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STOCHASTIC COMPARISONS OF THE LARGEST CLAIM AMOUNTS FROM TWO SETS OF INTERDEPENDENT HETEROGENEOUS PORTFOLIOS

HOSSEIN NADEB, HAMZEH TORABI* AND ALI DOLATI

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Abstract. Let $X_{\lambda_1},\dots,X_{\lambda_n}$ be continuous and dependent non-negative random variables and $Y_i = I_{p_i}X_{\lambda_i}$, $i = 1,\dots,n$, where $I_{p_1},\dots,I_{p_n}$ are independent Bernoulli random variables independent of X_{λ_i}'s, with $\mathrm{E}[I_{p_i}] = p_i$, $i = 1,\dots,n$. In actuarial sciences, Y_i corresponds to the claim amount in a portfolio of risks. In this paper, we compare the largest claim amounts of two sets of interdependent portfolios, in the sense of usual stochastic order, when the variables in one set have the parameters $\lambda_1,\dots,\lambda_n$ and $p_1,\dots,p_n$ and the variables in the other set have the parameters $\lambda_1^*,\dots,\lambda_n^*$ and $p_1^*,\dots,p_n^*$. For illustration, we apply the results to some important models in actuary.

1. Introduction

Suppose that X_{λ_i}, with survival function $\bar{F}(x;\lambda_i)$, denotes the total random severities of ith $(i = 1,\dots,n)$ policyholder in an insurance period, and let I_{p_i} be a Bernoulli random variable associated with X_{λ_i}, such that $I_{p_i} = 1$ whenever the ith policyholder makes random claim amount X_{λ_i} and $I_{p_i} = 0$ whenever does not make a claim. In this notation, $Y_i = I_{p_i}X_{\lambda_i}$ is the claim amount associated with ith policyholder and $(Y_1,\dots,Y_n)$ is said to be a portfolio of risks. Further, consider another portfolio of risks $(Y_1^*,\dots,Y_n^*)$ with the parameters $\lambda_1^*,\dots,\lambda_n^*$ and $p_1^*,\dots,p_n^*$.

The annual premium is the amount paid by the policyholder as the cost of the insurance cover being purchased. In fact, it is the primary cost to the policyholder for assigning the risk to the insurer which depends on the type of insurance. Determination of the annual premium is one of the important problems in insurance analysis. Deriving preferences between random future gains or losses is an appealing topic for the actuaries. For this purpose, stochastic orderings are very helpful. Stochastic orderings have been extensively used in some areas of sciences such as management science, financial economics, insurance, actuarial science, operation research, reliability theory, queuing theory and survival analysis. For more details on stochastic orderings, we refer to Müller and Stoyan [32], Shaked and Shanthikumar [34] and Li and Li [26].

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* Corresponding author.

The problem of stochastic comparisons of some important statistics in $(Y_1,\dots,Y_n)$ and $(Y_1^*,\dots,Y_n^*)$, such as the number of claims, $\sum_{i=1}^{n} I_{p_i}$, the aggregate claim amounts, $\sum_{i=1}^{n} Y_i$, the smallest, $Y_{1:n} = \min(Y_1,\dots,Y_n)$, and the largest claim amounts, $Y_{n:n} = \max(Y_1,\dots,Y_n)$ in two portfolios, have been discussed by many researchers in literature; see, e.g., Karlin and Novikoff [21], Ma [27], Frostig [17], Hu and Ruan [20], Denuit and Frostig [10], Khaledi and Ahmadi [22], Zhang and Zhao [36], Barmalzan et al. [4], Li and Li [24], Barmalzan et al. [7], Barmalzan and Najafabadi [3], Barmalzan et al. [5], Barmalzan et al. [6], Balakrishnan et al. [2] and Li and Li [25].

When the critical situations occur, such as earthquakes, tornadoes and epidemics, the role of the insurance companies is very highlighted. Usually, in these situations many of policies are simultaneously at risk and the severities have a positive dependence. The most of published articles consider the case that the severities are independent, while sometimes this assumption is not satisfied.

Assume that $X_{\lambda_1},\dots,X_{\lambda_n}$ are continuous and non-negative random variables with the joint distribution function $H(x_1,\dots,x_n)$, marginal distribution (survival) functions $F(x;\lambda_1),\dots,F(x;\lambda_n)$ $(\bar{F}(x;\lambda_1),\dots,\bar{F}(x;\lambda_n))$, and the copula C through the relation $H(x_1,\dots,x_n) = C(F(x;\lambda_1),\dots,F(x;\lambda_n))$ in the view of Sklar's Theorem; see Nelsen [33].

In this paper, we first focus on the stochastic comparisons of the largest claim amounts from two sets of heterogeneous portfolios in the sense of usual stochastic ordering, when the both portfolios include two policies. Then, some results in the case that the portfolios include more than two policies are provided.

The rest of the paper is organized as follows. In Section 2, we recall some definitions and lemmas which will be used in the sequel. In Section 3, stochastic comparisons of the largest claim amounts from two interdependent heterogeneous portfolios of risks in a general model in the sense of the usual stochastic ordering is discussed. Also, some examples are illustrated to show the validity of the results.

2. The basic definitions and some prerequisites

In this section, we recall some notions of stochastic orderings, majorization, weak majorization, copula and some useful lemmas which are helpful to prove the main results. Throughout the paper, we use the notations $\mathbb{R} = (-\infty,+\infty)$, $\mathbb{R}_+ = [0,+\infty)$ and $\mathbb{R}_{++} = (0,+\infty)$

DEFINITION 1. X is said to be smaller than Y in the usual stochastic ordering, denoted by $X \leqslant_{\text{st}} Y$, if $\bar{F}(x) \leqslant \bar{G}(x)$ for all $x \in \mathbb{R}$, which $\bar{F}(x)$ and $\bar{G}(x)$ denote the survival functions of X and Y, respectively.

For a comprehensive discussion of various stochastic orderings, we refer to Li and Li [26] and Shaked and Shanthikumar [34].

We also need the concept of majorization of vectors and the Schur-convexity and Schur-concavity of functions. For a comprehensive discussion of these topics, we refer to Marshall et al. [28]. We use the notation $x_{1:n} \leqslant x_{2:n} \leqslant \dots \leqslant x_{n:n}$ to denote the increasing arrangement of the components of the vector $\boldsymbol{x} = (x_1,\dots,x_n)$.

DEFINITION 2. The vector $\boldsymbol{x}$ is said to be

(i) weakly submajorized by the vector $\boldsymbol{y}$ (denoted by $\boldsymbol{x} \preceq_{\mathrm{w}} \boldsymbol{y}$) if $\sum_{i=j}^{n} x_{i:n} \leqslant \sum_{i=j}^{n} y_{i:n}$ for all $j = 1, \ldots, n$,

(ii) weakly supermajorized by the vector $\boldsymbol{y}$ (denoted by $\boldsymbol{x} \overset{\mathrm{w}}{\preceq} \boldsymbol{y}$) if $\sum_{i=1}^{j} x_{i:n} \geqslant \sum_{i=1}^{j} y_{i:n}$ for all $j = 1, \ldots, n$,

(iii) majorized by the vector $\boldsymbol{y}$ (denoted by $\boldsymbol{x} \overset{\mathrm{m}}{\preceq} \boldsymbol{y}$) if $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$ and $\sum_{i=1}^{j} x_{i:n} \geqslant \sum_{i=1}^{j} y_{i:n}$ for all $j = 1, \ldots, n-1$.

DEFINITION 3. A real valued function ϕ defined on a set $\mathscr{A} \subseteq \mathbb{R}^n$ is said to be Schur-convex (Schur-concave) on $\mathscr{A}$ if

$$\boldsymbol{x} \overset{\mathrm{m}}{\preceq} \boldsymbol{y} \quad \text{on} \quad \mathscr{A} \Longrightarrow \phi(\boldsymbol{x}) \leqslant (\geqslant) \phi(\boldsymbol{y}).$$

LEMMA 1. (Marshall et al. [28], Theorem 3.A.4) Let $\mathscr{A} \subseteq \mathbb{R}$ be an open set and let $\phi : \mathscr{A}^n \to \mathbb{R}$ be continuously differentiable. ϕ is Schur-convex (Schur-concave) on $\mathscr{A}^n$ if and only if, ϕ is symmetric on $\mathscr{A}^n$ and for all $i \neq j$,

$$(x_i - x_j)\left(\frac{\partial \phi(\boldsymbol{x})}{\partial x_i} - \frac{\partial \phi(\boldsymbol{x})}{\partial x_j}\right) \geqslant (\leqslant) 0, \quad \text{for all} \quad \boldsymbol{x} \in \mathscr{A}^n.$$

