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ON THE VARIATION OF THE DISCRETE MAXIMAL OPERATOR

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Abstract. In this note we study the endpoint regularity properties of the discrete nontangential fractional maximal operator

$$M_{\alpha,\beta}f(n) = \sup_{\substack{r\in\mathbb{N}\\ |m-n|\leqslant\beta r}} \frac{1}{(2r+1)^{1-\alpha}} \sum_{k=-r}^{r} |f(m+k)|,$$

where $\alpha \in [0,1)$, $\beta \in [0,\infty)$ and $\mathbb{N} = \{0,1,2,\dots,\}$, covering the discrete centered Hardy-Littlewood maximal operator and its fractional variant. More precisely, we establish the sharp boundedness and continuity for $M_{\alpha,\beta}$ from $\ell^1(\mathbb{Z})$ to $\mathrm{BV}(\mathbb{Z})$. When $\alpha=0$, we prove that the operator $M_{\alpha,\beta}$ is bounded and continuous on $\mathrm{BV}(\mathbb{Z})$. Here $\mathrm{BV}(\mathbb{Z})$ denotes the set of functions of bounded variation defined on $\mathbb{Z}$. Our main results represent generalizations as well as natural extensions of many previously known ones.

1. Introduction

The regularity theory of maximal operators has been the subject of many recent articles in harmonic analysis. The first work was contributed by Kinnunen [11], who studied the Sobolev regularity of the usual centered Hardy-Littlewood maximal function $\mathscr{M}$ and showed that $\mathscr{M}$ is bounded on the first order Sobolev spaces $W^{1,p}(\mathbb{R}^d)$ for all $1<p\leqslant\infty$. It was noticed that the $W^{1,p}$-bound for the uncentered maximal operator $\widetilde{\mathscr{M}}$ also holds by a simple modification of Kinnunen's arguments or [10, Theorem 1]. Later on, Kinnunen's results were extended to a local version in [12], to a fractional version in [13], to a multisublinear version in [7, 20] and to a one-sided version in [19]. Since $\mathscr{M}$ lacks the sublinear at the derivative level, the continuity of $\mathscr{M}: W^{1,p}(\mathbb{R}^d)\to W^{1,p}(\mathbb{R}^d)$ for $1<p<\infty$ is certainly a nontrivial issue. This problem was solved by Luiro [25] in the affirmative and was later extended to the local version in [26] and the multisublinear version in [7, 16].

Due to lack of L^1-boundedness, the $W^{1,1}$-regularity for the maximal operator is delicate. A crucial question was posed by Hajłasz and Onninen in [10]: Is the operator $f\mapsto|\nabla\mathscr{M}f|$ bounded from $W^{1,1}(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$? A complete solution was obtained only in dimension $d=1$ and partial progress on the general dimension $d\geqslant 2$ was given

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by Hajłasz and Malý [9] and Luiro [27]. Tanaka [31] first observed that if $f \in W^{1,1}(\mathbb{R})$, then the maximal function $\widetilde{\mathscr{M}} f$ is weakly differentiable and

$$\|(\widetilde{\mathscr{M}} f)'\|_{L^1(\mathbb{R})} \leqslant 2\|f'\|_{L^1(\mathbb{R})}. \tag{1}$$

This result was later sharpened by Aldaz and Pérez Lázaro [1] who proved that if f is of bounded variation on $\mathbb{R}$, then $\widetilde{\mathscr{M}} f$ is absolutely continuous and

$$\mathrm{Var}\big(\widetilde{\mathscr{M}} f\big) \leqslant \mathrm{Var}(f), \tag{2}$$

where $\mathrm{Var}(f)$ denotes the total variation of f on $\mathbb{R}$. This yields

$$\|(\widetilde{\mathscr{M}} f)'\|_{L^1(\mathbb{R})} \leqslant \|f'\|_{L^1(\mathbb{R})} \tag{3}$$

if $f \in W^{1,1}(\mathbb{R})$. Notice that the constant $C = 1$ in inequalities (2) and (3) is sharp. Later on, Liu et al. [18] gave a simple proof of inequality (3) by an adaptation of the methods in [31] and [1]. Recently, inequality (2) was extended to a fractional setting in [5, Theorem 1] and to a multisublinear fractional setting in [21, Theorems 1.3-1.4]. Very recently, Carneiro et al. [6] proved that the map $f \mapsto (\widetilde{\mathscr{M}} f)'$ is continuous from $W^{1,1}(\mathbb{R})$ to $L^1(\mathbb{R})$. In the centered setting, Kurka [14] showed that if f is of bounded variation on $\mathbb{R}$, then inequality (2) holds for $\mathscr{M}$ (with constant $C = 240,004$). It was also shown in [14] that if $f \in W^{1,1}(\mathbb{R})$, then $\mathscr{M} f$ is weakly differentiable and (1) holds for $\mathscr{M}$ with constant $C = 240,004$. It is currently unknown whether inequality (3) holds for $\mathscr{M}$ and the map $f \mapsto (\mathscr{M} f)'$ is continuous from $W^{1,1}(\mathbb{R})$ to $L^1(\mathbb{R})$. For other interesting works related to this theory, we refer the reader to [8, 17, 23, 30], among others. Specially, Ramos [30] investigated the total variation inequalities for a wider class of nontangential maximal operators

$$\mathscr{M}^{\alpha} f(x) = \sup_{|x-y| \leqslant \alpha t} \frac{1}{2t} \int_{y-t}^{y+t} |f(s)|ds,$$

where $\alpha \geqslant 0$, which cover the centered and uncentered Hardy-Littlewood maximal operators. More precisely, it is clear that $\mathscr{M}^0 = \mathscr{M}$ and $\mathscr{M}^1 = \widetilde{\mathscr{M}}$. In [30], Ramos proved that

$$\mathrm{Var}(\mathscr{M}^{\alpha} f) \leqslant \mathrm{Var}(f),$$

if $\alpha \in [\frac{1}{3}, \infty)$ and f is of bounded variation on $\mathbb{R}$.

