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NOTES ON THE COMPLETE ELLIPTIC INTEGRAL OF THE FIRST KIND

ZHEN-HANG YANG, WEI-MAO QIAN, WEN ZHANG AND YU-MING CHU*

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Abstract. In the article, we present several monotonicity properties and bounds for the complete elliptic integral of the first kind. As applications, we find sharp bounds for the arithmetic-geometric mean.

1. Introduction

The complete elliptic integral $\mathscr{K}(r)$ [22, 24, 50, 63, 64, 71, 72, 83, 84, 96] $(0 < r < 1)$ of the first kind is defined by

$$\mathscr{K}(r) = \int_0^{\pi/2} \frac{dt}{\sqrt{1-r^2\sin^2(t)}}, \quad \mathscr{K}(0^+) = \frac{\pi}{2}, \quad \mathscr{K}(1^-) = \infty.$$

It is well known that $\mathscr{K}(r)$ is the particular case of the Gaussian hypergeometric function [34, 36, 39, 49, 53, 59, 60, 61, 62, 65, 69, 74, 81, 95]

$$F(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)} \frac{x^n}{n!} \quad (-1 < x < 1), \tag{1.1}$$

where $(a,0) = 1$ for $a \neq 0$, $(a,n) = \Gamma(a+n)/\Gamma(a)$ and $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ $(x > 0)$ is the classical gamma function [27, 85, 86, 90, 93, 94, 98]. Indeed, we have the expression

$$\mathscr{K}(r) = \frac{\pi}{2}F\left(\frac{1}{2},\frac{1}{2};1;r^2\right) = \frac{\pi}{2}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2},n\right)^2}{(n!)^2}r^{2n}. \tag{1.2}$$

The Gaussian identity [20]

$$AG(a,b) = \frac{\pi a}{2\mathscr{K}\left(\sqrt{1-(\frac{b}{a})^2}\right)} \tag{1.3}$$

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* Corresponding author.

shows that the arithmetic-geometric mean $AG(a,b)$ [18, 19, 33, 38, 70, 77] of two positive real numbers a and b with $a > b$ can be expressed by the complete elliptic integral $\mathscr{K}(r)$ of the first kind, where the arithmetic-geometric mean $AG(a,b)$ is defined as the common limit of the sequences $\{a_n\}$ and $\{b_n\}$ as follows:

$$a_0 = a, \quad b_0 = b, \quad a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \sqrt{a_n b_n}.$$

Recently, the complete elliptic integral $\mathscr{K}(r)$ and Gaussian hypergeometric function $F(a,b;c;x)$ have attracted the attention of many researchers [2, 3, 4, 6, 7, 8, 25, 41, 42, 43, 44, 45, 48, 55, 68, 73, 75, 92]. In particular, many remarkable inequalities, properties and applications for $\mathscr{K}(r)$ and $F(a,b;c;x)$ can be found in the literature [5, 9, 13, 14, 16, 17, 21, 28, 29, 30, 31, 32, 37, 40, 47, 57, 66, 67, 82, 87, 91, 97, 99].

Carlson and Gustafson [15] proved that the double inequality

$$\log\frac{4}{r'} < \mathscr{K}(r) < \frac{4}{3+r^2}\log\frac{4}{r'} \tag{1.4}$$

holds for all $r \in (0,1)$. Here and in what follows we denote $r' = \sqrt{1-r^2}$.

The lower bound given in (1.4) was improved by Kühnau [46] as follows

$$\mathscr{K}(r) > \frac{9}{8+r^2}\log\frac{4}{r'}$$

for all $r \in (0,1)$.

Anderson, Vamanamurthy and Vourinen [11, Conjecture 3.1(1)] conjectured that the inequality

$$\mathscr{K}(r) < \log\left(1+\frac{4}{r'}\right) - \left(\log 5 - \frac{\pi}{2}\right)(1-r) \tag{1.5}$$

is valid for all $r \in (0,1)$.

Inequality (1.5) was proved by Qiu, Vamanamurthy and Vuorinen in [54, Theorem 1.7(2)]. Besides, they also provided the inequalities

$$\frac{\pi}{2} - \log 4 + \log\left(4-\pi+\frac{4}{r'}\right) < \mathscr{K}(r) < \log\left(\frac{16}{\pi}-4+\frac{4}{r'}\right), \tag{1.6}$$

$$\log\left[\left(e^{\pi/2}-4\right)r' + \frac{4}{r'}\right] < \mathscr{K}(r) < \log\left(e^{\pi/2}-4+\frac{4}{r'}\right) \tag{1.7}$$

for all all $r \in (0,1)$ (see [54, Theorem 1.6(1), Corollary 3.5(1)]).

Alzer and Qiu [10], and Yang, Song and Chu [88] independently established the inequality

$$\mathscr{K}(r) > \frac{\pi}{2}\left[\frac{\tanh^{-1}(r)}{r}\right]^{3/4}$$

for all $r \in (0,1)$.