LEMMA 2. (Marshall et al. [28], Theorem 3.A.7) *Let ϕ be a continuous real valued function on the set $\mathscr{D} = \{\boldsymbol{x} : x_1 \geqslant x_2 \geqslant \ldots \geqslant x_n\}$ and continuously differentiable on the interior of $\mathscr{D}$. Denote the partial derivative of ϕ with respect to ith argument by $\phi_{(i)}(\boldsymbol{z}) = \partial\phi(\boldsymbol{z})/\partial z_i$. Then,*

$$\phi(\boldsymbol{x}) \leqslant \phi(\boldsymbol{y}) \quad \textit{whenever } \boldsymbol{x} \preceq_{\mathrm{w}} \boldsymbol{y} \textit{ on } \mathscr{D}$$

if and only if

$$\phi_{(1)}(\boldsymbol{z}) \geqslant \phi_{(2)}(\boldsymbol{z}) \geqslant \ldots \geqslant \phi_{(n)}(\boldsymbol{z}) \geqslant 0,$$

i.e. the gradient $\triangledown\phi(\boldsymbol{z}) \in \mathscr{D}_+ = \{\boldsymbol{x} : x_1 \geqslant x_2 \geqslant \ldots \geqslant x_n \geqslant 0\}$, for all $\boldsymbol{z}$ in the interior of $\mathscr{D}$. Similarly,

$$\phi(\boldsymbol{x}) \leqslant \phi(\boldsymbol{y}) \quad \textit{whenever } \boldsymbol{x} \overset{\mathrm{w}}{\preceq} \boldsymbol{y} \textit{ on } \mathscr{D}$$

if and only if

$$0 \geqslant \phi_{(1)}(\boldsymbol{z}) \geqslant \phi_{(2)}(\boldsymbol{z}) \geqslant \ldots \geqslant \phi_{(n)}(\boldsymbol{z}),$$

i.e. the gradient $\triangledown\phi(\boldsymbol{z}) \in \mathscr{D}_- = \{\boldsymbol{x} : 0 \geqslant x_1 \geqslant x_2 \geqslant \ldots \geqslant x_n\}$, for all $\boldsymbol{z}$ in the interior of $\mathscr{D}$.

One of the needed concepts in this paper is Archimedean copula. The class of Archimedean copulas has a wide range of dependence structures including the independent copula. First, we consider the definition of Archimedean copula according to McNeil and Nešlehová [29] as the below.

DEFINITION 4. Let

$$C(v_1,\ldots,v_n)=\phi\left(\sum_{i=1}^{n}\phi^{-1}(v_i)\right),\quad (v_1,\ldots,v_n)\in[0,1]^n,$$

where $\phi:[0,\infty)\to[0,1]$ is called a generator function which satisfies $\phi(0)=1$, $\lim_{x\to\infty}\phi(x)=0$ and which is strictly decreasing on $[0,\inf\{x:\phi(x)=0\})$. Its inverse $\phi^{-1}:(0,1]\to[0,\infty)$ is extended at zero to $\phi^{-1}(0)=\inf\{x:\phi(x)=0\}$. For some given dimensions n, the function C is Archimedean copula if and only if ϕ is n-monotone; that is ϕ is differentiable on $(0,\infty)$ up to the order $n-2$, $(-1)^k\phi^{(k)}(x)\geqslant 0$, for $k=1,\ldots,n-2$, and $(-1)^{n-2}\phi^{(n-2)}(x)$ is decreasing and convex on $(0,\infty)$. In this setting, $\phi^{(i)}(x)$ denotes the ith derivative of function $\phi(x)$.

DEFINITION 5. A two dimentional copula C is positively quadrant dependent (PQD) if for all $(v_1,v_2)\in[0,1]^2$, we have $C(v_1,v_2)\geqslant v_1v_2$.

In the following, we state some useful definitions and lemmas related to copulas.

DEFINITION 6. Let C_1 and C_2 be two copulas. C_1 is less positively lower orthant dependent (PLOD) than C_2, denoted by $C_1\prec C_2$, if for all $\boldsymbol{v}\in[0,1]^n$, $C_1(\boldsymbol{v})\leqslant C_2(\boldsymbol{v})$.

We state the following lemmas from Durante [12] and Dolati and Dehghan Nezhad [11] related to Schur-concavity of copulas.

LEMMA 3. *Let C be a continuously differentiable copula. C is Schur-concave on $[0,1]^n$, if and only if,*

(i) *C is symmetric;*

(ii) $\frac{\partial C(\boldsymbol{v})}{\partial v_1}\geqslant\frac{\partial C(\boldsymbol{v})}{\partial v_2}$ *on the set* $\{\boldsymbol{v}\in[0,1]^n: v_1\leqslant\ldots\leqslant v_n\}$.

LEMMA 4. *Every Archimedean copula is Schur-concave.*

An important copula in application, is the Farlie-Gumbel-Morgenstern (FGM) copula which was introduced by Morgenstern [31] with a trace back to Eyraud [13] and was discussed by Gumbel [18] and Farlie [14], of the form $C_\theta(\boldsymbol{v})=\prod_{i=1}^{n}v_i+\theta\prod_{i=1}^{n}v_i(1-v_i)$, where $\theta\in[-1,1]$.

LEMMA 5. *The FGM copula is Schur-concave for any $\theta\in[-1,1]$.*

For a comprehensive discussion in the topic of copula and the different types of dependency, one may refer to Nelsen [33].

Also, we define a required space as below:

$$S=\left\{(\boldsymbol{x},\boldsymbol{y})=\begin{bmatrix}x_1 & x_2\\ y_1 & y_2\end{bmatrix}:(x_i-x_j)(y_i-y_j)\leqslant 0,\quad i,j=1,2\right\}.$$

3. Main results

In this section, we compare the largest claim amounts from two interdependent heterogeneous portfolios of risks in the sense of the usual stochastic ordering. Also, we present some examples to illustrate the validity of the results.

The following theorem provides a comparison between the largest claim amounts in two heterogeneous portfolios of risks, in terms of $\boldsymbol{p}$.

THEOREM 1. *Let X_{λ_1} and X_{λ_2} be non-negative random variables with $X_{\lambda_i} \sim \bar{F}(x;\lambda_i)$, $i=1,2$, and associated copula C. Further, suppose that I_{p_1}, I_{p_2} ($I_{p_1^*}, I_{p_2^*}$) is a set of independent Bernoulli random variables, independent of the X_{λ_i}'s, with $\mathrm{E}[I_{p_i}] = p_i$ ($\mathrm{E}[I_{p_i^*}] = p_i^*$), $i=1,2$. Assume that the following conditions hold:*

(i) *$h:(0,1] \to I \subset \mathbb{R}_{++}$ is a differentiable and strictly increasing concave function, with the log-concave inverse;*

(ii) *$\bar{F}(x;\lambda)$ is decreasing in λ for any $x \in \mathbb{R}_+$;*

(iii) *C is PQD.*

Then, for $(\boldsymbol{\lambda}, h(\boldsymbol{p})) \in S$ and $(\boldsymbol{\lambda}, h(\boldsymbol{p}^*)) \in S$, we have

$$(h(p_1^*), h(p_2^*)) \overset{\mathrm{m}}{\preceq} (h(p_1), h(p_2)) \Longrightarrow Y_{2:2}^* \leqslant_{\mathrm{st}} Y_{2:2}.$$

Proof. Without loss of generality, we suppose that $\lambda_1 \leqslant \lambda_2$. For $(\boldsymbol{\lambda}, h(\boldsymbol{p})) \in S$ and $(\boldsymbol{\lambda}, h(\boldsymbol{p}^*)) \in S$, we have $h(p_1) \geqslant h(p_2)$ and $h(p_1^*) \geqslant h(p_2^*)$. Let h^{-1} be the inverse of the function h, $u_i = h(p_i)$ and $u_i^* = h(p_i^*)$, for $i=1,2$. It can be easily verified that the distribution function of $Y_{2:2}$ is given by

$$\begin{aligned} G_{Y_{2:2}}(x) = & \prod_{i=1}^{2}\left(1 - h^{-1}(u_i)\bar{F}(x;\lambda_i)\right) \\ & + h^{-1}(u_1)h^{-1}(u_2)\Big[C\big(F(x;\lambda_1), F(x;\lambda_2)\big) - F(x;\lambda_1)F(x;\lambda_2)\Big]. \end{aligned} \tag{1}$$

Let

$$G_{Y_{2:2}}(x) = -\Psi_1(\boldsymbol{u}) - \Psi_2(\boldsymbol{u}),$$

where

$$\Psi_1(\boldsymbol{u}) = -\prod_{i=1}^{2}\left(1 - h^{-1}(u_i)\bar{F}(x;\lambda_i)\right),$$

and

$$\Psi_2(\boldsymbol{u}) = -h^{-1}(u_1)h^{-1}(u_2)\Big[C\big(F(x;\lambda_1), F(x;\lambda_2)\big) - F(x;\lambda_1)F(x;\lambda_2)\Big].$$

The partial derivative of $\Psi_1(\boldsymbol{u})$ with respect to u_i is given by

$$\frac{\partial \Psi_1(\boldsymbol{u})}{\partial u_i} = -\frac{\bar{F}(x;\lambda_i)\frac{\mathrm{d}h^{-1}(u_i)}{\mathrm{d}u_i}}{1-h^{-1}(u_i)\bar{F}(x;\lambda_i)}\Psi_1(\boldsymbol{u}) \geqslant 0.$$