The main purpose of this paper is to study the regularity properties of the discrete nontangential fractional maximal operator, which covers the discrete centered Hardy-Littlewood maximal operator and its fractional version. Precisely, let $\alpha \in [0,1)$ and $\beta \in [0,\infty)$, the discrete nontangential fractional maximal operator $M_{\alpha,\beta}$ is given by

$$M_{\alpha,\beta} f(n) = \sup_{\substack{r \in \mathbb{N} \\ |m-n| \leqslant \beta r}} \frac{1}{(2r+1)^{1-\alpha}} \sum_{k=-r}^{r} |f(m+k)|,$$

where $\mathbb{N} = \{0,1,2,3,\ldots,\}$. Clearly, $M_{\alpha,0}$ (resp., $M_{0,0}$) is just the classical discrete centered fractional (resp., Hardy-Littlewood) maximal operator. However, $M_{\alpha,1}$ (resp.,

$M_{0,1}$) is not the classical discrete uncentered fractional (resp., Hardy-Littlewood) maximal operator. Recall that the discrete uncentered fractional maximal operator is defined by

$$\widetilde{M}_\alpha f(n) = \sup_{r,s\in\mathbb{N}} \frac{1}{(r+s+1)^{1-\alpha}} \sum_{k=-r}^{s} f(n+k), \quad \forall n \in \mathbb{Z},$$

where $\alpha \in [0,1)$. Particularly, when $\alpha = 0$, $\widetilde{M}_\alpha$ reduces to the usual discrete uncentered Hardy-Littlewood maximal operator $\widetilde{M}$.

Before stating our main results, let us recall some pertinent definitions, notation and background. For a discrete function $f:\mathbb{Z}\to\mathbb{R}$ and $1 \leqslant p < \infty$, we define its $\ell^p(\mathbb{Z})$-norm by $\|f\|_{\ell^p(\mathbb{Z})} := (\sum_{n\in\mathbb{Z}} |f(n)|^p)^{1/p}$ and $\ell^\infty(\mathbb{Z})$-norm by $\|f\|_{\ell^\infty(\mathbb{Z})} := \sup_{n\in\mathbb{Z}} |f(n)|$. We also define the total variation of f by

$$\mathrm{Var}(f) = \|f'\|_{\ell^1(\mathbb{Z})} = \sum_{n\in\mathbb{Z}} |f(n+1) - f(n)|,$$

where $f'(n) = f(n+1) - f(n)$ is the first derivative of f. We also write

$$\mathrm{Var}(f;[a,b]) = \|f'\|_{\ell^1([a,b])} = \sum_{n=a}^{b-1} |f(n+1) - f(n)|$$

for the variation of f on the interval $[a,b]$, where a and b are integers (or possibly $a = -\infty$, or $b = \infty$). It is clear that $\mathrm{Var}(f;(-\infty,\infty)) = \mathrm{Var}(f)$. We denote by $\mathrm{BV}(\mathbb{Z})$ the set of functions of bounded variation defined on $\mathbb{Z}$, which is a Banach space with the norm

$$\|f\|_{\mathrm{BV}(\mathbb{Z})} := |f(-\infty)| + \mathrm{Var}(f),$$

where $f(-\infty) := \lim_{n\to-\infty} f(n)$. It is clear that

$$\|f\|_{\ell^\infty(\mathbb{Z})} \leqslant \|f\|_{\mathrm{BV}(\mathbb{Z})} \leqslant 3\|f\|_{\ell^1(\mathbb{Z})}. \tag{4}$$

Recently, the investigation of the regularity properties of the discrete maximal operators has also attracted the attention of many authors. A good start was due to Bober et al. [3] in 2012 when they proved that

$$\mathrm{Var}(\widetilde{M}f) \leqslant \mathrm{Var}(f) \tag{5}$$

if $\mathrm{Var}(f) < \infty$, and

$$\mathrm{Var}(M_{0,0}f) \leqslant \left(2 + \frac{146}{315}\right)\|f\|_{\ell^1(\mathbb{Z})} \tag{6}$$

if $f \in \ell^1(\mathbb{Z})$. Observe that inequality (5) is sharp. It was pointed out in [2] that

$$\mathrm{Var}(M_{0,1}f) \leqslant \mathrm{Var}(f) \tag{7}$$

if $\mathrm{Var}(f) < \infty$. Later on, inequality (5) for $M_{0,0}$ was established by Temur in [32] (with constant $C = 294{,}912{,}004$). From (4) we see that

$$\|M_{0,\beta}f\|_{\ell^\infty(\mathbb{Z})} \leqslant \|f\|_{\ell^\infty(\mathbb{Z})} \leqslant \|f\|_{\mathrm{BV}(\mathbb{Z})} \tag{8}$$

for all $\beta \in [0,\infty)$. (8) together with the above conclusions yields that $\widetilde{M}$, $M_{0,0}$ and $M_{0,1}$ are bounded from $\mathrm{BV}(\mathbb{Z})$ to $\mathrm{BV}(\mathbb{Z})$. Notice that inequality (6) is not optimal, and it was asked in [3] whether the sharp constant for (6) is in fact $C = 2$, which was resolved by Madrid in [28]. It's worth mentioning that the continuity of $\widetilde{M} : \mathrm{BV}(\mathbb{Z}) \to \mathrm{BV}(\mathbb{Z})$ and $M_{0,0} : \mathrm{BV}(\mathbb{Z}) \to \mathrm{BV}(\mathbb{Z})$ was established by Carneiro et al. [6] and Madrid [29] respectively. For the fractional case, Carneiro and Madrid [5] showed that if $0 \leqslant \alpha < 1$, $q = \frac{1}{1-\alpha}$, $f \in \mathrm{BV}(\mathbb{Z})$ and $\widetilde{M}_\alpha f \not\equiv \infty$, then

$$\|(\widetilde{M}_\alpha f)'\|_{\ell^q(\mathbb{Z})} \leqslant 4^{1/q}\mathrm{Var}(f).$$