The aim of the article is to provide the monotonicity properties and new bounds for the complete elliptic integral $\mathscr{K}(r)$.

2. Lemmas

In order to prove our main results we need several formulas and lemmas, which we present in this section.

Let $a,b\in\mathbb{R}$ with $a<b$, and $f,g:(a,b)\to\mathbb{R}$ be differentiable with $g'\neq 0$ on (a,b). Then the function $H_{f,g}$ is defined by

$$H_{f,g}=\frac{f'}{g'}g-f.$$

The hypergeometric function $F(a,b,c;x)$ has the following formulas (see [12, (1.16), 1.19(4), 1.20(10), 1.48] and [1, 15.3.10, 15.3.11]):

$$\frac{d^n}{dx^n}F(a,b,c;x)=\frac{(a,n)(b,n)}{(c,n)}F(a+n,b+n;c+n;x), \tag{2.1}$$

$$F(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\quad (c>a+b), \tag{2.2}$$

$$F(a,b;a+b+1;x)=(1-x)F(a+1,b+1;a+b+1;x), \tag{2.3}$$

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}F(a,b;a+b;x)+\log(1-x)+\psi(a)+\psi(b)+2\gamma=O\left((1-x)\log(1-x)\right) \tag{2.4}$$

as $x\to 1^-$, where

$$\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}$$

is the psi function and

$$\gamma=\lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n\right)=0.57721566\cdots$$

is the Euler-Mascheroni constant [26, 35, 56, 58, 76, 79].

$$F(a,b;a+b;x)=\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(n!)^2} \tag{2.5}$$

$$\times\left[2\psi(n+1)-\psi(a+n)-\psi(b+n)-\log(1-x)\right](1-x)^n,$$

$$F(a,b;a+b+m;x)=\frac{\Gamma(m)\Gamma(a+b+m)}{\Gamma(a+m)\Gamma(b+m)}\sum_{n=0}^{m-1}\frac{(a,n)(b,n)}{n!(1-m,n)}(1-x)^n \tag{2.6}$$

$$-\frac{\Gamma(a+b+m)}{\Gamma(a)\Gamma(b)}(x-1)^m\sum_{n=0}^{\infty}\frac{(a+m,n)(b+m,n)}{n!(n+m)!}(1-x)^n$$

$$\times\left[\log(1-x)-\psi(n+1)-\psi(n+m+1)+\psi(a+n+m)+\psi(b+n+m)\right]$$

for $m=1,2,3\cdots$.

LEMMA 2.1. *(see [12, Theorem 1.25], [23, Lemma 2.1], [51, Lemma 2.1], [78, Lemma 2.1]) Let* $a,b \in \mathbb{R}$ *with* $a<b$, $f,g:[a,b] \to \mathbb{R}$ *be continuous on* $[a,b]$ *and differentiable on* (a,b). *If* $g'(x) \neq 0$ *on* (a,b) *and* $f'(x)/g'(x)$ *is increasing (decreasing) on* (a,b), *then so are the functions*

$$\frac{f(x)-f(a)}{g(x)-g(a)}, \quad \frac{f(x)-f(b)}{g(x)-g(b)}.$$

If $f'(x)/g'(x)$ *is strictly monotone, then the monotonicity in the conclusion is also strict.*

LEMMA 2.2. *(see [52, Lemma 2.4], [80, Theorem 1.1]) Let* $A(t)=\sum_{k=0}^{\infty} a_k t^k$ *and* $B(t)=\sum_{k=0}^{\infty} b_k t^k$ *be two real power series converging on* $(-r,r)$ $(r>0)$ *with* $b_k>0$ *for all* k. *If the non-constant sequence* $\{a_k/b_k\}$ *is increasing (decreasing) for all* k, *then the function* $t \mapsto A(t)/B(t)$ *is strictly increasing (decreasing) on* $(0,r)$.

LEMMA 2.3. *(see [80, Theorem 2.1]) Let* $A(t)=\sum_{k=0}^{\infty} a_k t^k$ *and* $B(t)=\sum_{k=0}^{\infty} b_k t^k$ *be two real power series converging on* $(-r,r)$ *and* $b_k>0$ *for all* k. *Suppose that for certain* $m \in \mathbb{N}$, *the non-constant sequence* $\{a_k/b_k\}$ *is increasing (decreasing) for* $0 \leqslant k \leqslant m$ *and decreasing (increasing) for* $k \geqslant m$. *Then the function* A/B *is strictly increasing (decreasing) on* $(0,r)$ *if and only if* $H_{A,B}(r^-) \geqslant (\leqslant) 0$. *Moreover, if* $H_{A,B}(r^-) < (>) 0$, *then there exists* $t_0 \in (0,r)$ *such that the function* A/B *is strictly increasing (decreasing) on* $(0,t_0)$ *and strictly decreasing (increasing) on* (t_0,r).