Since $\bar{F}(x;\lambda)$ is decreasing in λ, by using the increasing and convexity properties of $h^{-1}(x)$ in $x \in \mathbb{R}_+$, for $\lambda_1 \leqslant \lambda_2$ and $u_1 \geqslant u_2$, we have

$$0 \leqslant 1-h^{-1}(u_1)\bar{F}(x;\lambda_1) \leqslant 1-h^{-1}(u_2)\bar{F}(x;\lambda_2), \tag{2}$$

and

$$\bar{F}(x;\lambda_1)\frac{\mathrm{d}h^{-1}(u_1)}{\mathrm{d}u_1} \geqslant \bar{F}(x;\lambda_2)\frac{\mathrm{d}h^{-1}(u_2)}{\mathrm{d}u_2} \geqslant 0. \tag{3}$$

Using (2) and (3), we obtain

$$\frac{\partial \Psi_1(\boldsymbol{u})}{\partial u_1} - \frac{\partial \Psi_1(\boldsymbol{u})}{\partial u_2} = -\left[\frac{\bar{F}(x;\lambda_1)\frac{\mathrm{d}h^{-1}(u_1)}{\mathrm{d}u_1}}{1-h^{-1}(u_1)\bar{F}(x;\lambda_1)} - \frac{\bar{F}(x;\lambda_2)\frac{\mathrm{d}h^{-1}(u_2)}{\mathrm{d}u_2}}{1-h^{-1}(u_2)\bar{F}(x;\lambda_2)}\right]\Psi_1(\boldsymbol{u}) \geqslant 0.$$

Applying the Lemma 2 and the assumption $(u_1^*,u_2^*) \overset{\mathrm{m}}{\preceq} (u_1,u_2)$, imply that

$$\Psi_1(\boldsymbol{u}^*) \leqslant \Psi_1(\boldsymbol{u}). \tag{4}$$

Now, the partial derivative of $\Psi_2(\boldsymbol{u})$ with respect to u_i is given by

$$\frac{\partial \Psi_2(\boldsymbol{u})}{\partial u_i} = \frac{\frac{\mathrm{d}h^{-1}(u_i)}{\mathrm{d}u_i}}{h^{-1}(u_i)}\Psi_2(\boldsymbol{u}) = \frac{\mathrm{d}\log h^{-1}(u_i)}{\mathrm{d}u_i}\Psi_2(\boldsymbol{u}) \leqslant 0.$$

Therefore, for $u_1 \geqslant u_2$, we obtain

$$\frac{\partial \Psi_2(\boldsymbol{u})}{\partial u_1} - \frac{\partial \Psi_2(\boldsymbol{u})}{\partial u_2} = \left[\frac{\mathrm{d}\log h^{-1}(u_1)}{\mathrm{d}u_1} - \frac{\mathrm{d}\log h^{-1}(u_2)}{\mathrm{d}u_2}\right]\Psi_2(\boldsymbol{u}) \geqslant 0,$$

where the inequality follows from log-concavity of h^{-1} and negativity of $\Psi_2(\boldsymbol{u})$ which is due to PQD property of C. Thus, applying Lemma 2 and the assumption $(u_1^*,u_2^*) \overset{\mathrm{m}}{\preceq} (u_1,u_2)$, imply that

$$\Psi_2(\boldsymbol{u}^*) \leqslant \Psi_2(\boldsymbol{u}). \tag{5}$$

By using (4) and (5), the proof is completed. □

The following theorem provides a comparison between the largest claim amounts in two heterogeneous portfolios of risks, in terms of $\boldsymbol{\lambda}$.

THEOREM 2. *Let X_{λ_1} and X_{λ_2} ($X_{\lambda_1^*}$ and $X_{\lambda_2^*}$) be non-negative random variables with $X_{\lambda_i} \sim \bar{F}(x;\lambda_i)$ ($X_{\lambda_i^*} \sim \bar{F}(x;\lambda_i^*)$), $i = 1,2$, and associated copula C. Further, suppose that I_{p_1}, I_{p_2} is a set of independent Bernoulli random variables, independent of the X_{λ_i}'s ($X_{\lambda_i^*}$'s), with $\mathrm{E}[I_{p_i}] = p_i$, $i = 1,2$. Assume that the following conditions hold:*

(i) $h:[0,1]\to I\subset\mathbb{R}_+$ *is a differentiable and strictly increasing function;*

(ii) $\bar F(x;\lambda)$ *is decreasing and convex in* λ *for any* $x\in\mathbb{R}_+$*;*

(iii) $\frac{\partial C(v_1,v_2)}{\partial v_1}\geqslant\frac{\partial C(v_1,v_2)}{\partial v_2}$*, for all* $0\leqslant v_1\leqslant v_2\leqslant 1$.

Then, for $(\boldsymbol{\lambda},h(\boldsymbol{p}))\in S$ *and* $(\boldsymbol{\lambda}^*,h(\boldsymbol{p}))\in S$*, we have*

$$(\lambda_1^*,\lambda_2^*)\overset{\text{w}}{\preceq}(\lambda_1,\lambda_2)\Longrightarrow Y_{2:2}^*\leqslant_{\text{st}}Y_{2:2}.$$

Proof. Without loss of generality, we suppose that $\lambda_1\leqslant\lambda_2$, $u_1\geqslant u_2$ and $u_1^*\geqslant u_2^*$. By some algebraic calculations in (1), the distribution function of $Y_{2:2}$ can be rewritten as the following form:

$$\begin{aligned}G_{Y_{2:2}}(x)&=(1-h^{-1}(u_1))(1-h^{-1}(u_2))+h^{-1}(u_1)h^{-1}(u_2)\\&\quad\times\left[C\big(F(x;\lambda_1),F(x;\lambda_2)\big)+\frac{1-h^{-1}(u_2)}{h^{-1}(u_2)}F(x;\lambda_1)+\frac{1-h^{-1}(u_1)}{h^{-1}(u_1)}F(x;\lambda_2)\right].\end{aligned}$$

Define $\Psi(\boldsymbol{\lambda})=-G_{Y_{2:2}}(x)$. The partial derivatives of $\Psi(\boldsymbol{\lambda})$ with respect to λ_i, $i=1,2$ are given by

$$\frac{\partial\Psi(\boldsymbol{\lambda})}{\partial\lambda_1}=-h^{-1}(u_1)h^{-1}(u_2)\frac{\mathrm{d}F(x;\lambda_1)}{\mathrm{d}\lambda_1}\left[\frac{\partial C\big(F(x;\lambda_1),F(x;\lambda_2)\big)}{\partial v_1}+\frac{1-h^{-1}(u_2)}{h^{-1}(u_2)}\right]\leqslant 0,$$

and

$$\frac{\partial\Psi(\boldsymbol{\lambda})}{\partial\lambda_2}=-h^{-1}(u_1)h^{-1}(u_2)\frac{\mathrm{d}F(x;\lambda_2)}{\mathrm{d}\lambda_2}\left[\frac{\partial C\big(F(x;\lambda_1),F(x;\lambda_2)\big)}{\partial v_2}+\frac{1-h^{-1}(u_1)}{h^{-1}(u_1)}\right]\leqslant 0,$$

where the inequalities are due to decreasing property of $\bar F(x;\lambda)$ in λ and positivity of $\frac{1-h^{-1}(x)}{h^{-1}(x)}$ in $x\in\mathbb{R}_+$. Since h^{-1} is increasing in $x\in\mathbb{R}_+$ and $\bar F(x;\lambda)$ is decreasing and convex in λ for any $x\in\mathbb{R}_+$, then for $\lambda_1\leqslant\lambda_2$ and $u_1\geqslant u_2$, we have

$$0\leqslant\frac{1-h^{-1}(u_1)}{h^{-1}(u_1)}\leqslant\frac{1-h^{-1}(u_2)}{h^{-1}(u_2)},\tag{6}$$

and

$$\frac{\mathrm{d}F(x;\lambda_1)}{\mathrm{d}\lambda_1}\geqslant\frac{\mathrm{d}F(x;\lambda_2)}{\mathrm{d}\lambda_2}\geqslant 0.\tag{7}$$

The decreasing property of $\bar F(x;\lambda)$ in λ and the condition (iii) imply that

$$\frac{\partial C\big(F(x;\lambda_1),F(x;\lambda_2)\big)}{\partial v_1}\geqslant\frac{\partial C\big(F(x;\lambda_1),F(x;\lambda_2)\big)}{\partial v_2}\geqslant 0.\tag{8}$$

Using (6), (7) and (8), we obtain

$$
\begin{aligned}
\frac{\partial\Psi(\boldsymbol{\lambda})}{\partial\lambda_2}-\frac{\partial\Psi(\boldsymbol{\lambda})}{\partial\lambda_1} &= -h^{-1}(u_1)h^{-1}(u_2)\\
&\quad\times\Bigg[\frac{\mathrm{d}F(x;\lambda_2)}{\mathrm{d}\lambda_2}\frac{\partial C\big(F(x;\lambda_1),F(x;\lambda_2)\big)}{\partial v_2}+\frac{\mathrm{d}F(x;\lambda_2)}{\mathrm{d}\lambda_2}\frac{1-h^{-1}(u_1)}{h^{-1}(u_1)}\\
&\quad-\frac{\mathrm{d}F(x;\lambda_1)}{\mathrm{d}\lambda_1}\frac{\partial C\big(F(x;\lambda_1),F(x;\lambda_2)\big)}{\partial v_1}-\frac{\mathrm{d}F(x;\lambda_1)}{\mathrm{d}\lambda_1}\frac{1-h^{-1}(u_2)}{h^{-1}(u_2)}\Bigg]\\
&\geqslant 0.
\end{aligned}
$$

Therefore, under the assumption $\boldsymbol{\lambda}^* \overset{\mathrm{w}}{\preceq} \boldsymbol{\lambda}$, Lemma 2 implies that

$$\Psi(\boldsymbol{\lambda}^*) \leqslant \Psi(\boldsymbol{\lambda}),$$

which completes the proof. □

The following theorem provides a comparison between the largest claim amounts in two heterogeneous portfolios of risks, in terms of $\boldsymbol{p}$ and $\boldsymbol{\lambda}$.

