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SOME COMMENTS ON CHEN XU, MENGMEI XI, XUEJUN WANG AND HAO XIA'S PAPER "L^r CONVERGENCE FOR WEIGHTED SUMS OF EXTENDED NEGATIVELY DEPENDENT RANDOM VARIABLES"

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Abstract. In this short note, we show that an assertion presented in the main result of Chen Xu, Mengmei Xi, Xuejun Wang and Hao Xia's paper, "L^r convergence for weighted sums of extended negatively dependent random variables", is false. A reformulation of this statement is announced making it valid.

1. Comments

In paper [2], the authors claimed that if $1 < r < 2$, $\{X_k, -\infty < k < \infty\}$ is a doubly infinite sequence of zero-mean extended negatively dependent random variables satisfying $x^r \sup_k \mathbb{P}\{|X_k| > x\} = o(1)$ as $x \to \infty$, and $\left\{a_{n,k}, -\infty < k < \infty, n \geqslant 1\right\}$ is an array of constants such that, for all $s \geqslant 1$, $\sup_{n\geqslant 1} \sum_{k=-\infty}^{\infty} \left|a_{n,k}\right|^s / n < \infty$ then

$$\frac{1}{n^{1/r}} \sum_{k=-\infty}^{\infty} a_{n,k} X_k \xrightarrow{\mathbb{P}} 0 \tag{1}$$

(see assertion (1) of Theorem 2.1). A simple counterexample shows that this statement is, in general, false. As we shall see, the convergence (1) holds true admitting extra conditions. Further, the authors of [2] consider throughout doubly infinite sequences of extended negatively dependent random variables omitting its definition (which is not exactly equal to the corresponding one for random sequences). Additionally, an indispensable auxiliary lemma to establish all assertions of Theorem 2.1 in [2] is not stated for doubly infinite arrays as required by the proof. We shall fulfill these voids.

We begin by introducing the notion of extended negatively dependence for doubly infinite sequences.

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DEFINITION 1. A doubly infinite sequence $\{X_k,\ -\infty < k < \infty\}$ of random variables is said to be *upper extended negatively dependent* (UEND) if for any nonnegative integers n and m, there is some $M > 0$ (not depending on n or m) such that

$$\mathbb{P}\left(\bigcap_{k=-m}^{n}\{X_k > x_k\}\right) \leqslant M\prod_{k=-m}^{n}\mathbb{P}\{X_k > x_k\} \tag{2}$$

holds for all real numbers $x_{-m},\dots,x_{-1},x_0,x_1,\dots,x_n$. A doubly infinite sequence $\{X_k,\ -\infty < k < \infty\}$ of random variables is said to be *lower extended negatively dependent* (LEND) if for any nonnegative integers n and m, there exists some $M > 0$ (not depending on n or m) such that

$$\mathbb{P}\left(\bigcap_{k=-m}^{n}\{X_k \leqslant x_k\}\right) \leqslant M\prod_{k=-m}^{n}\mathbb{P}\{X_k \leqslant x_k\} \tag{3}$$

holds for all real numbers $x_{-m},\dots,x_{-1},x_0,x_1,\dots,x_n$. A doubly infinite sequence $\{X_k,\ -\infty < k < \infty\}$ of random variables is said to be *extended negatively dependent* (END) if it is both UEND and LEND.

Let us point out that the preamble assumptions in Theorem 2.1 of [2], alone, are not enough to ensure the well-definiteness of $S_n = \sum_{k=-\infty}^{\infty} a_{n,k}X_k$ for all large n. For instance, let $\{a_{n,k},\ -\infty < k < \infty,\ n \geqslant 1\}$ be an array of constants not dependent on n given by $a_{n,k} = 1/(k+2)^2$ for any $k \geqslant 1$ and $a_{n,k} = 0$ otherwise, and $\{X_k,\ -\infty < k < \infty\}$ be a doubly infinite sequence of independent random variables defined by

$$\mathbb{P}\left\{X_k = (|k|+2)^2\right\} = \mathbb{P}\left\{X_k = -(|k|+2)^2\right\} = \frac{1}{2(|k|+2)}, \quad \mathbb{P}\{X_k = 0\} = 1 - \frac{1}{|k|+2}.$$

Thus, $\mathbb{E}X_k = 0$ for each k and $\sup_{n\geqslant 1}\sum_{k=-\infty}^{\infty}\left|a_{n,k}\right|^s/n = \sum_{k=1}^{\infty} 1/(k+2)^{2s} < \infty$ for all $s \geqslant 1$. However, $S_n = \sum_{k=1}^{\infty} X_k/(k+2)^2$ does not converge a.s. since $X_k/(k+2)^2 \overset{\text{a.s.}}{\not\to} 0$ according to Borel-Cantelli lemma.

Next, we give an example which meets all assumptions in assertion (1) of the referred Theorem 2.1, but does not verify (1).