LEMMA 2.4. *(see [89, Lemma 2.1]) Let* $-\infty \leqslant a < b \leqslant \infty$, $f,g:(a,b) \to \mathbb{R}$ *be differentiable on* (a,b) *with* $f(a^+)=g(a^+)=0$ *and* $g'(x)>0$ *on* (a,b), *and there* $\lambda_0 \in (a,b)$ *such that* $f'(x)/g'(x)$ *is strictly increasing on* (a,λ_0) *and strictly decreasing on* (λ_0,b). *Then the following statements are true:*

(1) $f(x)/g(x)$ *is strictly increasing on* (a,b) *if* $H_{f,g}(b^-) \geqslant 0$;

(2) *there exists* $\mu_0 \in (a,b)$ *such that* $f(x)/g(x)$ *is strictly increasing on* (a,μ_0) *and strictly decreasing on* (μ_0,b) *if* $H_{f,g}(b^-)<0$.

Let $a=b=1/2$ and $m=1$. Then equations (2.5) and (2.6) lead to Lemma 2.5 immediately.

LEMMA 2.5. *Let* $t=1-x$. *Then the asymptotic formulas*

$$F\left(\frac{1}{2},\frac{1}{2},;1;x\right)=\frac{\log\left(\frac{16}{t}\right)}{\pi}+\frac{t}{4\pi}\left[\log\left(\frac{16}{t}\right)-2\right]+O\left(t^2\log t\right), \tag{2.7}$$

$$F\left(\frac{1}{2},\frac{1}{2},;2;x\right)=\frac{4}{\pi}-\frac{t}{\pi}\left[\log\left(\frac{16}{t}\right)-3\right]+O\left(t^2\log t\right) \tag{2.8}$$

hold as $t \to 0^+$.

LEMMA 2.6. *Let $n \in \mathbb{N}$, $p \in (0,4)$, and a_n and b_n be defined by*

$$a_n = \frac{64n^2 + 8(p^2-8)n + 16 - p^2}{(n+1)(2n-1)^2}\left[\frac{\Gamma(n+1/2)}{\Gamma(1/2)\Gamma(n+1)}\right]^2, \tag{2.9}$$

$$b_n = \frac{4p\Gamma(n-1/2)}{\Gamma(1/2)\Gamma(n+1)} \quad (n \geqslant 1), \quad b_0 = 8(4-p). \tag{2.10}$$

Then the following statements are true:

(1) *the sequence $\{a_n/b_n\}_{n=0}^{\infty}$ is decreasing if $p \in [4/3,4)$;*

(2) *there exists $n_0 > 1$ such that the sequence $\{a_n/b_n\}_{n=0}^{\infty}$ is increasing for $n \leqslant n_0$ and decreasing for $n \geqslant n_0$ if $p \in (0,4/3)$.*

Proof. It follows from (2.9) and (2.10) that

$$\frac{a_0}{b_0} = \frac{p+4}{8}, \tag{2.11}$$

$$\frac{a_1}{b_1} - \frac{a_0}{b_0} = -\frac{(3p-4)(4-p)}{32p}, \tag{2.12}$$

$$\frac{a_2}{b_2} - \frac{a_1}{b_1} = -\frac{(3p-4)(3p+4)}{64p}, \tag{2.13}$$

$$\frac{a_3}{b_3} - \frac{a_2}{b_2} = -\frac{17\left(p - \frac{4}{\sqrt{17}}\right)\left(p + \frac{4}{\sqrt{17}}\right)}{512p}. \tag{2.14}$$

$$\frac{a_{n+1}}{b_{n+1}} - \frac{a_n}{b_n} = -\frac{\Gamma(n+1/2)\left[(24n+3)p^2 + 16(2n-1)(2n-5)\right]}{16p\Gamma(1/2)\Gamma(n+2)(n+2)(2n-1)} < 0 \tag{2.15}$$

for $n \geqslant 3$ and $p \in (0,4)$.

(1) If $p \in [4/3,4)$, then (2.12)-(2.15) lead to

$$\frac{a_0}{b_0} \geqslant \frac{a_1}{b_1} \geqslant \frac{a_2}{b_2} > \frac{a_3}{b_3} > \frac{a_4}{b_4} > \cdots > \frac{a_n}{b_n} > \frac{a_{n+1}}{b_{n+1}} > \cdots.$$

(2) If $p \in (0,4/3)$, then we divide the proof into two cases.