THEOREM 3. *Let X_{λ_1} and X_{λ_2} ($X_{\lambda_1^*}$ and $X_{\lambda_2^*}$) be non-negative random variables with $X_{\lambda_i} \sim \bar{F}(x;\lambda_i)$ ($X_{\lambda_i^*} \sim \bar{F}(x;\lambda_i^*)$), $i=1,2$, and associated copula C. Further, suppose that I_{p_1}, I_{p_2} ($I_{p_1^*}, I_{p_2^*}$) is a set of independent Bernoulli random variables, independent of the X_{λ_i}'s ($X_{\lambda_i^*}$'s), with $\mathrm{E}[I_{p_i}] = p_i$ ($\mathrm{E}[I_{p_i^*}] = p_i^*$), $i=1,2$. Assume that the following conditions hold:*

(i) *$h:(0,1] \to I \subset \mathbb{R}_{++}$ is a differentiable and strictly increasing concave function, with a log-concave inverse;*

(ii) *$\bar{F}(x;\lambda)$ is decreasing and convex in λ for any $x \in \mathbb{R}_+$;*

(iii) *C is PQD and $\frac{\partial C(v_1,v_2)}{\partial v_1} \geqslant \frac{\partial C(v_1,v_2)}{\partial v_2}$, for all $0 \leqslant v_1 \leqslant v_2 \leqslant 1$.*

Then, for $(\boldsymbol{\lambda}, h(\boldsymbol{p})) \in S$ and $(\boldsymbol{\lambda}^*, h(\boldsymbol{p}^*)) \in S$, we have

$$(h(p_1^*),h(p_2^*)) \overset{\mathrm{m}}{\preceq} (h(p_1),h(p_2)) \ \text{ and } \ (\lambda_1^*,\lambda_2^*) \overset{\mathrm{w}}{\preceq} (\lambda_1,\lambda_2) \Longrightarrow Y_{2:2}^* \leqslant_{\mathrm{st}} Y_{2:2}.$$

Proof. Let $V_{2:2}$, $Z_{2:2}$ and $W_{2:2}$ be the largest claim amounts from the portfolios $(I_{p_{1:2}^*}X_{\lambda_{2:2}^*}, I_{p_{2:2}^*}X_{\lambda_{1:2}^*})$, $(I_{p_{1:2}}X_{\lambda_{2:2}^*}, I_{p_{2:2}}X_{\lambda_{1:2}^*})$ and $(I_{p_{1:2}}X_{\lambda_{2:2}}, I_{p_{2:2}}X_{\lambda_{1:2}})$, respectively. It can be verified that $Y_{2:2}^* \overset{\mathrm{st}}{=} V_{2:2}$ and $Y_{2:2} \overset{\mathrm{st}}{=} W_{2:2}$. On the other hand, Theorem 1 and Theorem 2 imply that $V_{2:2} \leqslant_{\mathrm{st}} Z_{2:2}$ and $Z_{2:2} \leqslant_{\mathrm{st}} W_{2:2}$, respectively. Hence, the required result is obtained. □

The scale family is an applicable model in reliability theory and actuarial science. X_λ is said to follow the scale family, if its survival function can be expressed as $\bar{F}(x;\lambda) = \bar{F}(\lambda x)$, where $\bar{F}(x)$ is the baseline survival function with the corresponding density function $f(x)$ and $\lambda > 0$.

The following theorem provides a comparison between the largest claim amounts in two heterogeneous portfolio of risks, whenever the marginal distributions belonging to the scale family.

THEOREM 4. *Let* $\bar{F}(x;\lambda_i)=\bar{F}(\lambda_i x)$ *and* $\bar{F}(x;\lambda_i^*)=\bar{F}(\lambda_i^* x)$, *for* $i=1,2$. *Under the setup of Theorem 3, suppose that the following conditions hold:*

(i) $h:(0,1]\to I\subset\mathbb{R}_{++}$ *is a differentiable and strictly increasing concave function, with a log-concave inverse;*

(ii) $f(x)$ *is decreasing in* $x\in\mathbb{R}_+$;

(iii) C *is PQD and* $\frac{\partial C(v_1,v_2)}{\partial v_1}\geqslant\frac{\partial C(v_1,v_2)}{\partial v_2}$, *for all* $0\leqslant v_1\leqslant v_2\leqslant 1$.

Then, for $(\boldsymbol{\lambda},h(\boldsymbol{p}))\in S$ *and* $(\boldsymbol{\lambda}^*,h(\boldsymbol{p}^*))\in S$, *we have*

$$(h(p_1^*),h(p_2^*))\overset{\mathrm{m}}{\preceq}(h(p_1),h(p_2))\ \textit{and}\ (\lambda_1^*,\lambda_2^*)\overset{\mathrm{w}}{\preceq}(\lambda_1,\lambda_2)\Longrightarrow Y_{2:2}^*\leqslant_{\mathrm{st}}Y_{2:2}.$$

Proof. Note that the conditions (i) and (iii) are similar to the conditions (i) and (iii) of Theorem 3. Also, it can be easily verified that the condition (ii) of this theorem, satisfies the condition (ii) of Theorem 3, which holds the desired result. □

Gamma distribution is one of the most applicable distributions to depict the claim amounts whenever the shape parameter is less than 1. X has the gamma distribution with the shape parameter α and the rate parameter λ, denoted by $X\sim\Gamma(\alpha,\lambda)$, if its density function is given by

$$f(x;\alpha,\lambda)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\lambda x},\quad x\in\mathbb{R}_{++}.$$

The following example provides a numerical example to illustrate the validity of Theorem 4.

EXAMPLE 1. Let $X_{\lambda_i}\sim\Gamma(0.8,\lambda_i)$ ($X_{\lambda_i^*}\sim\Gamma(0.8,\lambda_i^*)$), for $i=1,2$, with the associated FGM copula. It is clear that this copula is PQD if $\theta\in[0,1]$. Further, suppose that I_{p_1},I_{p_2} ($I_{p_1^*},I_{p_2^*}$) is a set of independent Bernoulli random variables, independent of the X_{λ_i}'s ($X_{\lambda_i^*}$'s), with $\mathrm{E}[I_{p_i}]=p_i$ ($\mathrm{E}[I_{p_i^*}]=p_i^*$), for $i=1,2$. We take $h(p)=p$, $(\lambda_1,\lambda_2)=(0.26,0.74)$, $(p_1,p_2)=(0.03,0.02)$, $(\lambda_1^*,\lambda_2^*)=(0.4,0.6)$, $(p_1^*,p_2^*)=(0.026,0.024)$ and $\theta=0.5$. Using Lemma 3 and Lemma 5, we get the condition (iii) of Theorem 4, and obviously can be verified that the other conditions are also satisfied. So, we have $Y_{2:2}^*\leqslant_{\mathrm{st}}Y_{2:2}$. Figure 1 represents the survival functions of $Y_{2:2}$ and $Y_{2:2}^*$, which agrees with the intended result.

The following example illustrates that the conditions $(\boldsymbol{\lambda},h(\boldsymbol{p}))\in S$ and $(\boldsymbol{\lambda}^*,h(\boldsymbol{p}^*))\in S$ is an important condition and can not be dropped.

EXAMPLE 2. Under the same setup in Example 1, we take $(p_1,p_2)=(0.02,0.03)$ and $(p_1^*,p_2^*)=(0.028,0.022)$ with the other unchanged values. It is clear that $(\boldsymbol{\lambda},h(\boldsymbol{p}))\notin S$, but it can be easily verified that the other conditions of Theorem 4 are satisfied. Figure 2 represents the survival functions of $Y_{2:2}$ and $Y_{2:2}^*$, which cross each other.

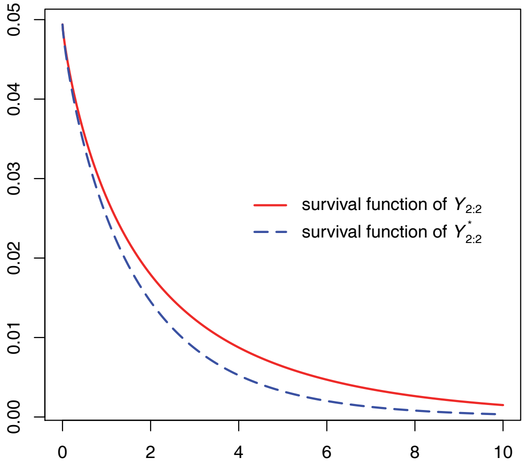

Figure 1: Plots of the survival functions of $Y_{2:2}$ and $Y^*_{2:2}$ in Example 1.

