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ON AN OPEN PROBLEM OF FENG QI AND BAI–NI GUO

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Abstract. In this work, we investigate an open problem posed by Feng Qi and Bai-Ni Guo in their paper "*Complete monotonicities of functions involving the gamma and digamma functions* [7]".

1. Introduction and statement of the main results

A function f is said to be *completely monotonic* on an interval I, if f has derivatives of all orders on I which alternate successively in sign, that is

$$(-1)^n f^{(n)}(x) \geqslant 0,$$

for $x \in I$ and $n \geqslant 0$. If the inequality above is strict for all $x \in I$ and for all $n \geqslant 0$, then f is said to be strictly completely monotonic. In recent years, there has been considerable interest in studying completely monotonic functions. The interest is due to their connection and applications to many different areas such as probability and potential theory. In [12] some interesting properties of these functions are exhibited. For more information and properties the interested reader may consult [1, 2, 3, 4, 8, 9, 10, 11]

Recently, in [7], the authors study a class of completely monotonic functions involving the gamma and digamma functions. In particular, they show that the function

$$f_s(x) = \frac{(x+1)^s}{\big(\Gamma(x+1)\big)^{1/x}},$$

is strictly completely monotonic on $(-1,+\infty)$ provided that $s \leqslant 1/(1+\alpha) < 1$. Here, α is the supremum of the function of two independent variables (n,t), defined on $\mathbb{N} \times [0,+\infty)$ by

$$\tau(n,t) = \frac{1}{n}\left(t - (t+n+1)\left(\frac{t}{1+t}\right)^{n+1}\right).$$

For $n = 2,3$, the authors compute numerically the maximum, they find that

$$\max_{t>0} \tau(2,t) \simeq 0.264076, \text{ and } \max_{t>0} \tau(3,t) \simeq 0.271807.$$

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Further progress are made in [5], where it was shown that $\alpha < 1$. A more precise upper bound is given in [6], where the authors showed that $\alpha < 3/10$. To the best of our knowledge, no analytic method was given until now to compute the value of α. In [7], the authors suggest the following problem:

PROBLEM 1. Find the exact value

$$\alpha = \sup_{(n,t)\in\mathbb{N}\times(0,+\infty)} \tau(n,t).$$

Using Mathematica, they conjecture that α is greater than $0,298$.
The main concern of this paper is to study Problem 1 and to show that their conjecture is true. More precisely we shall prove the following result.

THEOREM 1. *The function of two variables $(n,t) \mapsto \tau(n,t)$ admit a supremum*

$$\alpha = \sup_{(n,t)\in\mathbb{N}\times(0,+\infty)} \tau(n,t),$$

which is given by

$$\alpha = \frac{\ell}{1+\ell+\ell^2},$$

where ℓ is the unique solution of the equation

$$e^{\frac{1}{\ell}} - \frac{1}{\ell^2} - \frac{1}{\ell} - 1 = 0.$$

Moreover, numerically $\ell \simeq 0,5576367386...$, and $\alpha = 0,2984256075...$

The proof of Theorem 1 is given in Section 3. Its proof uses some intermediate results presented below.

2. Preliminary results

Define the function

$$\tau(x,t) = \frac{1}{x}\left(t - (t+x+1)\left(\frac{t}{1+t}\right)^{x+1}\right),$$

for any $(x,t) \in [1,+\infty)\times[0,+\infty)$. In the sequel, $\partial_1\tau(x,t)$, $\partial_2\tau(x,t)$ mean respectively the derivatives with respect to the first variable x and the second variable t.
We begin with the following proposition which shows that for every fixed $x \geqslant 1$, the function $t \mapsto \tau(x,t)$ attains its maximum at only one point $t(x)$ between 0 and x.

PROPOSITION 1. *For every $x \geqslant 1$, the function $t \mapsto \tau(x,t)$ attains its maximum at only one point $t(x) \in (0,x)$. Moreover, the value of the maximum $\alpha(x) := \tau(x,t(x))$ is given by*

$$\alpha(x) = \frac{(1+x)t(x)}{x^2+(1+t(x))^2+x(2+t(x))}$$

Proof. We give the proof in two steps. In the first one, we show the existence of the critical point. In the second step we compute the value of the maximum.