Case 1 $p \in (0,4/\sqrt{17})$. Then (2.12)-(2.15) lead to the conclusion that

$$\frac{a_0}{b_0} < \frac{a_1}{b_1} < \frac{a_2}{b_2} < \frac{a_3}{b_3} > \frac{a_4}{b_4} > \frac{a_5}{b_5} > \cdots > \frac{a_n}{b_n} > \frac{a_{n+1}}{b_{n+1}} > \cdots.$$

Case 2 $p \in [4/\sqrt{17},4/3)$. Then from (2.12)-(2.15) we clearly see that

$$\frac{a_0}{b_0} < \frac{a_1}{b_1} < \frac{a_2}{b_2} \geqslant \frac{a_3}{b_3} > \frac{a_4}{b_4} > \frac{a_5}{b_5} > \cdots > \frac{a_n}{b_n} > \frac{a_{n+1}}{b_{n+1}} > \cdots. \quad \square$$

LEMMA 2.7. *Let $p \in (0,4)$, $f_0(x)$ and $g_0(x)$ be defined by*

$$f_0(x) = \left(16 - p^2 + p^2 x\right) F\left(\frac{1}{2}, \frac{1}{2}; 2; x\right), \quad g_0(x) = 8(4 - p\sqrt{1-x}), \tag{2.16}$$

respectively. Then

$$H_{f_0,g_0}(1^-) = -\frac{64}{\pi}.$$

Proof. It follows from (2.1), (2.2), (2.4) and (2.16) that

$$\frac{f_0'(x)}{g_0'(x)} = \frac{8p^2\sqrt{1-x}F\left(\frac{1}{2}, \frac{1}{2}; 2; x\right) + \left(16 - p^2 + p^2 x\right)\sqrt{1-x}F\left(\frac{3}{2}, \frac{3}{2}; 3; x\right)}{32p}, \tag{2.17}$$

$$F\left(\frac{1}{2}, \frac{1}{2}; 2; 1^-\right) = \frac{\Gamma(2)\Gamma(1)}{\Gamma^2\left(\frac{3}{2}\right)} = \frac{4}{\pi}, \tag{2.18}$$

$$\lim_{x \to 1^-}\left[\sqrt{1-x}F\left(\frac{3}{2}, \frac{3}{2}; 3; x\right)\right] \tag{2.19}$$

$$= \frac{8}{\pi}\lim_{x \to 1^-}\left[4(\log 2 - 1) - \log(1-x) + O((1-x)\log(1-x))\right]\sqrt{1-x} = 0.$$

From (2.16)-(2.19) we clearly see that

$$H_{f_0,g_0}(1^-) = \lim_{x \to 1^-}\left(\frac{f_0'(x)}{g_0'(x)}g_0(x) - f_0(x)\right) = -\frac{64}{\pi}. \quad \square$$

LEMMA 2.8. *Let $p \in (0,4)$, $f(x)$ and $g(x)$ be defined by*

$$f(x) = F\left(\frac{1}{2}, \frac{1}{2}; 1; x\right) - 1, \quad g(x) = \log\left(p + \frac{4}{\sqrt{1-x}}\right) - \log(p+4), \tag{2.20}$$

respectively. Then

$$H_{f,g}(1^-) = 1 - \frac{2}{\pi}\log(p+4).$$

Proof. It follows from (2.1), (2.3) and (2.20) that

$$f'(x) = \frac{1}{4}F\left(\frac{3}{2}, \frac{3}{2}; 2; x\right) = \frac{1}{4(1-x)}F\left(\frac{1}{2}, \frac{1}{2}; 2; x\right). \tag{2.21}$$

$$g'(x) = \frac{2}{(1-x)(p\sqrt{1-x}+4)}, \tag{2.22}$$

$$H_{f,g}(x) = \frac{f'(x)}{g'(x)}g(x) - f(x) \tag{2.23}$$

$$= \frac{(16-p^2+p^2x)\,F\left(\frac{1}{2},\frac{1}{2};2;x\right)}{8(4-p\sqrt{1-x})}\log\left(\frac{p+\frac{4}{\sqrt{1-x}}}{p+4}\right) - F\left(\frac{1}{2},\frac{1}{2};1;x\right)+1.$$

Let $t=1-x$. Then Lemma 2.5 and (2.23) lead to

$$H_{f,g}(1^-) = \lim_{t\to 0^+}\left[\frac{(16-pt^2)\left(\frac{4}{\pi}-\frac{t}{\pi}(\log(16/t)-3)+O(t^2\log t)\right)}{8(4-p\sqrt{t})}\right.$$

$$\left.\times\log\left(\frac{p+4/\sqrt{t}}{p+4}\right) - \left(\frac{\log(16/t)}{\pi}+\frac{t}{4\pi}(\log(16/t)-2)+O(t^2\log t)\right)+1\right]$$

$$= 1+\lim_{t\to 0^+}\left[\frac{2}{\pi}\log\left(\frac{p+\frac{4}{\sqrt{t}}}{p+4}\right)-\frac{1}{\pi}\log\left(\frac{16}{t}\right)\right]$$

$$= 1-\frac{2}{\pi}\log(p+4). \quad \square$$

LEMMA 2.9. *Let $p\in(-4,\infty)$ and $x\in(0,1)$. Then the function*

$$p\mapsto U_p(x) = \frac{\pi}{2}+\frac{\pi(p+4)}{16}\log\left(\frac{p+\frac{4}{x}}{p+4}\right) \tag{2.24}$$

is strictly increasing on $(-4,\infty)$.