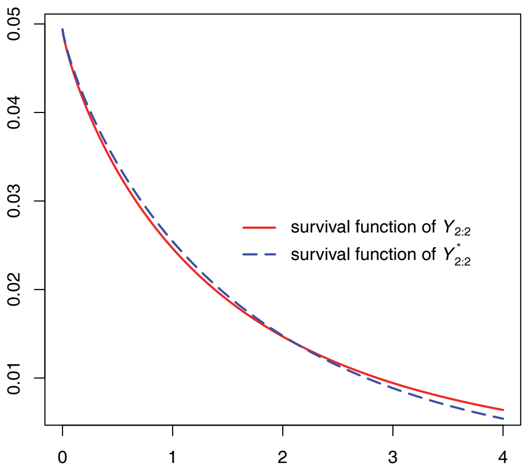

Figure 2: Plots of the survival functions of $Y_{2:2}$ and $Y^*_{2:2}$ in Example 2.

The proportional hazard rate (PHR) model is a flexible family of distributions with important role in reliability theory, actuarial science and other fields; see for example Cox [9], Finkelstein [15], Kumar and Klefsjö [23], Balakrishnan et al. [2] and Li and Li [25]. X_λ is said to follow the PHR model, if its survival function can be expressed as $\bar{F}(x;\lambda)=[\bar{F}(x)]^{\lambda}$, where $\bar{F}(x)$ is the baseline survival function and $\lambda>0$.

The following theorem provides a comparison between the largest claim amounts in two heterogeneous portfolio of risks, whenever the marginal distributions belonging to the PHR model.

THEOREM 5. *Let $\bar{F}(x;\lambda_i)=[\bar{F}(x)]^{\lambda_i}$ and $\bar{F}(x;\lambda_i^*)=[\bar{F}(x)]^{\lambda_i^*}$, for $i=1,2$. Under the setup of Theorem 3, suppose that the following conditions hold:*

(i) *$h:(0,1]\to I\subset\mathbb{R}_{++}$ is a differentiable and strictly increasing concave function, with the log-concave inverse;*

(ii) *C is PQD and $\frac{\partial C(v_1,v_2)}{\partial v_1}\geqslant\frac{\partial C(v_1,v_2)}{\partial v_2}$, for all $0\leqslant v_1\leqslant v_2\leqslant 1$.*

Then, for $(\boldsymbol{\lambda},h(\boldsymbol{p}))\in S$ and $(\boldsymbol{\lambda}^*,h(\boldsymbol{p}^*))\in S$, we have

$$(h(p_1^*),h(p_2^*))\overset{\mathrm{m}}{\preceq}(h(p_1),h(p_2))\ \text{ and }\ (\lambda_1^*,\lambda_2^*)\overset{\mathrm{w}}{\preceq}(\lambda_1,\lambda_2)\Longrightarrow Y_{2:2}^*\leqslant_{\mathrm{st}}Y_{2:2}.$$

Proof. Note that $\bar{F}(x;\lambda)=[\bar{F}(x)]^{\lambda}$ is decreasing and convex in λ, which satisfies the condition (ii) of Theorem 3. Therefore, applying Theorem 3 completes the proof. $\square$

The Pareto distribution is a special case of the PHR model, which is commonly used as the distribution of claim severity from policyholders in insurance. X has the Pareto distribution with parameters β and λ, denoted by $X\sim\text{Pareto}(\beta,\lambda)$, if its survival function is given by

$$\bar{F}(x;\beta,\lambda)=(\frac{\beta}{x})^{\lambda},\quad x\geqslant\beta.$$

The following example provides a numerical example to illustrate the validity of Theorem 5.

EXAMPLE 3. Let $X_{\lambda_i}\sim\text{Pareto}(1,\lambda_i)$ ($X_{\lambda_i^*}\sim\text{Pareto}(1,\lambda_i^*)$), for $i=1,2$, with the associated Ali-Mikhail-Haq copula, which was introduced by Ali et al. [1], of the form $C_\theta(v_1,v_2)=\frac{v_1v_2}{1-\theta(1-v_1)(1-v_2)}$, where $\theta\in[-1,1]$. According to Nelsen [33], this copula is Archimedean and obviously is PQD if $\theta\in[0,1]$. Further, suppose that I_{p_1},I_{p_2} ($I_{p_1^*},I_{p_2^*}$) is a set of independent Bernoulli random variables, independent of the X_{λ_i}'s ($X_{\lambda_i^*}$'s), with $\mathrm{E}[I_{p_i}]=p_i$ ($\mathrm{E}[I_{p_i^*}]=p_i^*$), for $i=1,2$. We take $h(p)=\log(p+2)$, $(\lambda_1,\lambda_2)=(4,2)$, $(p_1,p_2)=(0.02,0.06)$, $(\lambda_1^*,\lambda_2^*)=(4,6)$, $(p_1^*,p_2^*)=(0.0479,0.0319)$ and $\theta=0.3$. Lemma 3 and Lemma 4 imply the condition (ii) of Theorem 5, and it can be easily verified that the other condition is also satisfied. So, we have $Y_{2:2}^*\leqslant_{\mathrm{st}}Y_{2:2}$. Figure 3 represents the survival functions of $Y_{2:2}$ and $Y_{2:2}^*$, which agrees with the intended result.

The transmuted-G (TG) model, which was introduced by Mirhossaini and Dolati [30] and Shaw and Buckley [35], is an attractive model for constructing new flexible distributions by adding a new parameter. The random variable X_λ said to belong to the

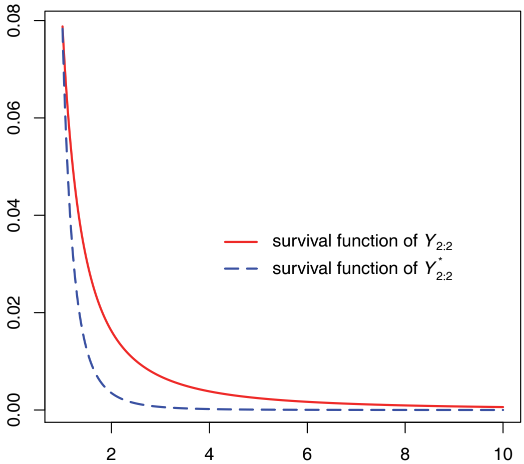

Figure 3: Plots of the survival functions of $Y_{2:2}$ and $Y^*_{2:2}$ in Example 3.

TG model with the baseline distribution function $F(x)$ and survival $\bar{F}(x)$, if its survival function can be expressed as $\bar{F}(x;\lambda) = \bar{F}(x)(1-\lambda F(x))$, where $\lambda \in [-1,1]$.

The following theorem provides a comparison between the largest claim amounts in two heterogeneous portfolios of risks, whenever the marginal distributions belonging to the TG model.

THEOREM 6. *Let $\bar{F}(x;\lambda_i) = \bar{F}(x)(1-\lambda_i F(x))$ and $\bar{F}(x;\lambda_i^*) = \bar{F}(x)(1-\lambda_i^* F(x))$, for $i = 1,2$. Under the setup of Theorem 3, suppose that the following conditions hold:*

(i) *$h : (0,1] \to I \subset \mathbb{R}_{++}$ is a differentiable and strictly increasing concave function, with the log-concave inverse;*

(ii) *C is PQD and $\frac{\partial C(v_1,v_2)}{\partial v_1} \geqslant \frac{\partial C(v_1,v_2)}{\partial v_2}$, for all $0 \leqslant v_1 \leqslant v_2 \leqslant 1$.*

Then, for $(\boldsymbol{\lambda}, h(\boldsymbol{p})) \in S$ and $(\boldsymbol{\lambda}^*, h(\boldsymbol{p}^*)) \in S$, we have

$$(h(p_1^*), h(p_2^*)) \overset{\mathrm{m}}{\preceq} (h(p_1), h(p_2)) \;\; and \;\; (\lambda_1^*, \lambda_2^*) \overset{\mathrm{w}}{\preceq} (\lambda_1, \lambda_2) \Longrightarrow Y^*_{2:2} \leqslant_{\mathrm{st}} Y_{2:2}.$$

Proof. Note that $\bar{F}(x;\lambda) = \bar{F}(x)(1-\lambda F(x))$ is decreasing and convex in λ, which satisfies the condition (ii) of Theorem 3. Therefore, applying Theorem 3 completes the proof. $\square$

The transmuted exponential distribution, which was introduced by Mirhossaini and Dolati [30] has non-negative support and can be used to simulate the claim severity from policyholders in insurance. X has the transmuted exponential distribution with parameters μ and λ, denoted by $X \sim \mathrm{TE}(\mu,\lambda)$, if its survival function is given by

$$\bar{F}(x,\mu,\lambda) = e^{-x/\mu}[1-\lambda(1-e^{-x/\mu})], \quad x \geqslant 0, \quad \mu > 0, \quad -1 \leqslant \lambda \leqslant 1.$$

The following example provides a numerical example to illustrate the validity of Theorem 6.