Step 1. Straightforward computations entails that the first and the second derivatives of the function $\tau(x,t)$ with respect to t are given by

$$\partial_2\tau(x,t) = \frac{1}{x}\left(1-\left(\frac{t}{1+t}\right)^{x+1} - \frac{(t+x+1)(x+1)}{t(1+t)}\left(\frac{t}{1+t}\right)^{x+1}\right), \tag{1}$$

$$\partial_2^2\tau(x,t) = (1+x)t^{-1+x}(1+t)^{-3-x}(-1-x+t). \tag{2}$$

Therefore, $\partial_2^2\tau(x,t) = 0$ on $(0,+\infty)$ if and only if $t = x+1$. Moreover, $\partial_2\tau(x,0) = \frac{1}{x}$, and

$$\partial_2\tau(x,x) = \frac{1}{x}\left(1-3\left(\frac{x}{1+x}\right)^{x+1} - \frac{1}{x}\left(\frac{x}{1+x}\right)^{x+1}\right).$$

We claim that, $\partial_2\tau(x,x) < 0$. To see this, consider for $x \geqslant 1$, the function $h(x) = 1-(3+\frac{1}{x})(\frac{x}{x+1})^{x+1}$. We have by successive differentiation

$$h'(x) = -x^x(1+x)^{-1-x}h_1(x),$$

where $h_1(x) = 3+(1+3x)\log(1+\frac{1}{x})$,

$$h_1'(x) = \frac{1}{x}+\frac{2}{1+x}+3\log(x)-3\log(1+x),$$

and

$$h_1''(x) = \frac{x-1}{x^2(1+x)^2} \geqslant 0.$$

Since, $\lim\limits_{x\to+\infty} h_1'(x) = 0$, therefore $h_1'(x) \leqslant 0$. Moreover, $\lim\limits_{x\to+\infty} h_1(x) = 0$, hence $h_1(x) \geqslant 0$. Accordingly, h is a non increasing function on $[1,+\infty)$, and $h(1) = 0$. Thus, for every $x > 1$

$$\partial_2\tau(x,x) < 0,$$

and the function $t \mapsto \partial_2\tau(x,t)$ decreases strictly on $(0,x+1)$. By the fact that the second derivative acroses the x-axis only one time on $(0,+\infty)$, we infer that there is a unique $t(x) \in (0,x+1)$ such that

$$\partial_2\tau(x,t(x)) = 0, \text{ and } t(x) \in (0,x),$$

in view of $\partial_2\tau(x,x) < 0$. Note that in the interval $(0,x)$, $\partial_2^2\tau(x,t) < 0$, hence the critical point $t(x)$ is a maximum for the function $t \mapsto \tau(x,t)$.

Step 2. For $x > 0$, if we denote by $\alpha(x)$ the value of the maximum at $t(x)$, $\alpha(x) = \tau(x,t(x))$. Then by equation (1), we have

$$1-\left(\frac{t(x)}{1+t(x)}\right)^{x+1} - \frac{(t(x)+x+1)(x+1)}{t(x)(1+t(x))}\left(\frac{t(x)}{1+t(x)}\right)^{x+1} = 0. \tag{3}$$

In view of the definition of the function $\tau(x,\cdot)$, we also have

$$\alpha(x)=\frac{1}{x}\left(t(x)-(t(x)+x+1)\left(\frac{t(x)}{1+t(x)}\right)^{x+1}\right). \tag{4}$$

Substituting equation (3) in (4), we get

$$\alpha(x)=\frac{(1+x)t(x)}{x^2+(1+t(x))^2+x(2+t(x))}. \tag{5}$$

The proof is thus complete. $\square$

LEMMA 1. *The function $x\mapsto t(x)$ is well defined, differentiable on $(0,+\infty)$, and satisfies*

$$\frac{(x+1)^2}{2x+3}\leqslant t(x)<x, \quad (x>0). \tag{6}$$

Proof. Firstly note that the uniqueness of $t(x)$ stated in Proposition 1 allows us to define the function $x\mapsto t(x)$. Now, let $a>0$. We showed in Proposition 1 that $\partial_2\tau(a,t(a))=0$ and $\partial_2^2\tau(a,t(a))<0$. Applying the implicit function theorem, one can find a neighborhood V_a of a and neighborhood $W_{t(a)}$ of $t(a)$ and C^1-diffeomorphism $\varphi: V_a\longrightarrow W_{t(a)}$ such that for every $x\in V_a$, and $y\in W_{t(a)}$