In [6], Carneiro et al. observed that the map $f \mapsto (\widetilde{M}_\alpha f)'$ is not continuous from $\mathrm{BV}(\mathbb{Z})$ to $\ell^q(\mathbb{Z})$ if $0 < \alpha < 1$ and $q = \frac{1}{1-\alpha}$. However, Carneiro and Madrid [5] established that both $M_{\alpha,0}$ and $\widetilde{M}_\alpha$ are bounded and continuous from $\ell^1(\mathbb{Z})$ to $\mathrm{BV}(\mathbb{Z})$ when $0 \leqslant \alpha < 1$ (also see [15, 24]). Particularly, Liu [15] proved that

$$\mathrm{Var}(M_{\alpha,0}f) \leqslant 2\|f\|_{\ell^1(\mathbb{Z})} \text{ and } \mathrm{Var}(\widetilde{M}_\alpha f) \leqslant 2\|f\|_{\ell^1(\mathbb{Z})} \text{ if } f \in \ell^1(\mathbb{Z}),$$

and the constants $C = 2$ are the best possible. Liu [15] also pointed out that both $M_{\alpha,0}$ and $\widetilde{M}_\alpha$ are not bounded from $\mathrm{BV}(\mathbb{Z})$ to $\mathrm{BV}(\mathbb{Z})$ when $0 < \alpha < 1$. For the general dimension $d \geqslant 2$, we refer the reader to [4, 5, 22].

In light of the aforementioned facts concerning the discrete Hardy-Littlewood maximal operator and its fractional version, it is natural to ask the following question:

QUESTION. *Whether $M_{\alpha,\beta}$ has the similar endpoint regularity properties as that of the discrete centered and uncentered Hardy-Littlewood maximal functions and their fractional variants when $\alpha \in (0,1)$ and $\beta \in (0,\infty)$?*

This problem is resolved by our main theorem:

THEOREM 1. *Let $\alpha \in [0,1)$ and $\beta \in [0,\infty)$. Then*

(i) *$M_{0,\beta}$ is bounded and continuous from* $\mathrm{BV}(\mathbb{Z})$ *to* $\mathrm{BV}(\mathbb{Z})$. *Moreover, if f is a function satisfying* $\mathrm{Var}(f) < \infty$, *then*

$$\mathrm{Var}(M_{0,\beta}f) \leqslant C\mathrm{Var}(f).$$

Specially, when $\beta \geqslant 1$, the constant $C = 1$.

(ii) *$M_{\alpha,\beta}$ is bounded and continuous from $\ell^1(\mathbb{Z})$ to* $\mathrm{BV}(\mathbb{Z})$. *Specially, if $f \in \ell^1(\mathbb{Z})$, then*

$$\mathrm{Var}(M_{\alpha,\beta}f) \leqslant 2\|f\|_{\ell^1(\mathbb{Z})}.$$

REMARK 1. (i) When $\alpha \in [0,1)$ and $\beta \in [0,1]$, then

$$\sup_{\substack{f\in\ell^1(\mathbb{Z})\\ f\not\equiv 0}} \frac{\mathrm{Var}(M_{\alpha,\beta}f)}{\|f\|_{\ell^1(\mathbb{Z})}} = 2.$$

This can be seen by (ii) of Theorem 1, [15, Theorem 1.2] and Proposition 1 in Section 2.

(ii) Ramos [30] proved that $\mathrm{Var}(\mathscr{M}^{\alpha} f) \leqslant \mathrm{Var}(f)$ if $\alpha \in [\frac{1}{3}, \infty)$ and f is of bounded variation on $\mathbb{R}$. This poses a natural question that whether Ramos's result can be extended to the discrete setting, which is very interesting.

The paper is organized as follows. After presenting some auxiliary lemmas in Section 2, we shall prove Theorem 1 in Section 3. The proof of Theorem 1 is based on some previous results followed from [2, 3, 15, 29, 32] and some technical results (see Proposition 1 and Lemma 2). It should be pointed out that Proposition 1 is the key that allows to extend some known results to the nontangential operators. Throughout the paper, the letter C or c, sometimes with certain parameters, will stand for positive constants not necessarily the same one at each occurrence, but are independent of the essential variables.

2. Preliminary Lemmas

This section is devoted to presenting some preliminary results, which will play key roles in the proof of Theorem 1. For an integer $r \in \mathbb{N}$ and a discrete function $f : \mathbb{Z} \to \mathbb{R}$, we define the fractional average function $A_r(f)$ by

$$A_r(f)(n) = \frac{1}{(2r+1)^{1-\alpha}} \sum_{k=-r}^{r} |f(n+k)|.$$

LEMMA 1. *Let $\alpha \in [0,1)$ and $\beta \in [0,\infty)$. Let $f \in \ell^{\infty}(\mathbb{Z})$ such that $M_{\alpha,\beta} f \not\equiv \infty$ and $n \in \mathbb{Z}$.*

(i) *If $M_{\alpha,\beta} f(n) > M_{\alpha,\beta} f(a)$ for some $a \in \mathbb{Z}$, then $M_{\alpha,\beta} f(n)$ is attained by a average $A_r(f)(z)$ with $(z,r) \in \mathbb{Z} \times \mathbb{N}$ and $|n-z| \leqslant \beta r$. Moreover, we have that $z < a - \beta r$ whenever $n < a$ and $z > a + \beta r$ whenever $a < n$.*

(ii) *Assume that there exist two integers a_1, b_1 such that $n \in (a_1, b_1)$ and*

$$M_{\alpha,\beta} f(n) > \max\{M_{\alpha,\beta} f(a_1), M_{\alpha,\beta} f(b_1)\}.$$