Proof. It follows from (2.24) that

$$\frac{\partial U_p(x)}{\partial p} = \frac{\pi}{16}\log\left(\frac{p+\frac{4}{x}}{p+4}\right)+\frac{\pi(x-1)}{4(px+4)}, \tag{2.25}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial U_p(x)}{\partial p}\right) = \frac{\pi(x-1)}{x(px+4)^2} < 0 \tag{2.26}$$

for $p\in(-4,\infty)$ and $x\in(0,1)$.

Equation (2.25) and inequality (2.26) lead to

$$\frac{\partial U_p(x)}{\partial p} > \left.\frac{\partial U_p(x)}{\partial p}\right|_{x=1} = 0,$$

which shows that the function $p\mapsto U_p(x)$ is strictly increasing on $(-4,\infty)$. $\square$

LEMMA 2.10. *(See* [87, Lemma 2.1]*) Let $\{a_k\}_{k=0}^{\infty}$ be a nonnegative real sequence with $a_m>0$ and $\sum_{k=m+1}^{\infty}a_k>0$, and*

$$S(t) = -\sum_{k=0}^{m}a_kt^k+\sum_{k=m+1}^{\infty}a_kt^k$$

be a convergent power series on the interval $(0,r)$ $(r>0)$. If $S(r^-)>0$, then there exists $t_0\in(0,r)$ such that $S(t)<0$ for $t\in(0,t_0)$ and $S(t)>0$ for $t\in(t_0,r)$.

LEMMA 2.11. *Let* $a = e^{\pi/2} - 4 = 0.8104\cdots$ *and* $b = \log 5 - \pi/2 = 0.0386\cdots$. *Then*

$$\log\left(a + \frac{4}{x}\right) < \log\left(1 + \frac{4}{x}\right) - bx^2 \tag{2.27}$$

for all $x \in (0,1)$.

Proof. Let

$$f_1(x) = \log\left(a + \frac{4}{x}\right) - \log\left(1 + \frac{4}{x}\right) + bx^2, \tag{2.28}$$

$$f_2(x) = \frac{(x+4)(ax+4)}{2} f_1'(x). \tag{2.29}$$

Then elaborated computations lead to

$$f_1(0^+) = f_1(1^-) = 0, \tag{2.30}$$

$$f_2(x) = -2(1-a) + 16bx + 4b(a+1)x^2 + abx^3, \tag{2.31}$$

$$f_2(1^-) = 2a + 20b + 5ab - 2 > 2a + 20b - 2 = 0.3937\cdots > 0. \tag{2.32}$$

From Lemma 2.10, (2.29), (2.31) and (2.32) we clearly see that there exists $\eta_0 \in (0,1)$ such that $f_1(x)$ is strictly decreasing on $(0,\eta_0)$ and strictly increasing on $(\eta_0,1)$.

It follows from (2.30) and the piecewise monotonicity of $f_1(x)$ on the interval $(0,1)$ that

$$f_1(x) < \max\{f_1(0^+), f_1(1^-)\} = 0 \tag{2.33}$$

for all $x \in (0,1)$.

Therefore, inequality (2.27) follows from (2.28) and (2.33). □

3. Main Results

THEOREM 3.1. *Let* $p \in (0,4)$, $r \in (0,1)$ *and* $\mathscr{F}(r)$ *be defined by*

$$\mathscr{F}(r) = \frac{\frac{2}{\pi}\mathscr{K}(r) - 1}{\log\left(p + \frac{4}{r'}\right) - \log(p+4)}. \tag{3.1}$$

Then the following statements are true:

(1) *If* $4/3 \leqslant p < 4$, *then* $\mathscr{F}(r)$ *is strictly decreasing on* $(0,1)$, *and the double inequality*

$$\frac{\pi}{2} + \log\left(\frac{p + \frac{4}{r'}}{p+4}\right) < \mathscr{K}(r) < \frac{\pi}{2} + \frac{\pi(p+4)}{16}\log\left(\frac{p + \frac{4}{r'}}{p+4}\right) \tag{3.2}$$

holds for all $r \in (0,1)$.

(2) *If* $0 < p \leqslant e^{\pi/2} - 4 = 0.8104\cdots$, *then* $\mathscr{F}(r)$ *is strictly increasing on* $(0,1)$, *and the two-sided inequality*

$$\frac{\pi}{2} + \frac{\pi(p+4)}{16}\log\left(\frac{p+\frac{4}{r'}}{p+4}\right) < \mathscr{K}(r) < \frac{\pi}{2} + \log\left(\frac{p+\frac{4}{r'}}{p+4}\right) \tag{3.3}$$

is valid for all $r \in (0,1)$.

