_{11}(x) + J_{11}(z) + J_{21}(x) + J_{21}(z) + J_{31} + J_{32} + J_{32} \\ &\leqslant C\left(\log(e+\delta^{-1})\right)^{-\tau+1/q'}, \end{aligned}$$

which gives the theorem. □

In the same way as above, we obtain the following result.

THEOREM 5. *Suppose* $1 < p < q < \infty$, $\tau \geqslant 0$, $0 < \alpha + \theta - N/p < \theta$ *and* $1/p - 1/q = \theta/N > 0$. *Then there is a constant* $C > 0$ *such that*

$$|b(x)I_\alpha f(x) - b(z)I_\alpha f(z)| \leqslant C|x-z|^{\alpha+\theta-N/p}(\log(e+|x-z|^{-1}))^{-\tau}$$

*for all* $x, z \in \mathbf{R}^N$ *with* $0 < |x-z| < 1/2$ *and* $f \in L^\Phi(\mathbf{R}^N)$ *with* $\|f\|_{L^\Phi(\mathbf{R}^N)} \leqslant 1$.

## 6. Appendix

For reader's convenience, we shall give direct proofs of Theorems 1 and 2 by the boundedness of the maximal operator on $L^p(\mathbf{R}^N)$ and $L^q(\mathbf{R}^N)$.

THEOREM 6. *Suppose* $1 < p \leqslant q < \infty$, $\tau \geqslant 0$ *and* $1/p - 1/q = \theta/N \geqslant 0$. *Then there is a constant* $C > 0$ *such that*

$$\int_{\mathbf{R}^N} \Phi(x, Mf(x))\,dx < C$$

*for all* $f \in L^\Phi(\mathbf{R}^N)$ *with* $\|f\|_{L^\Phi(\mathbf{R}^N)} \leqslant 1$.

*Proof.* Let $f$ be a measurable function on $\mathbf{R}^N$ with $\|f\|_{L^\Phi(\mathbf{R}^N)} \leqslant 1$. For $x \in \mathbf{R}^N$ and $r > 0$, we have

$$\begin{aligned} I &= b(x)\frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)|\,dy \\ &= \frac{1}{|B(x,r)|}\int_{B(x,r)} (b(x)-b(y))|f(y)|\,dy + \frac{1}{|B(x,r)|}\int_{B(x,r)} b(y)|f(y)|\,dy \\ &\leqslant Cr^\theta(\log(e+r^{-1}))^{-\tau}\frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)|\,dy + \frac{1}{|B(x,r)|}\int_{B(x,r)} b(y)|f(y)|\,dy \\ &= I_1 + I_2. \end{aligned}$$

For $0 < r < \delta$

$$I_1 \leqslant Cr^\theta(\log(e+r^{-1}))^{-\tau}Mf(x) \leqslant C\delta^\theta(\log(e+\delta^{-1}))^{-\tau}Mf(x)$$

and for $0 < \delta \leqslant r$ by Hölder's inequality

$$I_1 \leqslant Cr^{\theta}(\log(e+r^{-1}))^{-\tau}\left(\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)|^p\,dy\right)^{1/p} \leqslant Cr^{\theta-N/p}(\log(e+r^{-1}))^{-\tau}$$
$$\leqslant C\delta^{\theta-N/p}(\log(e+\delta^{-1}))^{-\tau},$$

since $\theta - N/p = -N/q < 0$. Thus

$$I_1 \leqslant C\left\{\delta^{\theta}(\log(e+\delta^{-1}))^{-\tau}Mf(x)+\delta^{\theta-N/p}(\log(e+\delta^{-1}))^{-\tau}\right\}.$$

Now, letting $\delta^{-N/p} = Mf(x)$, we obtain

$$I_1 \leqslant CMf(x)^{1-\theta p/N}(\log(e+Mf(x)))^{-\tau} = CMf(x)^{p/q}(\log(e+Mf(x)))^{-\tau}.$$

Moreover, for $\delta > 0$ we find from Lemma 1 with $E = B(x,r)$ and $r = \delta$ and the boundedness of $b$

$$I_2 \leqslant C\left\{\left(\log(e+\delta^{-1})\right)^{-\tau}Mh(x)+\delta^{-N/q}(\log(e+\delta^{-1}))^{-\tau}\right\},$$

where $h(y) = b(y)|f(y)|\,(\log(e+|f(y)|))^{\tau}$. Now, letting $\delta^{-N/q} = Mh(x)$, we obtain

$$I_2 \leqslant CMh(x)(\log(e+Mh(x)))^{-\tau}.$$

Now we establish

$$b(x)Mf(x) \leqslant C\left\{Mf(x)^{p/q}(\log(e+Mf(x)))^{-\tau}+Mh(x)\,(\log(e+Mh(x)))^{-\tau}\right\}.$$