Then $M_{\alpha,\beta} f(n)$ is attained by a average $A_r(f)(z)$ with $(z,r) \in \mathbb{Z} \times \mathbb{N}$ and $|n-z| \leqslant \beta r$. Moreover,

$$[z - \beta r, z + \beta r] \subset [a_1, b_1]. \tag{9}$$

(iii) *Let $[a,b]$ be an interval with $a, b \in \mathbb{Z} \cup \{-\infty, \infty\}$ and $n \in (a,b)$. Assume that $M_{\alpha,\beta} f$ is monotonically non-decreasing on $[a,n]$ and is monotonically non-increasing on $[n,b]$. Suppose also that there exist two integers a_1, b_1 such that $a_1 \in [a,n)$, $b_1 \in (n,b]$ and $M_{\alpha,\beta} f(n) > \max\{M_{\alpha,\beta} f(a_1), M_{\alpha,\beta} f(b_1)\}$, then, for any $0 \leqslant \gamma < \beta$, it holds that*

$$\mathrm{Var}(M_{\alpha,\beta} f; [a,b]) \leqslant \mathrm{Var}(M_{\alpha,\gamma} f; [a,b]). \tag{10}$$

(iv) *Let $b \in \mathbb{Z}$ and $M_{\alpha,\beta} f$ be monotonically non-increasing on $(-\infty, b]$, then*

$$\mathrm{Var}(M_{\alpha,\beta} f; (-\infty, b]) \leqslant \mathrm{Var}(M_{\alpha,\gamma} f; (-\infty, b]). \tag{11}$$

(v) *Let* $a \in \mathbb{Z}$ *and* $M_{\alpha,\beta}f$ *be monotonically non-decreasing on* $[a,\infty)$*, then*

$$\mathrm{Var}(M_{\alpha,\beta}f;[a,\infty)) \leqslant \mathrm{Var}(M_{\alpha,\gamma}f;[a,\infty)).$$

Proof. We first prove (i). Assume that $M_{\alpha,\beta}f(n)$ is not attained for any pair $(z,r) \in \mathbb{Z}\times\mathbb{N}$. There exists an increasing sequence of positive integer numbers $\{r_i\}_{i\geqslant 1}$ satisfying $r_1 \geqslant \frac{|n-a|}{2}$ and $\lim_{i\to\infty} r_i = \infty$ such that

$$M_{\alpha,\beta}f(n) = \sup_{\substack{N\in\mathbb{N},N\geqslant r_i\\ |z-n|\leqslant\beta N}} \frac{1}{(2N+1)^{1-\alpha}} \sum_{k=-N}^{N} |f(z+k)|, \quad \forall i \geqslant 1.$$

Fix $i \geqslant 1$, $N \geqslant r_i$ and z with $|z-n| \leqslant \beta N$. Notice that $[z-N,z+N]\cap[z+a-n-N,z+a-n+N] = [z+a-n-N,z+N]$ if $a>n$ and $[z-N,z+N]\cap[z+a-n-N,z+a-n+N] = [z-N,z+a-n+N]$ if $a<n$. This yields that

$$\begin{aligned}&\Big|\frac{1}{(2N+1)^{1-\alpha}}\sum_{k=-N}^{N}|f(z+k)| - \frac{1}{(2N+1)^{1-\alpha}}\sum_{k=-N}^{N}|f(z+a-n+k)|\Big|\\ &\leqslant \frac{2\|f\|_{\ell^\infty(\mathbb{Z})}|n-a|}{(2N+1)^{1-\alpha}}.\end{aligned}$$

This together with the fact that $|(z+a-n)-a| \leqslant \beta N$ implies that

$$M_{\alpha,\beta}f(n) \leqslant M_{\alpha,\beta}f(a) + \frac{2\|f\|_{\ell^\infty(\mathbb{Z})}|n-a|}{(2r_i+1)^{1-\alpha}}, \quad \forall i \geqslant 1.$$

This leads to $M_{\alpha,\beta}f(n) \leqslant M_{\alpha,\beta}f(a)$ by letting $i\to\infty$, which is a contradiction. Hence, $M_{\alpha,\beta}f(n)$ is attained by a average $A_r(f)(z)$ with $(z,r)\in\mathbb{Z}\times\mathbb{N}$ and $|n-z|\leqslant\beta r$. It follows that $|a-z|>\beta r$ since $M_{\alpha,\beta}f(n) = A_r(f)(z) > M_{\alpha,\beta}f(a)$. So, $z > a+\beta r$ if $n>a$ and $z<a-\beta r$ if $n<a$ since $|n-z|\leqslant\beta r$.

Next we verify (ii). It follows from (i) that there exists $(z,r)\in\mathbb{Z}\times\mathbb{N}$ such that $M_{\alpha,\beta}f(n) = A_r(f)(z)$ and $|n-z|\leqslant\beta r$. Assume that (9) doesn't hold. Since $n\in[z-\beta r, z+\beta r]\cap(a_1,b_1)$, then either $a_1\in[z-\beta r,z+\beta r]$ or $b_1\in[z-\beta r,z+\beta r]$. Without loss of generality we assume that $a_1\in[z-\beta r,z+\beta r]$. Then $M_{\alpha,\beta}f(a_1)\geqslant A_r(f)(z) = M_{\alpha,\beta}f(n)$, which is a contradiction. Thus (9) holds.

It remains to show (iii). By (ii), there exist $z\in[a,b]$ and $r\in\mathbb{N}$ such that $M_{\alpha,\beta}f(n) = A_r(f)(z) \leqslant M_{\alpha,\gamma}f(z)$. We now prove (10) by considering the following different cases. If $a,b\in\mathbb{Z}$. Then

$$\begin{aligned}\mathrm{Var}(M_{\alpha,\beta}f;[a,b]) &= 2M_{\alpha,\beta}f(n) - M_{\alpha,\beta}f(a) - M_{\alpha,\beta}f(b)\\ &\leqslant 2M_{\alpha,\gamma}f(z) - M_{\alpha,\gamma}f(a) - M_{\alpha,\gamma}f(b)\\ &\leqslant \mathrm{Var}(M_{\alpha,\gamma}f;[a,b]).\end{aligned}$$