EXAMPLE 4. Let $X_{\lambda_i} \sim \mathrm{TE}(3,\lambda_i)$ $(X_{\lambda_i^*} \sim \mathrm{TE}(3,\lambda_i^*))$, for $i=1,2$, with the associated Gumbel-Hougaard copula, which was first introduced by Gumbel [19], of the form

$$C_\theta(v_1,v_2) = \exp\left(-\left[(-\log v_1)^\theta + (-\log v_2)^\theta\right]^{1/\theta}\right),$$

where $\theta \in [1,\infty)$. According to Nelsen [33], this copula is Archimedean and is PQD. Further, suppose that I_{p_1}, I_{p_2} $(I_{p_1^*}, I_{p_2^*})$ is a set of independent Bernoulli random variables, independent of the X_{λ_i}'s ($X_{\lambda_i^*}$'s), with $\mathrm{E}[I_{p_i}] = p_i$ ($\mathrm{E}[I_{p_i^*}] = p_i^*$), for $i=1,2$. We take $h(p)=\sqrt{p}$, $(\lambda_1,\lambda_2)=(0.6,-0.2)$, $(p_1,p_2)=(0.04,0.09)$, $(\lambda_1^*,\lambda_2^*)=(0.1,0.4)$, $(p_1^*,p_2^*)=(0.0676,0.0576)$ and $\theta=10$. Lemma 3 and Lemma 4 imply the condition (ii) of Theorem 6, and it can be easily verified that the other condition is also satisfied. So, we have $Y_{2:2}^* \leqslant_{\mathrm{st}} Y_{2:2}$. Figure 4 represents the survival functions of $Y_{2:2}$ and $Y_{2:2}^*$, which coincides with the intended result.

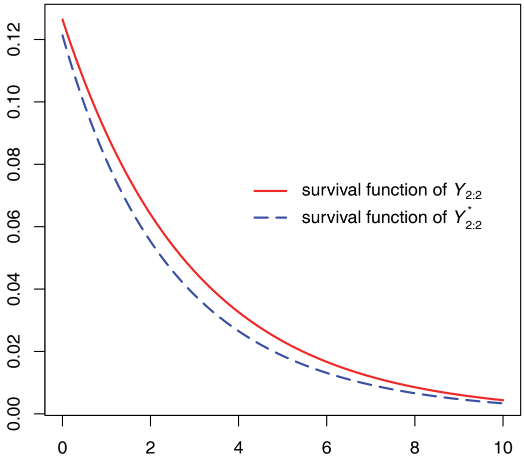

Figure 4: Plots of the survival functions of $Y_{2:2}$ and $Y_{2:2}^*$ in Example 4.

Next, we consider the case that the occurrence probabilities are also interdependent. Here, we denote $\boldsymbol{I}=(I_1,I_2)$ and $P(\boldsymbol{I}=\boldsymbol{\mu})=p(\boldsymbol{\mu})$. The following lemma considers the concept of weakly stochastic arrangement increasing through left tail probability (LWSAI) for $\boldsymbol{I}$, which is a particular case of Lemma 5.3 of Cai and Wei [8].

LEMMA 6. *A bivariate Bernoulli random vector* $\boldsymbol{I}$ *is* LWSAI, *if and only if* $p(1,0) \leqslant p(0,1)$.

The following theorem gives a comparison between the largest claim amounts in two heterogeneous portfolios of risks, whenever the occurrence probabilities are interdependent.

THEOREM 7. *Let* X_{λ_1} *and* X_{λ_2} *(*$X_{\lambda_1^*}$ *and* $X_{\lambda_2^*}$*) be non-negative random variables with* $X_{\lambda_i} \sim \bar{F}(x;\lambda_i)$ *(*$X_{\lambda_i^*} \sim \bar{F}(x;\lambda_i^*)$*),* $i=1,2$, *and associated copula C. Further, suppose that* $\boldsymbol{I}$ *is* LWSAI, *and independent of the* X_{λ_i}*'s (*$X_{\lambda_i^*}$*'s). Assume that the following conditions hold:*

(i) $\bar{F}(x;\lambda)$ *is decreasing and convex in* λ *for any* $x \in \mathbb{R}_+$*;*

(ii) $(\lambda_1^*,\lambda_2^*) \overset{\mathrm{m}}{\preceq} (\lambda_1,\lambda_2)$, *such that* $\lambda_1 \geqslant \lambda_2$ *and* $\lambda_1^* \geqslant \lambda_2^*$*;*

(iii) C *is Schur-concave.*

Then, we have $Y_{2:2}^* \leqslant_{\mathrm{st}} Y_{2:2}$.

Proof. Let $X_{2:2} = \max(X_{\lambda_1}, X_{\lambda_2})$ and $X_{2:2}^* = \max(X_{\lambda_1^*}, X_{\lambda_2^*})$. First, we prove that $X_{2:2}^* \leqslant_{\mathrm{st}} X_{2:2}$. It is enough to show that the function

$$F_{X_{2:2}}(x) = C(F(x;\lambda_1), F(x;\lambda_2)),$$

is Schur-concave in $\boldsymbol{\lambda}$. According to Marshall et al. [28, Table 2, Page 91], Schur-concavity of C and increasing and concavity properties of $F(x;\lambda)$ in λ, implies that $F_{X_{2:2}}(x)$ is increasing and Schur-concave in $\boldsymbol{\lambda}$. Thus, condition (ii) implies

$$X_{2:2}^* \leqslant_{\mathrm{st}} X_{2:2}. \tag{9}$$

Also, according to Marshall et al. [28], the convexity of $\bar{F}(x;\lambda_i)$ in λ_i, implies the Schur-convexity of $\bar{F}(x;\lambda_1) + \bar{F}(x;\lambda_2)$ in $\boldsymbol{\lambda}$. Thus, the condition (ii) implies that

$$\bar{F}(x;\lambda_1^*) + \bar{F}(x;\lambda_2^*) \leqslant \bar{F}(x;\lambda_1) + \bar{F}(x;\lambda_2). \tag{10}$$

Note that

$$G_{Y_{2:2}}(x) = p(0,0) + p(1,1)F_{X_{2:2}}(x) + p(0,1)F(x;\lambda_2) + p(1,0)F(x;\lambda_1),$$

and similarly,

$$G_{Y_{2:2}^*}(x) = p(0,0) + p(1,1)F_{X_{2:2}^*}(x) + p(0,1)F(x;\lambda_2^*) + p(1,0)F(x;\lambda_1^*).$$

Thus, we have

$$
\begin{aligned}
G_{Y_{2:2}}(x)-G_{Y^*_{2:2}}(x) &= p(1,1)[F_{X_{2:2}}(x)-F_{X^*_{2:2}}(x)]+p(0,1)[F(x;\lambda_2)-F(x;\lambda_2^*)]\\
&\quad +p(1,0)[F(x;\lambda_1)-F(x;\lambda_1^*)]\\
&= p(1,1)[\bar F_{X^*_{2:2}}(x)-\bar F_{X_{2:2}}(x)]+p(0,1)[\bar F(x;\lambda_2^*)-\bar F(x;\lambda_2)]\\
&\quad +p(1,0)[\bar F(x;\lambda_1^*)-\bar F(x;\lambda_1)]\\
&\leqslant p(0,1)[\bar F(x;\lambda_2^*)-\bar F(x;\lambda_2)]+p(1,0)[\bar F(x;\lambda_1^*)-\bar F(x;\lambda_1)]\\
&\leqslant p(0,1)[\bar F(x;\lambda_2^*)-\bar F(x;\lambda_2)]+p(0,1)[\bar F(x;\lambda_1^*)-\bar F(x;\lambda_1)]\\
&= p(0,1)[\bar F(x;\lambda_1^*)+\bar F(x;\lambda_2^*)-\bar F(x;\lambda_1)-\bar F(x;\lambda_2)]\\
&\leqslant 0,
\end{aligned}
$$

where the first inequality is due to (9), the second inequality is according to Lemma 6 and the last inequality is based on (10). Hence, it is proved that $G_{Y_{2:2}}(x)\leqslant G_{Y^*_{2:2}}(x)$ which completes the proof. □

In the following, three special cases of Theorem 7 with respect to the scale, PHR and TG models, are represented.

THEOREM 8. *Let $\bar F(x;\lambda_i)=\bar F(\lambda_i x)$ and $\bar F(x;\lambda_i^*)=\bar F(\lambda_i^* x)$, for $i=1,2$. Under the setup of Theorem 7, suppose that the following conditions hold:*

(i) *$f(x)$ is decreasing in $x\in\mathbb{R}_+$;*

(ii) *$(\lambda_1^*,\lambda_2^*)\overset{\mathrm{m}}{\preceq}(\lambda_1,\lambda_2)$, such that $\lambda_1\geqslant\lambda_2$ and $\lambda_1^*\geqslant\lambda_2^*$;*

(iii) *C is Schur-concave.*

Then, we have $Y^*_{2:2}\leqslant_{\mathrm{st}} Y_{2:2}$.