When $Mf(x)^{p/q} \geqslant Mh(x)$, we have

$$\begin{aligned}&\left\{b(x)Mf(x)\,(\log(e+Mf(x)))^{\tau}\right\}^q\\ \leqslant &C\,(Mf(x))^p\,(\log(e+Mf(x)))^{-\tau q}\,(\log(e+Mf(x)))^{\tau q} \leqslant CMf(x)^p\end{aligned}$$

and when $Mf(x)^{p/q} \leqslant Mh(x)$, we have

$$\begin{aligned}&\left\{b(x)Mf(x)\,(\log(e+Mf(x)))^{\tau}\right\}^q\\ &\leqslant C\,(Mh(x))^q\,(\log(e+Mh(x)))^{-\tau q}\,(\log(e+Mf(x)))^{\tau q} \leqslant CMh(x)^q.\end{aligned}$$

Hence we obtain

$$\left\{b(x)Mf(x)\,(\log(e+Mf(x)))^{\tau}\right\}^q \leqslant C\left\{Mf(x)^p+Mh(x)^q\right\}.$$

Therefore, the boundedness of the maximal operator on $L^p(\mathbf{R}^N)$ and $L^q(\mathbf{R}^N)$ gives the theorem. □

Recall that

$$\Psi(x,t) = t^{p^*}+\left\{b(x)t\,(\log(e+t))^{\tau}\right\}^{q^*}.$$

THEOREM 7. *Suppose* $1 < p \leqslant q < \infty$, $\tau \geqslant 0$, $\alpha + \theta < N/p$ *and* $1/p - 1/q = \theta/N \geqslant 0$. *Then there is a constant* $C > 0$ *such that*

$$\int_{\mathbf{R}^N} \Psi(x, |I_\alpha f(x)|)\, dx < C$$

*for all* $f \in L^\Phi(\mathbf{R}^N)$ *with* $\|f\|_{L^\Phi(\mathbf{R}^N)} \leqslant 1$.

*Proof.* Let $f$ be a measurable function on $\mathbf{R}^N$ with $\|f\|_{L^\Phi(\mathbf{R}^N)} \leqslant 1$. For $x \in \mathbf{R}^N$ and $r > 0$, we have

$$\begin{aligned}
&b(x) \int_{\mathbf{R}^N} |x-y|^{\alpha-N} |f(y)|\, dy \\
&= \int_{\mathbf{R}^N} |x-y|^{\alpha-N} (b(x) - b(y)) |f(y)|\, dy + \int_{\mathbf{R}^N} |x-y|^{\alpha-N} b(y) |f(y)|\, dy \\
&\leqslant C \int_{\mathbf{R}^N} |x-y|^{\alpha-N+\theta} \left(\log(e + |x-y|^{-1})\right)^{-\tau} |f(y)|\, dy + \int_{\mathbf{R}^N} |x-y|^{\alpha-N} b(y) |f(y)|\, dy \\
&= J_1 + J_2.
\end{aligned}$$

For $\delta > 0$, we have

$$\int_{B(x,\delta)} |x-y|^{\alpha-N+\theta} \left(\log(e + |x-y|^{-1})\right)^{-\tau} |f(y)|\, dy \leqslant C\delta^{\alpha+\theta} \left(\log(e+\delta^{-1})\right)^{-\tau} Mf(x)$$

and by Hölder's inequality

$$\begin{aligned}
&\int_{\mathbf{R}^N \setminus B(x,\delta)} |x-y|^{\alpha-N+\theta} \left(\log(e + |x-y|^{-1})\right)^{-\tau} |f(y)|\, dy \\
&\leqslant C \int_\delta^\infty r^{\alpha+\theta} (\log(e + r^{-1}))^{-\tau} \left( \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy \right) \frac{dr}{r} \\
&\leqslant C \int_\delta^\infty r^{\alpha+\theta} (\log(e + r^{-1}))^{-\tau} \left( \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|^p\, dy \right)^{1/p} \frac{dr}{r} \\
&\leqslant C \int_\delta^\infty r^{\alpha+\theta-N/p} (\log(e + r^{-1}))^{-\tau} \frac{dr}{r} \leqslant C\delta^{\alpha+\theta-N/p} (\log(e+\delta^{-1}))^{-\tau},
\end{aligned}$$

since $\alpha + \theta - N/p < 0$. Hence

$$J_1 \leqslant C \left\{ \delta^{\alpha+\theta} (\log(e+\delta^{-1}))^{-\tau} Mf(x) + \delta^{\alpha+\theta-N/p} (\log(e+\delta^{-1}))^{-\tau} \right\}.$$