This proves (10) in this case. If $a=-\infty$ or $b=\infty$. Without loss of generality we may assume that $a=-\infty$ and $b=\infty$ (since other cases can be obtained similarly). It is obvious that both $\lim_{m\to-\infty}M_{\alpha,\beta}f(m)$ and $\lim_{n\to+\infty}M_{\alpha,\beta}f(n)$ exist. Moreover,

$$\lim_{m\to-\infty} M_{\alpha,\beta}f(m) \geqslant \liminf_{m\to-\infty} M_{\alpha,\gamma}f(m);$$

$$\lim_{p\to+\infty} M_{\alpha,\beta}f(p) \geqslant \liminf_{p\to+\infty} M_{\alpha,\gamma}f(p).$$

Then we have

$$\begin{aligned}\mathrm{Var}(M_{\alpha,\beta}f;[-\infty,\infty]) &\leqslant 2M_{\alpha,\beta}f(n) - \lim_{m\to-\infty} M_{\alpha,\beta}f(m) - \lim_{p\to+\infty} M_{\alpha,\beta}f(p)\\ &\leqslant 2M_{\alpha,\gamma}f(z) - \liminf_{m\to-\infty} M_{\alpha,\gamma}f(m) - \liminf_{p\to+\infty} M_{\alpha,\gamma}f(p)\\ &\leqslant \mathrm{Var}(M_{\alpha,\gamma}f;[-\infty,\infty]).\end{aligned}$$

This proves inequality (10).

Finally, we only prove (iv) and (v) is analogous. If $M_{\alpha,\beta}f(m) \equiv M_{\alpha,\beta}(b)$ for all $m \in (-\infty,b]$, the inequality (11) is trivial. Otherwise, there exists $n_0 < b$ such that $M_{\alpha,\beta}f(m) > M_{\alpha,\beta}f(b)$ for all $m \leqslant n_0$. Fix $m \leqslant n_0$. By (i), there exists a pair $(z_m,r_m) \in \mathbb{Z}\times\mathbb{N}$ such that $M_{\alpha,\beta}f(m) = A_{r_m}(f)(z_m)$ and $z_m < b-\beta r_m$. Thus we have

$$\begin{aligned}M_{\alpha,\beta}f(m) - M_{\alpha,\beta}f(b) &= A_{r_m}(f)(z_m) - M_{\alpha,\gamma}f(b)\\ &\leqslant M_{\alpha,\gamma}f(z_m) - M_{\alpha,\gamma}f(b) \leqslant \mathrm{Var}(M_{\alpha,\gamma}f;(-\infty,b]).\end{aligned}$$

It follows that

$$\mathrm{Var}(M_{\alpha,\beta}f;(-\infty,b]) = \limsup_{m\to-\infty} M_{\alpha,\beta}f(m) - M_{\alpha,\beta}f(b) \leqslant \mathrm{Var}(M_{\alpha,\gamma}f;(-\infty,b]),$$

which gives (11) and completes the proof of Lemma 1.

In order to formulate the forthcoming results, let us recall the definitions of strings of local maxima and local minima.

DEFINITION 1. ([5]). For a discrete function $g:\mathbb{Z}\to\mathbb{R}$, we say that an interval $[n,m]$ is a *string of local maxima* of g if

$$g(n-1) < g(n) = \ldots = g(m) > g(m+1).$$

If $n=-\infty$ or $m=\infty$ (but not both simultaneously) we modify the definition accordingly, eliminating one of the inequalities. The rightmost point m of such a string is a *right local maximum* of g, while the leftmost point n is a *left local maximum* of g. We define *string of local minima*, *right local minimum* and *left local minimum* analogously.

In [30], Ramos observed that $\mathrm{Var}(\mathscr{M}^{\beta}f) \leqslant \mathrm{Var}(\mathscr{M}^{\alpha}f)$ if $0\leqslant\alpha<\beta$. Motivated by the idea in [30], we show that the maximal operator $M_{\alpha,\beta}$ possess similar characteristic of variation, which plays a key role in the proof of the boundedness part in Theorem 1.

PROPOSITION 1. Let $0\leqslant\alpha<1$ and $0\leqslant\gamma<\beta$. If $f\in\ell^{\infty}(\mathbb{Z})$ such that $M_{\alpha,\beta}f \not\equiv \infty$, then

$$\mathrm{Var}(M_{\alpha,\beta}f) \leqslant \mathrm{Var}(M_{\alpha,\gamma}f). \tag{12}$$

Proof. Assume that $M_{\alpha,\beta}f$ is not constant, since the conclusion is obvious in case $M_{\alpha,\beta}f$ is a constant. Let us consider the alternating sequence of strings of local maximas $\{[a_i^-,a_i^+]\}_{i\in\mathbb{Z}}$ and strings of local minima $\{[b_i^-,b_i^+]\}_{i\in\mathbb{Z}}$ of $M_{\alpha,\beta}f$, satisfying

$$\cdots < a_{-1}^- \leqslant a_{-1}^+ < b_{-1}^- \leqslant b_{-1}^+ < a_0^- \leqslant a_0^+ < b_0^- \leqslant b_0^+ < a_1^- \leqslant a_1^+ < b_1^- \leqslant b_1^+ < \cdots \quad (13)$$

In what follows, we consider the different cases.

Case 1. The sequence (13) does not end on both sides.

In this case we can write

$$\mathrm{Var}(M_{\alpha,\beta}f) = \sum_{i=-\infty}^{\infty} \mathrm{Var}(M_{\alpha,\beta}f;[b_{i-1}^+,b_i^+]). \quad (14)$$

Fix $i\in\mathbb{Z}$, note that $M_{\alpha,\beta}f(a_i^+) > \max\{M_{\alpha,\beta}f(b_{i-1}^+), M_{\alpha,\beta}f(b_i^-)\}$ and $M_{\alpha,\beta}f$ is monotonically non-decreasing on $[b_{i-1}^+,a_i^+]$ and is monotonically non-increasing on $[a_i^+,b_i^-]$. Invoking part (iii) of Lemma 1, we have $\mathrm{Var}(M_{\alpha,\beta}f;[b_{i-1}^+,b_i^-]) \leqslant \mathrm{Var}(M_{\alpha,\gamma}f;[b_{i-1}^+,b_i^-])$. It follows that

$$\mathrm{Var}(M_{\alpha,\beta}f;[b_{i-1}^+,b_i^+]) = \mathrm{Var}(M_{\alpha,\beta}f;[b_{i-1}^+,b_i^-]) \leqslant \mathrm{Var}(M_{\alpha,\gamma}f;[b_{i-1}^+,b_i^+]). \quad (15)$$