$$\partial_2\tau(x,y)=0\Leftrightarrow y=\varphi(x).$$

By uniqueness we have $\varphi(x)=t(x)$. Which implies that the function $x\mapsto t(x)$ is differentiable at each point of $(0,+\infty)$. Finally, let us show that Eq. (6) holds. Note that we have already shown that $t(x)<x$. To prove the left-hand side, it is enough to show that

$$\partial_2\tau\Big(x,\frac{(x+1)^2}{2x+3}\Big)\geqslant 0,$$

and use the decrease of the function $t\mapsto\partial_2\tau(x,t)$ on $[0,x+1]$ and the fact that $t(x)$ is a maximum on $[0,x]$. To do this, set

$$\phi(x)=\partial_2\tau(x,\frac{(x+1)^2}{2x+3}).$$

Easy computations yield

$$\phi(x)=\frac{(x+1)^{2x+2}}{x(x+2)^{2x+4}}\Big(\frac{(x+2)^{2x+4}}{(x+1)^{2x+2}}-7x^2-21x-16\Big).$$

Observe that $\phi\geqslant 0$ if and only if the function

$$\Phi(x)=2\log(x+2)+2(x+1)\log\left(\frac{x+2}{x+1}\right)-\log(7x^2+21x+16).$$

is also positive. Take the first and second derivatives of Φ, then

$$\Phi'(x) = 2\log\left(\frac{x+2}{x+1}\right) - \frac{21+14x}{16+21x+7x^2},$$

and

$$\Phi''(x) = \frac{-35x^2-105x-78}{(1+x)(2+x)(16+7x(3+x))^2} < 0.$$

Furthermore, $\lim_{x\to+\infty}\Phi'(x) = 0$, therefore, $\Phi'(x) > 0$ for every $x \geqslant 0$. Thus,

$$\Phi(x) \geqslant \Phi(0) = 0,$$

and the result follows. □

PROPOSITION 2. *The function $x \mapsto \alpha(x)$ is differentiable, non decreasing on $[1,+\infty)$ with derivative*

$$\alpha'(x) = \partial_1\tau(x,t(x)).$$

Moreover, for every $x \geqslant 1$ the following inequality holds

$$0 \leqslant \alpha(x) < \frac{x}{3x+1}.$$

Proof. For the sake of readability, we divide the proof into two steps. Firstly we compute the derivative of α, and in the second step we show the inequality satisfied by the function α.

Step 1. Recall that

$$\alpha(x) = \tau(x,t(x)).$$

Therefore, $\alpha(x)$ is differentiable as composed of the differentiable functions $x \mapsto (x,t(x))$ and $(x,t) \mapsto \tau(x,t)$. Since $\partial_2\tau(x,t(x)) = 0$, differentiating α we get

$$\alpha'(x) = \partial_1\tau(x,t(x)) + t'(x)\partial_2\tau(x,t(x)) = \partial_1\tau(x,t(x)).$$

Now take the derivative of the function $\tau(x,t)$ with respect to x, it yields

$$\partial_1\tau(x,t(x)) = \frac{t(x)\left(-1-t(x)+\left(\frac{t(x)}{1+t(x)}\right)^x\left(1+t(x)+x(1+t(x)+x)\log(1+\frac{1}{t(x)})\right)\right)}{(1+t(x))x^2}$$

$$= \frac{t(x)}{x^2}\left(-1+\frac{\Big(1+t(x)\Big)\Big(1+t(x)+x\big(1+t(x)+x\big)\log(1+\frac{1}{t(x)})\Big)}{t(x)(1+t(x))+(t(x)+x+1)(x+1)}\right),$$

where we used equation (3). For $0 < u \leqslant x$, and $x \geqslant 1$, let's define

$$k(u) = -1+\frac{(1+u)(1+u+x(1+u+x)\log(1+\frac{1}{u}))}{u(1+u)+(u+x+1)(x+1)}. \tag{7}$$

On the one hand
$$\alpha'(x) = \partial_1 \tau(x,t(x)) = \frac{t(x)}{x^2} k(t(x)).$$
So, it is enough to show that $k(t(x)) \geqslant 0$.