Now, letting $\delta^{-N/p} = Mf(x)$, we obtain

$$J_1 \leqslant CMf(x)^{1-(\alpha+\theta)p/N} (\log(e + Mf(x)))^{-\tau} = CMf(x)^{p/q^*} (\log(e + Mf(x)))^{-\tau}.$$

Moreover, for $\delta > 0$,

$$\int_{B(x,\delta)} |x-y|^{\alpha-N} b(y) |f(y)|\, dy$$

$$\leqslant \int_{B(x,\delta)} |x-y|^{\alpha-N} b(y)|f(y)| \left( \frac{\log(e+f(y))}{\log\left(e+\delta^{-N/q}(\log(e+\delta^{-1}))^{-\tau}\right)} \right)^{\tau} dy$$
$$+ C\delta^{-N/q}(\log(e+\delta^{-1}))^{-\tau} \int_{B(x,\delta)} |x-y|^{\alpha-N} dy$$
$$\leqslant C\left\{\delta^{\alpha}\left(\log(e+\delta^{-1})\right)^{-\tau} Mh(x) + \delta^{\alpha-N/q}(\log(e+\delta^{-1}))^{-\tau}\right\},$$

where $h(y) = b(y)|f(y)|\,(\log(e+|f(y)|))^{\tau}$. Similarly, we have by Corollary 1 with $E = B(x,r)$

$$\begin{aligned}\int_{\mathbf{R}^N\setminus B(x,\delta)} |x-y|^{\alpha-N}[b(y)|f(y)|]\,dy &\leqslant C\int_{\delta}^{\infty} r^{\alpha}\left(\frac{1}{|B(x,r)|}\int_{B(x,r)}[b(y)|f(y)|]\,dy\right)\frac{dr}{r} \\ &\leqslant C\int_{\delta}^{\infty} r^{\alpha-N/q}(\log(e+r^{-1}))^{-\tau}\frac{dr}{r} \\ &\leqslant C\delta^{\alpha-N/q}(\log(e+\delta^{-1}))^{-\tau}\end{aligned}$$

since $\alpha - N/q < 0$. Thus

$$J_2 \leqslant C\left\{\delta^{\alpha}(\log(e+\delta^{-1}))^{-\tau}Mh(x) + \delta^{\alpha-N/q}(\log(e+\delta^{-1}))^{-\tau}\right\}.$$

Now, letting $\delta^{-N/q} = Mh(x)$, we obtain

$$J_2 \leqslant CMh(x)^{1-q\alpha/N}(\log(e+Mh(x)))^{-\tau} = CMh(x)^{q/q^*}\,(\log(e+Mh(x)))^{-\tau}.$$

Now we establish

$$b(x)|I_{\alpha}f(x)| \leqslant C\left\{Mf(x)^{p/q^*}\,(\log(e+Mf(x)))^{-\tau} + Mh(x)^{q/q^*}\,(\log(e+Mh(x)))^{-\tau}\right\}.$$

As in the final discussions of the previous proof, we have

$$\left\{b(x)|I_{\alpha}f(x)|\,(\log(e+|I_{\alpha}f(x)|))^{\tau}\right\}^{q^*} \leqslant C\left\{Mf(x)^p + Mh(x)^q\right\}.$$

Hence we obtain the required result by the boundedness of the maximal operator on $L^p(\mathbf{R}^N)$ and $L^q(\mathbf{R}^N)$. $\square$

## REFERENCES


[1] E. ACERBI AND G. MINGIONE, *Regularity results for stationary electro-rheological fluids*, Arch. Ration. Mech. Anal. **164** (2002), 213–259.

[2] D. R. ADAMS AND L. I. HEDBERG, *Function Spaces and Potential Theory*, Springer, 1996.

[3] P. BARONI, M. COLOMBO AND G. MINGIONE, *Non-autonomous functionals, borderline cases and relatedfunction classes*, St Petersburg Math. J. **27** (2016), 347–379.

[4] P. BARONI, M. COLOMBO AND G. MINGIONE, *Regularity for general functionals with double phase*, Calc. Var. (2018) **57**: 62.