(3) *If* $e^{\pi/2} - 4 < p < 4/3$, *then there exists* $r_0 \in (0,1)$ *such that* $\mathscr{F}(r)$ *is strictly increasing on* $(0, r_0]$ *and strictly decreasing on* $[r_0, 1)$, *and the inequality*

$$\mathscr{K}(r) > \frac{\pi}{2} + \min\left\{\frac{\pi(p+4)}{16}, 1\right\}\log\left(\frac{p+\frac{4}{r'}}{p+4}\right) \tag{3.4}$$

takes place for all $r \in (0,1)$. *Moreover, one has*

$$\mathscr{K}(r) > \frac{\pi}{2} + \frac{\pi(p+4)}{16}\log\left(\frac{p+\frac{4}{r'}}{p+4}\right) \tag{3.5}$$

for $r \in (0,1)$ *if* $e^{\pi/2} - 4 < p \leqslant 16/\pi - 4 = 1.0929\cdots$, *and*

$$\mathscr{K}(r) > \frac{\pi}{2} + \log\left(\frac{p+\frac{4}{r'}}{p+4}\right) \tag{3.6}$$

for $r \in (0,1)$ *if* $16\pi - 4 \leqslant p < 4/3$.

Proof. Let $x = r^2 \in (0,1)$, $f_0(x)$, $g_0(x)$, $f(x)$ and $g(x)$ be defined by (2.16) and (2.10), respectively. Then from (1.1), (1.2), (2.16), (2.20)-(2.22) and (3.1) we clearly see that

$$f(0^+) = g(0^+) = 0, \tag{3.7}$$

$$\mathscr{F}(r) = \frac{f(x)}{g(x)}, \tag{3.8}$$

$$\frac{f'(x)}{g'(x)} = \frac{(16 - p^2 + p^2 x)F\left(\frac{1}{2}, \frac{1}{2}; 2; x\right)}{8(4 - p\sqrt{1-x})} = \frac{f_0(x)}{g_0(x)}. \tag{3.9}$$

It follows from (1.1) and (3.9) that

$$\frac{f'(x)}{g'(x)} = \frac{(16 - p^2 + p^2 x)\sum_{n=0}^{\infty} \frac{\left(\frac{\Gamma(n+1/2)}{\Gamma(1/2)\Gamma(n+1)}\right)^2}{n+1} x^n}{8(4-p) + 4p\sum_{n=1}^{\infty} \frac{\Gamma(n-1/2)}{\Gamma(1/2)\Gamma(n+1)} x^n} = \frac{\sum_{n=0}^{\infty} a_n x^n}{\sum_{n=0}^{\infty} b_n x^n}, \tag{3.10}$$

where a_n and b_n are defined by (2.9) and (2.10), respectively.

From (2.2), (2.11), (2.21), (2.22), (3.7), (3.8) and (3.10) we get

$$\mathscr{F}(0^+) = \frac{a_0}{b_0} = \frac{p+4}{8}, \tag{3.11}$$

$$\mathscr{F}(1^-) = \lim_{x\to 1^-}\frac{f'(x)}{g'(x)} = \lim_{x\to 1^-}\frac{\frac{1}{4(1-x)}F(1/2,1/2;2;x)}{\frac{2}{(1-x)(p\sqrt{1-x}+4)}}$$
$$= \frac{1}{2}\lim_{x\to 1^-} F\left(\frac{1}{2},\frac{1}{2};2;x\right) = \frac{1}{2}\frac{\Gamma(2)\Gamma(1)}{\Gamma^2(3/2)} = \frac{2}{\pi}. \tag{3.12}$$

(1) If $4/3 \leqslant p < 4$, then Lemma 2.6(1) and (3.10) lead to the conclusion that the function $f'(x)/g'(x)$ is strictly decreasing on the interval $(0,1)$. Therefore, $\mathscr{F}(r)$ is strictly decreasing on $(0,1)$ follows from Lemma 2.1, (3.7) and (3.8) together with the monotonicity of the function $f'(x)/g'(x)$ on the interval $(0,1)$, and inequality (3.2) follows easily from (3.1), (3.11) and (3.12) together with the monotonicity of $\mathscr{F}(r)$.

Next, we suppose that $0 < p < 4/3$, then from Lemma 2.6(2) and Lemma 2.7 together with (3.9) we know that there exists $n_0 > 0$ such that the non-constant sequence $\{a_n/b_n\}_{n=0}^{\infty}$ is increasing for $n \leqslant n_0$ and decreasing for $n \geqslant n_0$, and

$$H_{f',g'}(1^-) = H_{f_0,g_0}(1^-) = -\frac{64}{\pi} < 0. \tag{3.13}$$

It follows from Lemma 2.3, (3.10), (3.13) and the piecewise monotonicity of the sequence $\{a_n/b_n\}_{n=0}^{\infty}$ that there exists $\lambda_0 \in (0,1)$ such that the function $f'(x)/g'(x)$ is strictly increasing on $(0,\lambda_0)$ and strictly decreasing on $(\lambda_0,1)$.

We divide the proof into (2) and (3) two cases as follows.