Combining (15) with (14) implies that

$$\mathrm{Var}(M_{\alpha,\beta}f) = \sum_{i=-\infty}^{\infty} \mathrm{Var}(M_{\alpha,\beta}f;[b_{i-1}^+,b_i^-]) \leqslant \sum_{i=-\infty}^{\infty} \mathrm{Var}(M_{\alpha,\gamma}f;[b_{i-1}^+,b_i^+]) = \mathrm{Var}(M_{\alpha,\gamma}f),$$

which gives (12) in this case.

Case 2. The sequence (13) terminates on one (or both) side(s).

In this case several different behaviors might occur. We only consider the case that the sequence terminates on both sides, since the other cases can be obtained similarly by making minor modifications. We consider the following two cases:

(i) The sequence (13) does not exist any string of local minima. Let us consider the following subcases:

(a) The sequence (13) does not exist any string of local maxima. In this case we have that $M_{\alpha,\beta}f$ is monotonically non-increasing or non-decreasing on $(-\infty,\infty)$. Without loss of generality we may assume that $M_{\alpha,\beta}f$ is monotonically non-increasing on $(-\infty,\infty)$. Fix $n\in\mathbb{Z}$, we get by (iv) of Lemma 1 that

$$\mathrm{Var}(M_{\alpha,\beta}f;(-\infty,n]) \leqslant \mathrm{Var}(M_{\alpha,\gamma}f;(-\infty,n]) \leqslant \mathrm{Var}(M_{\alpha,\gamma}f),$$

which yields (12) in this case.

(b) The sequence (13) exists an unique string of local maxima. Without loss of generality we may assume that the string of local maxima is $[a_0^-,a_0^+]$. Then $M_{\alpha,\beta}f$ is monotonically non-decreasing on $(-\infty,a_0^-)$ and is monotonically non-increasing on $[a_0^+,\infty)$. There are two integers $c_1\in(-\infty,a_0^-)$ and $c_2\in(a_0^-,\infty)$ such that

$$M_{\alpha,\beta}f(a_0^-) > \max\{M_{\alpha,\beta}f(c_1), M_{\alpha,\beta}f(c_2)\}.$$

Then we get (12) by using (iii) of Lemma 1 in this case.

(ii) The sequence (13) exists string of local minima. In this case without loss of generality we may assume that the first string of local minima is $[b_0^-, b_0^+]$ and the last one is $[b_l^-, b_l^+]$ for some $l \geqslant 0$. We consider the following subcases:

(c) The sequence (13) does not have the strings of local maxima $[a_0^-, a_0^+]$ and $[a_{l+1}^-, a_{l+1}^+]$. Then $M_{\alpha,\beta}f$ is monotonically non-increasing on $(-\infty, b_0^-]$ and is monotonically non-decreasing on $[b_l^+, \infty)$. By (iv) and (v) of Lemma 1, we obtain

$$\mathrm{Var}(M_{\alpha,\beta}f; (-\infty, b_0^-]) \leqslant \mathrm{Var}(M_{\alpha,\gamma}f; (-\infty, b_0^-]), \tag{16}$$

$$\mathrm{Var}(M_{\alpha,\beta}f; [b_l^+, \infty)) \leqslant \mathrm{Var}(M_{\alpha,\gamma}f; [b_l^+, \infty)). \tag{17}$$

Next we shall prove

$$\mathrm{Var}(M_{\alpha,\beta}f; [b_0^-, b_l^+]) \leqslant \mathrm{Var}(M_{\alpha,\gamma}f; [b_0^-, b_l^+]). \tag{18}$$

If $l = 0$, then (18) is obvious. If $l \geqslant 1$, then we get from (15) that

$$\begin{aligned}\mathrm{Var}(M_{\alpha,\beta}f; [b_0^-, b_l^+]) &= \sum_{j=1}^{l} \mathrm{Var}(M_{\alpha,\beta}f; [b_{j-1}^+, b_j^+]) \\ &\leqslant \sum_{j=1}^{l} \mathrm{Var}(M_{\alpha,\gamma}f; [b_{j-1}^+, b_j^+]) \leqslant \mathrm{Var}(M_{\alpha,\gamma}f; [b_0^-, b_l^+]).\end{aligned}$$

This proves (18). (18) together with (16)-(17) yields (12) in this case.

(d) The sequence (13) exists the strings of local maxima $[a_0^-, a_0^+]$ and $[a_{l+1}^-, a_{l+1}^+]$. Then $M_{\alpha,\beta}f$ is monotonically non-decreasing on $(-\infty, a_0^-]$ and is monotonically non-increasing on $[a_0^-, b_0^-]$. Moreover, $M_{\alpha,\beta}f(a_0^-) > \max\{M_{\alpha,\beta}f(a_0^- - 1), M_{\alpha,\beta}f(b_0^-)\}$. Applying (iii) of Lemma 1, we get

$$\mathrm{Var}(M_{\alpha,\beta}f; (-\infty, b_0^-]) \leqslant \mathrm{Var}(M_{\alpha,\gamma}f; (-\infty, b_0^-]). \tag{19}$$

Similarly, we can get

$$\mathrm{Var}(M_{\alpha,\beta}f; [b_l^+, \infty)) \leqslant \mathrm{Var}(M_{\alpha,\gamma}f; [b_l^+, \infty)). \tag{20}$$

Then (12) follows immediately from (18)-(20) in this case.

(e) The sequence (13) exists the string of local maxima $[a_0^-, a_0^+]$, but the string of local maxima $[a_{l+1}^-, a_{l+1}^+]$ does not exist. Then by (17)-(19) we can get (12) in this case.