[5] B. BOJARSKI AND P. HAJŁASZ, *Pointwise inequalities for Sobolev functions and some applications*, Studia Math. **106**(1) (1993), 77–92.

[6] C. CAPONE, D. CRUZ-URIBE AND A. FIORENZA, *The fractional maximal operator and fractional integrals on variable $L^p$ spaces*, Rev. Mat. Iberoamericana **23** (2007), no.3, 743–770.

[7] Y. CHEN, S. LEVINE AND M. RAO, *Variable exponent, linear growth functionals in image restoration*, SIAM J. Appl. Math. **66** (2006), no. 4, 1383–1406.
[8] F. COLASUONNO AND M. SQUASSINA, *Eigenvalues for double phase variational integrals*, Ann. Mat. Pura Appl. (4) **195** (2016), no. 6, 1917–1959.
[9] M. COLOMBO AND G. MINGIONE, *Regularity for double phase variational problems*, Arch. Rat. Mech. Anal. **215** (2015), 443–496.
[10] M. COLOMBO AND G. MINGIONE, *Bounded minimizers of double phase variational integrals*, Arch. Rat. Mech. Anal. **218** (2015), 219–273.
[11] D. CRUZ-URIBE AND A. FIORENZA, *Variable Lebesgue spaces. Foundations and harmonic analysis. Applied and Numerical Harmonic Analysis.* Birkhauser/Springer, Heidelberg, 2013.
[12] L. DIENING, *Riesz potential and Sobolev embeddings of generalized Lebesgue and Sobolev spaces* $L^{p(\cdot)}$ *and* $W^{k,p(\cdot)}$, Math. Nachr. **263** (2004), no. 1, 31–43.
[13] L. DIENING, P. HARJULEHTO, P. HÄSTÖ AND M. RŮŽIČKA, *Lebesgue and Sobolev spaces with variable exponents*, Lecture Notes in Mathematics, **2017**, Springer, Heidelberg, 2011.
[14] T. FUTAMURA, Y. MIZUTA AND T. SHIMOMURA, *Sobolev embeddings for variable exponent Riesz potentials on metric spaces*, Ann. Acad. Sci. Fenn. Math. **31** (2006), 495–522.
[15] P. HÄSTÖ, *The maximal operator on generalized Orlicz spaces*, J. Funct. Anal. **269** (2015), no. 12, 4038–4048; Corrigendum to "*The maximal operator on generalized Orlicz spaces*", J. Funct. Anal. **271** (2016), no. 1, 240–243.
[16] P. HARJULEHTO AND P. HÄSTÖ, *Boundary regularity under generalized growth conditions*, Z. Anal. Anwendungen. **38** (2019), no. 1, 73–96.
[17] P. HARJULEHTO AND P. HÄSTÖ, *Orlicz Spaces and Generalized Orlicz Spaces*, Lecture Notes in Mathematics, vol. 2236, Springer-Verlag, Berlin, 2019, to appear.
[18] P. HARJULEHTO, P. HÄSTÖ AND A. KARPPINEN, *Local higher integrability of the gradient of a quasiminimizer under generalized Orlicz growth conditions*, Nonlinear Anal. **177** (2018), 543–552.
[19] P. HARJULEHTO, P. HÄSTÖ, V. LATVALA AND O. TOIVANEN, *Critical variable exponent functionals in image restoration*, Appl. Math. Letters **26** (2013), 56–60.
[20] V. KOKILASHVILI AND S. SAMKO, *On Sobolev theorem for Riesz type potentials in the Lebesgue spaces with variable exponent*, Z. Anal. Anwendungen **22** (2003), no. 4, 899–910.
[21] J. L. LEWIS, *On very weak solutions of certain elliptic systems*, Comm. Partial Differential Equations **18**(9) (10) (1993), 1515–1537.
[22] F.-Y. MAEDA, Y. MIZUTA, T. OHNO AND T. SHIMOMURA, *Boundedness of maximal operators and Sobolev's inequality on Musielak-Orlicz-Morrey spaces*, Bull. Sci. math. **137** (2013), 76–96.
[23] F.-Y. MAEDA, Y. MIZUTA, T. OHNO AND T. SHIMOMURA, *Sobolev's inequality for double phase functionals with variable exponents*, Forum. Math. **31** (2019), no. 2, 517–527.
[24] M. RŮŽIČKA, *Electrorheological Fluids: Modeling and Mathematical Theory*, Springer-Verlag, Berlin, 2000.
[25] E. M. STEIN, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, 1970.

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