(2) If $0 < p \leqslant e^{\pi/2} - 4$, then Lemma 2.8 leads to

$$H_{f,g}(1^-) = 1 - \frac{2}{\pi}\log(p+4) \geqslant 0. \tag{3.14}$$

Therefore, $\mathscr{F}(r)$ is strictly increasing on $(0,1)$ follows from Lemma 2.4(1), (2.22), (3.7), (3.8), (3.14) and the piecewise monotonicity of the function $f'(x)/g'(x)$ on the interval $(0,1)$, and inequality (3.3) follows easily from (3.1), (3.11) and (3.12) together with the monotonicity of $\mathscr{F}(r)$ on the interval $(0,1)$.

(3) If $e^{\pi/2} - 4 < p < 4/3$, then from Lemma 2.8 one has

$$H_{f,g}(1^-) = 1 - \frac{2}{\pi}\log(p+4) < 0. \tag{3.15}$$

Therefore, there exists $\mu_0 \in (0,1)$ such that $\mathscr{F}(r)$ is strictly increasing on $(0,\mu_0)$ and strictly decreasing on $(\mu_0,1)$ follows from Lemma 2.4(2), (2.22), (3.7), (3.8), (3.15) and the piecewise monotonicity of the function $f'(x)/g'(x)$ on the interval $(0,1)$, inequality (3.4) follows from (3.1), (3.11), (3.12) and the piecewise monotonicity of $\mathscr{F}(r)$ on the interval $(0,1)$, and inequalities (3.5) and (3.6) can be derived from inequality (3.4) immediately. □

THEOREM 3.2. *Let $p,q \in (-4,\infty)$. Then the double inequality*

$$\frac{\pi}{2} + \frac{\pi(p+4)}{16}\log\left(\frac{p+\frac{4}{r'}}{p+4}\right) < \mathscr{K}(r) < \frac{\pi}{2} + \frac{\pi(q+4)}{16}\log\left(\frac{q+\frac{4}{r'}}{q+4}\right) \tag{3.16}$$

holds for all $r \in (0,1)$ if and only if $p \leqslant 16/\pi - 4$ and $q \geqslant 4/3$.

Proof. If $p \leqslant 16/\pi - 4$ and $q \geqslant 4/3$, then inequality (3.16) follows from the second inequality of (3.2) and inequality (3.5) together with Lemma 2.9.

If the first inequality of (3.16) holds for all $r \in (0,1)$, then from the limit formula

$$\lim_{r\to 1^-}\left(\mathscr{K}(r) - \log\left(\frac{4}{r'}\right)\right) = 0 \tag{3.17}$$

given in [12, (3.25)] we get

$$\lim_{r\to 1^-}\frac{\frac{\pi}{2} + \frac{\pi(p+4)}{16}\log\left(\frac{p+\frac{4}{r'}}{p+4}\right)}{\mathscr{K}(r)} = \frac{\pi(p+4)}{16} \leqslant 1,$$

which leads to $p \leqslant 16/\pi - 4$.

If the second inequality of (3.16) holds for all $r \in (0,1)$, then

$$\lim_{r\to 0+}\frac{\mathscr{K}(r) - \frac{\pi}{2} + \frac{\pi(q+4)}{16}\log\left(\frac{q+\frac{4}{r'}}{q+4}\right)}{r^4} = \frac{\pi(3q-4)}{128(q+4)} \geqslant 0,$$

which implies that $q \geqslant 4/3$. □

Let $p = 16/\pi - 4, 1, 0^+$ and $q = 4/3, 2, 4$. Then Lemma 2.9 and Theorem 3.2 lead to Corollary 3.3 immediately.

COROLLARY 3.3. *The following inequalities*

$$\frac{\pi}{2} + \frac{\pi}{4}\log\left(\frac{1}{r'}\right) < \frac{\pi}{2} - \frac{5\pi}{16}\log 5 + \frac{5\pi}{16}\log\left(1 + \frac{4}{r'}\right)$$
$$< \frac{\pi}{2} - \log 4 + \log\left(4 - \pi + \frac{\pi}{r'}\right) < \mathscr{K}(r) < \frac{\pi}{2} + \frac{\pi}{3}\log\left(\frac{1}{4} + \frac{3}{4r'}\right)$$
$$< \frac{\pi}{2} + \frac{3\pi}{8}\log\left(\frac{1}{3} + \frac{2}{3r'}\right) < \frac{\pi}{2} + \frac{\pi}{2}\log\left(\frac{1}{2} + \frac{1}{2r'}\right)$$

hold for all $r \in (0,1)$.

REMARK 3.4. The third inequality in Corollary 3.3 gives the same lower bound for $\mathscr{K}(r)$ in (1.6).

THEOREM 3.5. *Let* $p, q \in (-4, \infty)$. *Then the double inequality*

$$\frac{\pi}{2} + \log\left(\frac{p+\frac{4}{r'}}{p+4}\right) < \mathscr{K}(r) < \frac{\pi}{2} + \log\left(\frac{q+\frac{4}{r'}}{q+4}\right) \tag{3.18}$$

holds for all $r \in (0,1)$ *if and only if* $p \geqslant 16/\pi - 4$ *and* $q \leqslant e^{\pi/2} - 4$.