(g) The sequence (13) exists the string of local maxima $[a_{l+1}^-, a_{l+1}^+]$, but the string of local maxima $[a_0^-, a_0^+]$ does not exist. Then (12) follows from (16), (18) and (20) in this case.

This completes the proof of Proposition 1.

We also need the following lemma, which will play a key role in the proof of the continuity part in Theorem 1.

LEMMA 2. *Let* $0 \leqslant \alpha < 1$ *and* $0 \leqslant \gamma < \beta$. *Let* $\{f_j\}_{j\geqslant 1} \subset \mathrm{BV}(\mathbb{Z})$ *and* $f \in \mathrm{BV}(\mathbb{Z})$ *be such that* $\|f_j - f\|_{\mathrm{BV}(\mathbb{Z})} \to 0$ *as* $j \to \infty$. *Suppose that the following conditions hold:*

(i) *Given* $\varepsilon > 0$, *there exist* $\Lambda = \Lambda(\varepsilon) > 0$ *and* $N = N(\varepsilon) > 0$ *such that*

$$\max\{\mathrm{Var}(M_{\alpha,\gamma}f_j;(-\infty,-\Lambda]),\mathrm{Var}(M_{\alpha,\gamma}f_j;[\Lambda,\infty))\} \leqslant C\varepsilon$$

for any $j \geqslant N$. *Here* $C > 0$ *is independent of* ε.

(ii) $\mathrm{Var}(M_{\alpha,\beta}f) < \infty$.

(iii) $M_{\alpha,\beta}f_j \to M_{\alpha,\beta}f$ *in* $\ell^\infty(\mathbb{Z})$ *as* $j \to \infty$.

Then we have

$$\|M_{\alpha,\beta}f_j - M_{\alpha,\beta}f\|_{\mathrm{BV}(\mathbb{Z})} \to 0 \text{ as } j \to \infty. \tag{21}$$

Proof. By assumption (iii), to prove (21), it suffices to show that

$$\mathrm{Var}(M_{\alpha,\beta}f_j - M_{\alpha,\beta}f) \to 0 \text{ as } j \to \infty. \tag{22}$$

Given $\varepsilon > 0$. By our assumption (i), there exist $\Lambda_1 = \Lambda_1(\varepsilon) > 0$ and $N_1 = N_1(\varepsilon) > 0$ such that

$$\max\{\mathrm{Var}(M_{\alpha,\gamma}f_j;(-\infty,-\Lambda_1]),\mathrm{Var}(M_{\alpha,\gamma}f_j;[\Lambda_1,\infty))\} \leqslant C\varepsilon \tag{23}$$

for all $j \geqslant N_1$. Here $C > 0$ is independent of ε. By assumption (ii), there exists $\Lambda_2 = \Lambda_2(\varepsilon) > 0$ such that

$$\max\{\mathrm{Var}(M_{\alpha,\beta}f;(-\infty,-\Lambda_2]),\mathrm{Var}(M_{\alpha,\beta}f;[\Lambda_2,\infty))\} \leqslant \varepsilon. \tag{24}$$

By assumption (iii), there exists $N_2 = N_2(\varepsilon) > 0$ such that

$$|M_{\alpha,\beta}f_j(n) - M_{\alpha,\beta}f(n)| \leqslant \|M_{\alpha,\beta}f_j - M_{\alpha,\beta}f\|_{\ell^\infty(\mathbb{Z})} \leqslant \varepsilon \tag{25}$$

for any $n \in \mathbb{Z}$ and $j \geqslant N_2$. Let $\Lambda = \max\{\Lambda_1,\Lambda_2\}$ and fix $j \geqslant \max\{N_1,N_2\}$. We shall prove that there exists a constant $C > 0$ independent of j and ε such that

$$\mathrm{Var}(M_{\alpha,\beta}f_j;[\Lambda,\infty)) \leqslant C\varepsilon, \tag{26}$$

$$\mathrm{Var}(M_{\alpha,\beta}f_j;(-\infty,-\Lambda]) \leqslant C\varepsilon. \tag{27}$$

We only prove (26) since (27) can be obtained similarly. The arguments are similar to the proof of Proposition 1. Let $\{[a_i^-,a_i^+]\}_{i\in\mathbb{Z}}$ and $\{[b_i^-,b_i^+]\}_{i\in\mathbb{Z}}$ be the sequences of all strings of local maxima and local minima of $M_{\alpha,\beta}f_j$ ordered as follows:

$$\cdots < a_{-1}^- \leqslant a_{-1}^+ < b_{-1}^{-1} \leqslant b_{-1}^+ < a_0^- \leqslant a_0^+ < b_0^- \leqslant b_0^+ < a_1^- \leqslant a_1^+ < b_1^- \leqslant b_1^+ < \cdots \tag{28}$$

We allow the possibilities of a_i^- or $b_i^- = -\infty$ and a_i^+ or $b_i^+ = \infty$. By the argument similar to those used to derive (15), we have that if $[b_i^+,b_{i+1}^+]$ exists for some $i \in \mathbb{Z}$, then

$$\mathrm{Var}(M_{\alpha,\beta}f_j;[b_i^+,b_{i+1}^+]) \leqslant \mathrm{Var}(M_{\alpha,\gamma}f_j;[b_i^+,b_{i+1}^+]). \tag{29}$$

Before proving (26), let us point out the following simple observations: Let $j \geqslant \max\{N_1, N_2\}$ and a, b be two integers with $b > a \geqslant \Lambda$. If $M_{\alpha,\beta} f_j$ is monotonically non-increasing or non-decreasing on $[a,b]$, then

$$\mathrm{Var}(M_{\alpha,\beta} f_j; [a,b]) \leqslant 3\varepsilon. \tag{30}$$

Without loss of generality we may assume that $M_{\alpha,\beta} f_j$ is monotonically non-increasing on $[a,b]$, then by (24) and (25) we get

$$\begin{aligned}\mathrm{Var}(M_{\alpha,\beta} f_j; [a,b]) &= M_{\alpha,\beta} f_j(a) - M_{\alpha,\beta} f_j(b) \leqslant (M_{\alpha,\beta} f(a) + \varepsilon) - (M_{\alpha,\beta} f(b) - \varepsilon) \\ &\leqslant \mathrm{Var}(M_{\alpha,\beta} f; [a,b]) + 2\varepsilon \leqslant 3\varepsilon,\end{aligned}$$

which proves (30).