EXAMPLE 1. Fix $n \geqslant 1$ and suppose $\{X_k,\ -\infty < k < \infty\}$ given by $X_k = 0$ for all $k \neq n$, and X_n having probability density function

$$f_n(x) = \begin{cases} n^2/|x|^3, & |x| \geqslant n \\ 0, & \text{otherwise} \end{cases}.$$

This doubly infinite sequence clearly satisfies (2) and (3); thus, it is END. We also have $\mathbb{E}X_k = 0$ for all k. Moreover, for any $1 < r < 2$,

$$x^r \sup_k \mathbb{P}(|X_k| > x) = x^r\mathbb{P}(|X_n| > x) = \frac{n^2}{x^{2-r}}, \quad x > n.$$

Nevertheless, taking $a_{n,k}$ given by $a_{n,n} = 1$ and $a_{n,k} = 0$ whenever $k \neq n$, we can conclude that, for each $\varepsilon > 0$, there is some positive integer $n_0 = n_0(\varepsilon, r)$ such that for all $n \geqslant n_0$,

$$\mathbb{P}\left\{\left|\sum_{k=-\infty}^{\infty} a_{n,k}X_k\right| > \varepsilon n^{1/r}\right\} = \mathbb{P}\left\{|X_n| > \varepsilon n^{1/r}\right\} = 1$$

implying that (1) does not hold.

An array $\{X_{n,k},\, -\infty < k < \infty, n \geqslant 1\}$ of random variables is said to be *extended negatively dependent* (END) if for each $n \geqslant 1$, the doubly infinite sequence $\{X_{n,k},\, -\infty < k < \infty\}$ is END (in the sense of Definition 1) with the same constant M not depending on n. The following lemma announced for doubly infinite arrays allows us to proceed with the demonstration of (1). Assuring that all terms involved are well-defined, its proof is analogous to the one presented in Lemma 2 of [1] by noting that $a_{n,k}X_{n,k} = a^+_{n,k}X_{n,k} - a^-_{n,k}X_{n,k}$, where $a^+_{n,k}$ and $a^-_{n,k}$ are the positive and negative parts of $a_{n,k}$, respectively.

LEMMA 1. *If* $p \geqslant 2$, $\{a_{n,k},\, -\infty < k < \infty, n \geqslant 1\}$ *is an array of constants satisfying* $\sum_{k=-\infty}^{\infty} |a_{n,k}| < \infty$ *for all* n *and* $\{X_{n,k},\, -\infty < k < \infty, n \geqslant 1\}$ *is an array of zero-mean END random variables such that* $\sup_k \mathbb{E}|X_{n,k}|^p < \infty$ *for all* n *then*

$$\mathbb{E}\left|\sum_{k=-\infty}^{\infty} a_{n,k}X_{n,k}\right|^p \leqslant (1+M)C(p)\left[\sum_{k=-\infty}^{\infty} \mathbb{E}\left|a_{n,k}X_{n,k}\right|^p + \left(\sum_{k=-\infty}^{\infty} \mathbb{E}\left|a_{n,k}X_{n,k}\right|^2\right)^{p/2}\right]$$

where $C(p)$ *is a positive constant depending only on* p.

The statement below allows us to obtain (1) under extra assumptions; its proof follows exactly the same steps as the proof of assertion (1) in Theorem 2.1 of [2], and so will be omitted.

THEOREM 1. *Let* $1 < r < 2$ *and* $\{a_{n,k},\, -\infty < k < \infty, n \geqslant 1\}$ *be an array of constants satisfying* $\sup_n \sum_{k=-\infty}^{\infty} |a_{n,k}|^s / n < \infty$ *for any* $s \geqslant 1$. *If* $\{X_k,\, -\infty < k < \infty\}$ *is a doubly infinite sequence of zero-mean END random variables such that*

(a) $x^r \sup_k \mathbb{P}\{|X_k| > x\} = o(1)$ *as* $x \to \infty$,

(b) $\int_0^1 yn \sup_k \mathbb{P}\{|X_k| > yn^{1/r}\}\,dy = o(1)$ *as* $n \to \infty$,

(c) $\int_1^\infty n \sup_k \mathbb{P}\{|X_k| > yn^{1/r}\}\,dy = o(1)$ *as* $n \to \infty$

then

$$\frac{1}{n^{1/r}} \sum_{k=-\infty}^{\infty} a_{n,k}X_k \xrightarrow{\mathbb{P}} 0.$$

It should be noted that assumption $\sup_k \mathbb{E}|X_k|^r < \infty$ is stronger than above conditions (b) and (c) together. Indeed, if $\sup_k \mathbb{E}|X_k|^r < \infty$ then the sequence of functions $\varphi_n \colon [0,1] \longrightarrow \mathbb{R}$ defined by $\varphi_n(y) := yn \sup_k \mathbb{P}\left(|X_k| > yn^{1/r}\right)$, $n \geqslant 1$ converges pointwise on $[0,1]$ to zero, $|\varphi_n(y)| \leqslant g(y)$ for all n and every $0 \leqslant y \leqslant 1$, where $g \colon [0,1] \longrightarrow \mathbb{R}$ is given by $g(y) = y^{1-r} \sup_k \mathbb{E}|X_k|^r$, $0 < y \leqslant 1$ and $g(0) = 0$. Hence, $\int_0^1 yn \sup_k \mathbb{P}\left\{|X_k| > yn^{1/r}\right\} dy = o(1)$ as $n \to \infty$ via Lebesgue's dominated convergence theorem. Similarly, one can prove that $\int_1^\infty n \sup_k \mathbb{P}\left\{|X_k| > yn^{1/r}\right\} dy = o(1)$ as $n \to \infty$.

REFERENCES

[1] J. LITA DA SILVA, *Convergence in p-mean for arrays of row-wise extended negatively dependent random variables*, Acta Math. Hungar., **150**, 2 (2016), 346–362.

[2] CHEN XU, MENGMEI XI, XUEJUN WANG AND HAO XIA, *L^r convergence for weighted sums of extended negatively dependent random variables*, J. Math. Inequal., **10**, 4 (2016), 1157–1167.

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