Differentiate the function k, then
$$k'(u) = \frac{x\Big(u^2 - (1+x)^3 - ux(3+2x) + ux(1+x)(2+2u+x)\log\big(1+\frac{1}{u}\big)\Big)}{u\big(u^2+(1+x)^2+u(2+x)\big)^2}.$$
For $u > 0$, using the inequality $\log(1+\frac{1}{u}) \leqslant \frac{1}{u}$, we get
$$k'(u) \leqslant \frac{(1+u)(u-1-x)x}{u\,(u^2+(1+x)^2+u(2+x))^2} < 0.$$
Thus, $k(u)$ decreases, since $t(x) < x$. Hence, for every $x \geqslant 1$
$$k(t(x)) \geqslant k(x) = -1 + \frac{(1+x)(1+x+x(1+x+x)\log(1+\frac{1}{x}))}{x(1+x)+(x+x+1)(x+1)}.$$
Which is equivalent to
$$k(t(x)) \geqslant -1 + \frac{1+x+x(1+2x)\log(1+\frac{1}{x})}{3x+1}. \tag{8}$$
Set
$$\Theta(x) = -2x + x(1+2x)\log(1+\frac{1}{x}).$$
A straightforward computations give
$$\Theta'(x) = -4 + \frac{1}{1+x} + (1+4x)\log\left(1+\frac{1}{x}\right),$$
$$\Theta''(x) = \frac{-1-2x(3+2x)}{x(1+x)^2} + 4\log\left(1+\frac{1}{x}\right),$$
and
$$\Theta'''(x) = \frac{1-x}{x^2(1+x)^3}.$$
For $x \geqslant 1$, $\Theta'''(x) < 0$, and $\Theta''(x) \geqslant \lim_{x\to+\infty}\Theta''(x) = 0$. Hence $\Theta'(x) \leqslant \lim_{x\to+\infty}\Theta'(x) = 0$. Then, Θ decreases along $[1,+\infty)$, moreover, for large x, $\Theta(x) = 1/(6x) + o(1/x)$, therefore $\Theta(x) \geqslant 0$. By equation (8), one deduces that $k(t(x))$ is positive for every $x \geqslant 1$, and then
$$\alpha'(x) = \partial_1 \tau(x,t(x)) \geqslant 0.$$
Step 2. We saw by Proposition 1 that, $\alpha(x) = \psi(t(x))$, where
$$\psi(u) = \frac{(x+1)u}{x^2+(u+1)^2+x(u+2)}.$$

Differentiate with respect to u, yields

$$\psi'(u) = \frac{(1+x)((1+x)^2-u^2)}{\left(x^2+(1+u)^2+x(2+u)\right)^2},$$

Hence, the function ψ increases on $[0,1+x]$ for every $x \geqslant 1$.
Since, by Lemma 1, $t(x) < x$ and $\psi(x) = x/(3x+1)$. One deduces that, for every $x \geqslant 1$

$$0 \leqslant \alpha(x) \leqslant \frac{x}{3x+1}.$$

□

3. Proof of Theorem 1

In the sequel we use the following notations: for $n \in \mathbb{N}$, $t_n := t(n)$, the unique maximum of the function $t \mapsto \tau(n,t)$, and $\alpha_n := \tau(n,t_n)$ as stated early in Proposition 2.1.

CLAIM 1. *The sequence $(t_n)_n$ increases, and an accumulation point ℓ of the sequence $(t_n/n)_n$ is the unique solution of the equation*

$$e^{\frac{1}{\ell}} - \frac{1}{\ell^2} - \frac{1}{\ell} - 1 = 0.$$

Moreover, numerical computations give $\ell \simeq 0.5577\ldots$.

Proof. Let t_{n+1} denotes the unique maximum of the function $t \mapsto \tau(n+1,t)$. Straightforward computation gives

$$\partial_2\tau(n,t_{n+1}) = \frac{1}{n}\left(1-\left(1+\frac{(t_{n+1}+n+1)(n+1)}{t_{n+1}(1+t_{n+1})}\right)\left(\frac{t_{n+1}}{1+t_{n+1}}\right)^{n+1}\right).$$

Furthermore, using the following equation $\partial_2\tau(n+1,t_{n+1}) = 0$, we get

$$\left(\frac{t_{n+1}}{1+t_{n+1}}\right)^{n+2} = \frac{t_{n+1}(1+t_{n+1})}{t_{n+1}(1+t_{n+1})+(t_{n+1}+n+2)(n+2)},$$

which implies that

$$\partial_2\tau(n,t_{n+1}) = \frac{1}{n}\left(1-\frac{t_{n+1}(t_{n+1}+1)+(t_{n+1}+n+1)(n+1)}{t_{n+1}(t_{n+1}+1)+(t_{n+1}+n+2)(n+2)}\frac{t_{n+1}+1}{t_{n+1}}\right).$$