Proof. We clearly see that the function $p \to \log((p+4/r')/(p+4))$ is strictly decreasing on $(-4,\infty)$.

If $p \geqslant 16/\pi - 4$ and $q \leqslant e^{\pi/2} - 4$, then inequality (3.18) follows from the second inequality of (3.3) and inequality (3.6) together with the monotonicity of the function $p \to \log((p+4/r')/(p+4))$ on the interval $(-4,\infty)$.

If the first inequality of (3.18) holds for all $r \in (0,1)$, then we have

$$\lim_{r\to 0^+} \frac{\mathscr{K}(r) - \frac{\pi}{2} + \log\left(\frac{p+\frac{4}{r'}}{p+4}\right)}{r^2} = \frac{\pi p - 16 + 4\pi}{8(p+4)} \geqslant 0,$$

that is $p \geqslant 16/\pi - 4$.

If the second inequality of (3.18) holds for all $r \in (0,1)$, then it follows from (3.17) that

$$\begin{aligned}&\lim_{r\to 1^-}\left(\mathscr{K}(r) - \frac{\pi}{2} + \log\left(\frac{q+\frac{4}{r'}}{q+4}\right)\right)\\ &= \lim_{r\to 1^-}\left(\log\left(\frac{4}{r'}\right) - \frac{\pi}{2} + \log\left(\frac{q+\frac{4}{r'}}{q+4}\right)\right)\\ &= \log(q+4) - \frac{\pi}{2} \leqslant 0,\end{aligned}$$

which leads to $q \leqslant e^{\pi/2} - 4$. □

Let $p = 16/\pi - 4, 4/3$ and $q = e^{\pi/2} - 4, 0^+$, then Theorem 3.5 leads to Corollary 3.6 immediately.

COROLLARY 3.6. *The following inequalities*

$$\begin{aligned}\frac{\pi}{2} + \log\left(\frac{1}{4} + \frac{3}{4r'}\right) &< \frac{\pi}{2} - \log 4 + \log\left(4 - \pi + \frac{\pi}{r'}\right)\\ &< \mathscr{K}(r) < \log\left(e^{\pi/2} - 4 + \frac{4}{r'}\right) < \frac{\pi}{2} + \log\left(\frac{1}{r'}\right)\end{aligned}$$

hold for all $r \in (0,1)$.

REMARK 3.7. The third inequality

$$\mathscr{K}(r) < \log\left(e^{\pi/2} - 4 + \frac{4}{r'}\right) \tag{3.19}$$

in Corollary 3.6 gives the same upper bound for $\mathscr{K}(r)$ in (1.7). In particular, inequality (3.19) is an improvement of inequality (1.5), indeed from Lemma 2.11 one has

$$\log\left(e^{\pi/2} - 4 + \frac{4}{r'}\right) < \log\left(1 + \frac{4}{r'}\right) - \left(\log 5 - \frac{\pi}{2}\right)(1 - r^2)$$

$$< \log\left(1+\frac{4}{r'}\right) - \left(\log 5 - \frac{\pi}{2}\right)(1-r).$$

From (1.3) and Theorems 3.2 and 3.5 we get Corollary 3.8 immediately.

COROLLARY 3.8. *Let $p,q,\lambda,\mu \in (-4,\infty)$. Then the double inequalities*

$$\frac{1}{1+\frac{p+4}{8}\log\left(\frac{p+\frac{4}{r}}{p+4}\right)} < AG(1,r) < \frac{1}{1+\frac{q+4}{8}\log\left(\frac{q+\frac{4}{r}}{q+4}\right)},$$

$$\frac{1}{1+\frac{2}{\pi}\log\left(\frac{\lambda+\frac{4}{r}}{\lambda+4}\right)} < AG(1,r) < \frac{1}{1+\frac{2}{\pi}\log\left(\frac{\mu+\frac{4}{r}}{\mu+4}\right)}$$

hold for all $r \in (0,1)$ if and only if $p \geqslant 4/3$, $q \leqslant 16/\pi - 4$, $\lambda \leqslant e^{\pi/2} - 4$ and $\mu \geqslant 16/\pi - 4$.

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Zhen-Hang Yang
Department of Mathematics
Huzhou University
Huzhou 313000, P. R. China
e-mail: yzhkm@163.com

Wei-Mao Qian
School of Continuing Education
Huzhou Broadcast and TV University
Huzhou 313000, P. R. China
e-mail: qwm661977@126.com

Wen Zhang
Friedman Brain Institute
Icahn School of Medicine at Mount Sinai
New York NY 10029, USA
e-mail: zhang.wen81@gmail.com

Yu-Ming Chu
College of Science
Hunan City University
Yiyang 413000, P. R. China
School of Mathematics and Statistics
Changsha University of Science & Technology
Changsha 410114, P. R. China
e-mail: chuyuming2005@126.com