Proof. Obviously, the condition (i) of Theorem 8 implies the condition (i) of Theorem 7 which completes the proof. □

THEOREM 9. *Let $\bar F(x;\lambda_i)=[\bar F(x)]^{\lambda_i}$ and $\bar F(x;\lambda_i^*)=[\bar F(x)]^{\lambda_i^*}$, for $i=1,2$. Under the setup of Theorem 7, suppose that the following conditions hold:*

(i) *$(\lambda_1^*,\lambda_2^*)\overset{\mathrm{m}}{\preceq}(\lambda_1,\lambda_2)$, such that $\lambda_1\geqslant\lambda_2$ and $\lambda_1^*\geqslant\lambda_2^*$;*

(ii) *C is Schur-concave.*

Then, we have $Y^*_{2:2}\leqslant_{\mathrm{st}} Y_{2:2}$.

Proof. Obviously, $\bar F(x;\lambda)=[\bar F(x)]^{\lambda}$ satisfies the condition (i) of Theorem 7 which completes the proof. □

THEOREM 10. *Let $\bar F(x;\lambda_i)=\bar F(x)(1-\lambda_i F(x))$ and $\bar F(x;\lambda_i^*)=\bar F(x)(1-\lambda_i^* F(x))$, for $i=1,2$. Under the setup of Theorem 7, suppose that the following conditions hold:*

(i) $(\lambda_1^*,\lambda_2^*)\overset{\text{m}}{\preceq}(\lambda_1,\lambda_2)$, *such that* $\lambda_1 \geqslant \lambda_2$ *and* $\lambda_1^* \geqslant \lambda_2^*$*;*

(ii) C *is Schur-concave.*

Then, we have $Y^*_{2:2} \leqslant_{\text{st}} Y_{2:2}$.

Proof. Obviously, $\bar{F}(x;\lambda) = \bar{F}(x)(1-\lambda F(x))$ satisfies the condition (i) of Theorem 7 which completes the proof. □

The following example provides a numerical example to illustrate the validity of Theorem 9.

EXAMPLE 5. Let $X_{\lambda_i} \sim \text{Pareto}(1,\lambda_i)$ ($X_{\lambda_i^*} \sim \text{Pareto}(1,\lambda_i^*)$), for $i=1,2$, with the associated FGM copula, with $\theta = 0.7$. Let $(\lambda_1,\lambda_2) = (7,2)$, $(\lambda_1^*,\lambda_2^*) = (5.5,3.5)$, $p(0,0) = 0.89$, $p(0,1) = 0.06$, $p(1,0) = 0.04$ and $p(1,1) = 0.01$. Using Lemma 5, we get the condition (ii) of Theorem 9, and obviously can be verified that the other conditions are also satisfied. So, we have $Y^*_{2:2} \leqslant_{\text{st}} Y_{2:2}$. Figure 5 represents the survival functions of $Y_{2:2}$ and $Y^*_{2:2}$, which approves with the intended result.

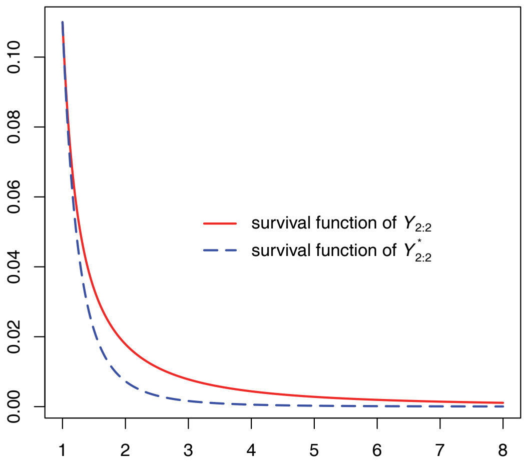

Figure 5: Plots of the survival functions of $Y_{2:2}$ and $Y^*_{2:2}$ in Example 5.

The following example illustrates that the conditions (ii) of Theorem 7 can not be dropped.

EXAMPLE 6. Under the same setup in Example 5, we take $(\lambda_1,\lambda_2) = (2,7)$ with the other unchanged values. It is clear that $\lambda_1 \ngeqslant \lambda_2$, but it can be easily verified that the other conditions of Theorem 7 are satisfied. Figure 6 represents the survival functions of $Y_{2:2}$ and $Y^*_{2:2}$, which cross each other.

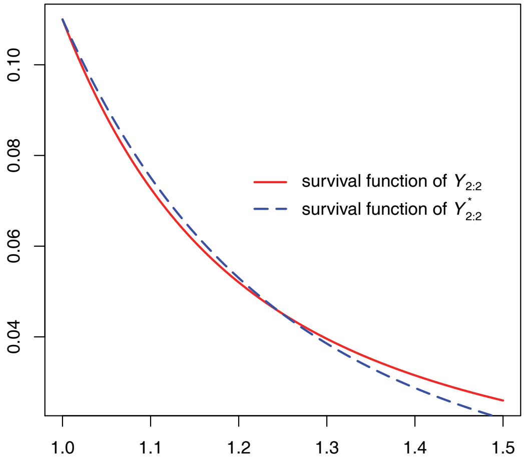

Figure 6: Plots of the survival functions of $Y_{2:2}$ and $Y^*_{2:2}$ in Example 6.

The following theorem provides a comparison between the largest claim amounts in two heterogeneous portfolios of risks, in terms of $\boldsymbol{\lambda}$.

THEOREM 11. *Let $X_{\lambda_1},\ldots,X_{\lambda_n}$ ($X_{\lambda_1^*},\ldots,X_{\lambda_n^*}$) be non-negative random variables with $X_{\lambda_i} \sim \bar{F}(x;\lambda_i)$ ($X_{\lambda_i^*} \sim \bar{F}(x;\lambda_i^*)$), $i=1,\ldots,n$, and associated copula C. Further, suppose that $I_{p_1},\ldots,I_{p_n}$ is a set of independent Bernoulli random variables, independent of the X_{λ_i}'s ($X_{\lambda_i^*}$'s), with $\mathrm{E}[I_{p_i}] = p_i$, $i=1,\ldots,n$. Assume that $\bar{F}(x;\lambda)$ is decreasing in λ for any $x \in \mathbb{R}_+$. Then, we have*

$$\lambda_i \leqslant \lambda_i^*,\ \forall\, i=1,\ldots,n \implies Y^*_{n:n} \leqslant_{\mathrm{st}} Y_{n:n}.$$

Proof. Denote $p(\boldsymbol{\mu}) = \mathrm{P}(I_{p_1}=\mu_1,\ldots,I_{p_n}=\mu_n)$. The distribution function of $Y_{n:n}$ can be obtained as follows:

$$\begin{aligned}
G_{Y_{n:n}}(x) &= \mathrm{P}\Big(Y_1 \leqslant x,\ldots,Y_n \leqslant x\Big)\\
&= \mathrm{P}\Big(I_{p_1}X_{\lambda_1} \leqslant x,\ldots,I_{p_n}X_{\lambda_n} \leqslant x\Big)\\
&= \sum_{\boldsymbol{\mu}\in\{0,1\}^n} p(\boldsymbol{\mu})\ \mathrm{P}\Big(I_{p_1}X_{\lambda_1} \leqslant x,\ldots,I_{p_n}X_{\lambda_n} \leqslant x | I_{p_1}=\mu_1,\ldots,I_{p_n}=\mu_n\Big)\\
&= \sum_{\boldsymbol{\mu}\in\{0,1\}^n} p(\boldsymbol{\mu})\ \mathrm{P}\Big(\mu_1X_{\lambda_1} \leqslant x,\ldots,\mu_nX_{\lambda_n} \leqslant x | I_{p_1}=\mu_1,\ldots,I_{p_n}=\mu_n\Big)
\end{aligned}$$

$$= \sum_{\boldsymbol{\mu}\in\{0,1\}^n} p(\boldsymbol{\mu})\, \mathrm{P}\left(\mu_1 X_{\lambda_1} \leqslant x, \ldots, \mu_n X_{\lambda_n} \leqslant x\right)$$

$$= \sum_{\boldsymbol{\mu}\in\{0,1\}^n} p(\boldsymbol{\mu})\, C\left([F(x;\lambda_1)]^{\mu_1}, \ldots, [F(x;\lambda_n)]^{\mu_n}\right). \tag{11}$$

Based on decreasing property of $\bar{F}(x;\lambda)$ in λ and the nature of copula, we immediately conclude that $G_{Y_{n:n}}(x)$ is increasing in λ_i, for $i = 1, \ldots, n$. Hence, the desired result holds. □

The following theorem represents the impact due to degree of dependence in comparison the largest claim amounts in two heterogeneous portfolios of risks.