Which is equivalent to,

$$\partial_2\tau(n,t_{n+1}) = \frac{(1+n)(t_{n+1}-n-1)}{nt_{n+1}\left((2+n)^2+(3+n)t_{n+1}+t_{n+1}^2\right)} < 0,$$

in view of $t_{n+1} \leqslant n+1$.
Using the fact that $t \mapsto \partial_2 \tau(n,t)$ decreases on $(0,n+1)$, $t_n, t_{n+1} \in (0,n+1)$ and $\partial_2 \tau(n,t_n) = 0$, we get $t_n < t_{n+1}$
By Lemma 1, we saw that $(n+1)^2/n(2n+3) \leqslant t_n/n \leqslant 1$, and the left hand side is bounded. Then t_n/n is bounded too. Up to subtract a subsequence, one may assume that t_n/n converges to ℓ. Then, by equation (3) with $x = n$, one deduces that, as $n \longrightarrow +\infty$

$$\frac{e^{-1/\ell}(-1-\ell+(-1+e^{\frac{1}{\ell}})\ell^2)}{\ell^2} = 0,$$

which is equivalent, by setting $a = \frac{1}{\ell}$, with $a \geqslant 1$, to

$$e^a = a^2 + a + 1. \tag{9}$$

Let $\eta(a) = e^a - a^2 - a - 1$, then $\eta'(a) = e^a - 2a - 1$, and $\eta''(a) = e^a - 2 > 0$ for $a \geqslant 1$. Hence, η' increases on $(1,+\infty)$. Since, $\eta'(1) = e - 3 < 0$, and $\eta'(2) = e^2 - 5 > 0$. Whence, there is a unique $a_0 \in]1,2[$ such that $\eta'(a_0) = 0$, ($a_0 \sim 1.26$). Observe that $\eta(1) = e - 3 < 0$, and $\eta(a_0) = 2a_0 + 1 - a_0^2 - a_0 - 1 = a_0(1-a_0) < 0$. Moreover, on the interval $[1,a_0]$ the function η decreases and is strictly negative, and increases on $(a_0,+\infty)$ with $\lim_{a\to+\infty)} \eta(a) = +\infty$. Thus, equation (9) admits a unique solution x_0 in $(a_0,+\infty)$. $\square$

CLAIM 2. *The sequence $(\alpha_n)_n$ is bounded. Moreover, α_n converges to α, where*

$$\alpha = \frac{\ell}{1+\ell+\ell^2} \quad \textit{and} \quad \sup_{(n,t)\in\mathbb{N}\times(0,+\infty)} \tau(n,t) = \alpha.$$

Numerically $\alpha \simeq 0.298438....$

Proof. In the proof of Proposition 1, with $x = n$, one has

$$\alpha_n = \frac{(1+n)t_n}{n^2+(1+t_n)^2+n(2+t_n)},$$

Now, using Proposition 2, since the sequence α_n increases and is bounded by $1/3$, hence it converges to some $\alpha \geqslant 0$. As n goes to $+\infty$, and up to subtracting a subsequence we saw that $t_n/n \to \ell$, hence

$$\alpha := \lim_{n\to\infty} \alpha_n = \frac{\ell}{1+\ell+\ell^2}.$$

Moreover, for every $n \geqslant 1$, the function $t \mapsto \tau(n,t)$ attains its maximum at t_n, and $\max_{t>0} \tau(n,t) = \tau(n,t_n) = \alpha_n$. Therefore,

$$\tau(n,t) \leqslant \alpha_n \leqslant \alpha.$$

Hence, $\sup_{(n,t)\in\mathbb{N}\times(0,+\infty)} \tau(n,t)$ is well defined and

$$\sup_{(n,t)\in\mathbb{N}\times(0,+\infty)} \tau(n,t) \leqslant \alpha.$$

Moreover,

$$\alpha_n = \tau(n,t_n) \leqslant \sup_{(n,t)\in\mathbb{N}\times(0,+\infty)} \tau(n,t).$$

One deduces that

$$\sup_{(n,t)\in\mathbb{N}\times(0,+\infty)} \tau(n,t) = \sup_{n\geqslant 1} \alpha_n = \alpha.$$

This completes the proof. □

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