THEOREM 12. *Let $X_{\lambda_1}, \ldots, X_{\lambda_n}$ be non-negative random variables with $X_{\lambda_i} \sim \bar{F}(x;\lambda_i)$, $i = 1, \ldots, n$, and associated copula C (C^*). In addition, suppose that $I_{p_1}, \ldots, I_{p_n}$ is a set of independent Bernoulli random variables, independent of the X_{λ_i}'s, with $\mathrm{E}[I_{p_i}] = p_i$, $i = 1, \ldots, n$. Then, we have*

$$C \prec C^* \implies Y^*_{n:n} \leqslant_{\mathrm{st}} Y_{n:n}.$$

Proof. By (11) and Definition 6, the proof is immediately completed. □

The following theorem provides a comparison between the largest claim amounts in two heterogeneous portfolios of risks, in terms of $\boldsymbol{\lambda}$ and degree of dependence.

THEOREM 13. *Let $X_{\lambda_1}, \ldots, X_{\lambda_n}$ ($X_{\lambda_1^*}, \ldots, X_{\lambda_n^*}$) be non-negative random variables with $X_{\lambda_i} \sim \bar{F}(x;\lambda_i)$ ($X_{\lambda_i^*} \sim \bar{F}(x;\lambda_i^*)$), $i = 1, \ldots, n$, and associated copula C (C^*). Furthermore, suppose that $I_{p_1}, \ldots, I_{p_n}$ is a set of independent Bernoulli random variables, independent of the X_{λ_i}'s ($X_{\lambda_i^*}$'s), with $\mathrm{E}[I_{p_i}] = p_i$, $i = 1, \ldots, n$. Assume that $\bar{F}(x;\lambda)$ is decreasing in λ for any $x \in \mathbb{R}_+$. Then, we have*

$$C \prec C^* \text{ and } \lambda_i \leqslant \lambda_i^*,\ \forall\, i = 1, \ldots, n \implies Y^*_{n:n} \leqslant_{\mathrm{st}} Y_{n:n}.$$

Proof. Let $V_{n:n}$, $Z_{n:n}$ and $W_{n:n}$ be the largest claim amounts from the portfolios $(I_{p_1}X_{\lambda_1^*}, \ldots, I_{p_n}X_{\lambda_n^*})$ with associated copula C^*, $(I_{p_1}X_{\lambda_1}, \ldots, I_{p_n}X_{\lambda_n})$ with associated copula C^*, and $(I_{p_1}X_{\lambda_1}, \ldots, I_{p_n}X_{\lambda_n})$ with associated copula C, respectively. It is easily seen that $Y^*_{n:n} \overset{\mathrm{st}}{=} V_{n:n}$ and $Y_{n:n} \overset{\mathrm{st}}{=} W_{n:n}$. On the other hand, Theorem 12 and Theorem 13 imply that $V_{n:n} \leqslant_{\mathrm{st}} Z_{n:n}$ and $Z_{n:n} \leqslant_{\mathrm{st}} W_{n:n}$, respectively. Hence, the proof is completed. □

The three following theorems consider the scale, PHR and TG models as the special cases of Theorem 13.

THEOREM 14. *Let $\bar{F}(x;\lambda_i) = \bar{F}(\lambda_i x)$ and $\bar{F}(x;\lambda_i^*) = \bar{F}(\lambda_i^* x)$, for $i = 1, \ldots, n$. Under the setup of Theorem 13, Then, we have $Y^*_{n:n} \leqslant_{\mathrm{st}} Y_{n:n}$.*

THEOREM 15. *Let $\bar{F}(x;\lambda_i) = [\bar{F}(x)]^{\lambda_i}$ and $\bar{F}(x;\lambda_i^*) = [\bar{F}(x)]^{\lambda_i^*}$, for $i = 1, \ldots, n$. Under the setup of Theorem 13, we have $Y^*_{n:n} \leqslant_{\mathrm{st}} Y_{n:n}$.*

THEOREM 16. *Let $\bar{F}(x;\lambda_i)=\bar{F}(x)(1-\lambda_i F(x))$ and $\bar{F}(x;\lambda_i^*)=\bar{F}(x)(1-\lambda_i^* F(x))$, for $i=1,\ldots,n$. Under the setup of Theorem 13, we have $Y_{n:n}^* \leqslant_{\rm st} Y_{n:n}$.*

Another important distribution used as the distribution of claim severity from policyholders is Weibull distribution, which is a special case of the scale model. X has the Weibull distribution with parameters α and λ, denoted by $X \sim \mathrm{Wei}(\alpha,\lambda)$, if its survival function is given by

$$\bar{F}(x;\alpha,\lambda)=e^{-(\lambda x)^{\alpha}}, \quad x\in\mathbb{R}_{++}.$$

The following example provides a numerical example to illustrate the validity of Theorem 14.

EXAMPLE 7. Let $X_{\lambda_i}\sim \mathrm{Wei}(3,\lambda_i)$ ($X_{\lambda_i^*}\sim \mathrm{Wei}(3,\lambda_i^*)$), for $i=1,2,3$, with the associated Frank copula, which was introduced by Frank [16], of the form

$$C_\theta(v_1,v_2,v_3)=-\frac{1}{\theta}\log\left(1+\frac{(e^{-\theta v_1}-1)(e^{-\theta v_2}-1)(e^{-\theta v_3}-1)}{(e^{-\theta}-1)^2}\right),$$

where $\theta\in(0,\infty)$. Further, suppose that I_{p_1},I_{p_2},I_{p_3} is a set of independent Bernoulli random variables, independent of the X_{λ_i}'s ($X_{\lambda_i^*}$'s), with $\mathrm{E}[I_{p_i}]=p_i$, for $i=1,2,3$. We take $(\lambda_1,\lambda_2,\lambda_3)=(0.5,0.7,0.3)$, $(\lambda_1^*,\lambda_2^*,\lambda_3^*)=(0.51,0.7,0.33)$, $(p_1,p_2,p_3)=(0.01,0.02,0.07)$ and $\theta=0.6$. Obviously, the conditions of Theorem 14 are satisfied. So, we have $Y_{3:3}^*\leqslant_{\rm st} Y_{3:3}$. Figure 7 represents the survival functions of $Y_{3:3}$ and $Y_{3:3}^*$, which coincides with the intended result.

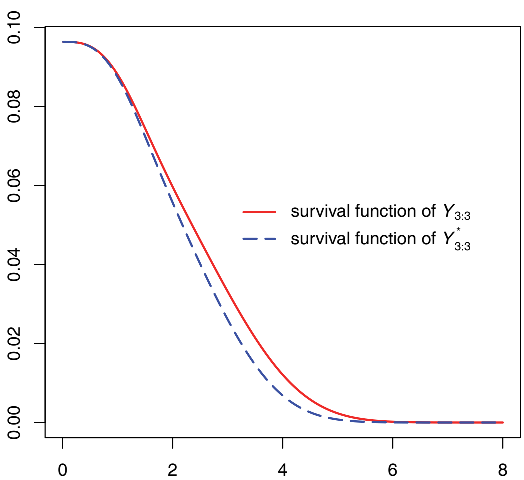

Figure 7: Plots of the survival functions of $Y_{3:3}$ and $Y_{3:3}^*$ in Example 7.

We recall that all the proven results of the paper hold under some sufficient conditions. The following example shows that the mentioned conditions in Theorem 14 are not necessary.

EXAMPLE 8. Under the same setup in Example 7, we take $(\lambda_1^*, \lambda_2^*, \lambda_3^*) = (0.51, 0.68, 0.33)$ with the other unchanged values. It is clear that $\lambda_2 \not\leq \lambda_2^*$, and consequently the conditions of Theorem 14 are not fulfilled but anyhow the desired property is satisfied; that is $Y_{3:3}^* \leqslant_{\text{st}} Y_{3:3}$. Figure 8 represents this fact.

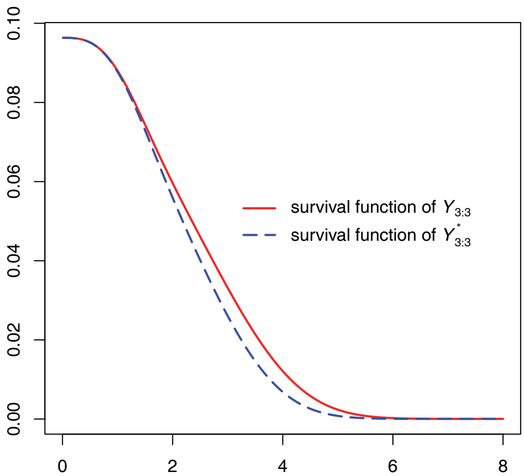

Figure 8: Plots of the survival functions of $Y_{3:3}$ and $Y_{3:3}^*$ in Example 8.

Conclusion

In this paper, under some certain conditions, we discussed stochastic comparisons between the largest claim amounts under dependency of severities in the sense of usual stochastic ordering in a general model, which particularly includes some important models such as the scale, PHR and TG models. However, we applied some distributions to illustrate the results.

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Hossein Nadeb
Department of Statistics
Yazd University
Yazd, Iran
e-mail: honadeb@yahoo.com

Hamzeh Torabi
Department of Statistics
Yazd University
Yazd, Iran
e-mail: htorabi@yazd.ac.ir

Ali Dolati
Department of Statistics
Yazd University
Yazd, Iran
e-mail: adolati@yazd.ac.ir