We now prove (26) by considering the following two cases:

Case 1. The sequence (28) does not end on both sides.

In this case there exists $i_0 \geqslant 1$ such that $b^+_{i_0-1} \leqslant \Lambda \leqslant b^+_{i_0}$. We first prove that there exists a constant $C > 0$ independent of ε such that

$$\mathrm{Var}(M_{\alpha,\beta} f_j; [\Lambda, b^+_{i_0}]) \leqslant C\varepsilon. \tag{31}$$

If $\Lambda \in [b^+_{i_0-1}, a^+_{i_0}]$, then $M_{\alpha,\beta} f_j$ is monotonically non-decreasing on $[\Lambda, a^+_{i_0}]$ and is monotonically non-increasing on $[a^+_{i_0}, b^+_{i_0}]$. In this case (31) follows easily from (30). If $\Lambda \in (a^+_{i_0}, b^+_{i_0}]$, then $M_{\alpha,\beta} f_j$ is monotonically non-increasing on $[\Lambda, b^+_{i_0}]$. Then we get from (30) that (31) holds. Hence, we get from (23), (29) and (31) that

$$\begin{aligned}\mathrm{Var}(M_{\alpha,\beta} f_j; [\Lambda, \infty)) &= \mathrm{Var}(M_{\alpha,\beta} f_j; [\Lambda, b^+_{i_0}]) + \sum_{i=i_0}^{\infty} \mathrm{Var}(M_{\alpha,\beta} f_j; [b^+_i, b^+_{i+1}]) \\ &\leqslant C\varepsilon + \sum_{i=i_0}^{\infty} \mathrm{Var}(M_{\alpha,\gamma} f_j; [b^+_i, b^+_{i+1}]) \\ &\leqslant C\varepsilon + \mathrm{Var}(M_{\alpha,\gamma} f_j; [\Lambda, \infty)) \leqslant C\varepsilon,\end{aligned} \tag{32}$$

where the constant $C > 0$ is independent of ε. This proves (26) in this case.

Case 2. The sequence (28) terminates on one (or both) side(s).

In this case we only consider the case that the sequence terminates on both sides, since the other cases can be obtained similarly by making minor modifications. We consider the following two cases:

(i) There doesn't exist any string of local minima in $[\Lambda, \infty)$. Let us consider the following subcases:

(a) There doesn't exist any string of local maxima in $[\Lambda, \infty)$. In this case we have that $M_{\alpha,\beta} f_j$ is monotonically non-increasing or non-decreasing on $[\Lambda, \infty)$. Then (26) follows easily from (30).

(b) There exists an unique string of local maxima in $[\Lambda, \infty)$. Without loss of generality we may assume that the string of local maxima is $[a^-_0, a^+_0]$. It is clear that $M_{\alpha,\beta} f_j$ is monotonically non-decreasing on $[\Lambda, a^-_0)$ and is monotonically non-increasing on $[a^+_0, \infty)$. Then by (30) we have

$$\mathrm{Var}(M_{\alpha,\beta} f_j; [\Lambda, \infty)) = \mathrm{Var}(M_{\alpha,\beta} f_j; [\Lambda, a^-_0]) + \mathrm{Var}(M_{\alpha,\beta} f_j; [a^+_0, \infty)) \leqslant 6\varepsilon,$$

which gives (26).

(ii) There exists string of local minima in $[\Lambda,\infty)$. In this case without loss of generality we may assume that the first string of local minima is $[b_0^-,b_0^+]$ and the last one is $[b_l^-,b_l^+]$ for some $l\geqslant 0$. We consider the following subcases:

(c) There do not exist the strings of local maxima $[a_0^-,a_0^+]$ and $[a_{l+1}^-,a_{l+1}^+]$ in $[\Lambda,\infty)$. Then $M_{\alpha,\beta}f_j$ is monotonically non-increasing on $[\Lambda,b_0^-]$ and is monotonically non-decreasing on $[b_l^+,\infty)$. By (30), we obtain

$$\max\{\mathrm{Var}(M_{\alpha,\beta}f_j;[\Lambda,b_0^-]),\mathrm{Var}(M_{\alpha,\beta}f_j;[b_l^+,\infty))\}\leqslant 3\varepsilon, \tag{33}$$

By (23) and the arguments similar to those used in (18),

$$\mathrm{Var}(M_{\alpha,\beta}f_j;[b_0^-,b_l^+])\leqslant \mathrm{Var}(M_{\alpha,\gamma}f_j;[b_0^-,b_l^+])\leqslant \mathrm{Var}(M_{\alpha,\gamma}f_j;[\Lambda,\infty))\leqslant C\varepsilon. \tag{34}$$

(34) together with (33) implies (26) in this case.

(d) There exist the strings of local maxima $[a_0^-,a_0^+]$ and $[a_{l+1}^-,a_{l+1}^+]$ in $[\Lambda,\infty)$. Then $M_{\alpha,\beta}f_j$ is monotonically non-decreasing on $[\Lambda,a_0^-]$ and $[b_l^+,a_{l+1}^+]$ and is monotonically non-increasing on $[a_0^+,b_0^-]$ and $[a_{l+1}^+,\infty)$. In this case we get by (30) and (34) that

$$\begin{aligned}&\mathrm{Var}(M_{\alpha,\beta}f_j;[\Lambda,\infty))\\ =&\mathrm{Var}(M_{\alpha,\beta}f_j;[\Lambda,a_0^-])+\mathrm{Var}(M_{\alpha,\beta}f_j;[a_0^+,b_0^-])+\mathrm{Var}(M_{\alpha,\beta}f_j;[b_0^-,b_l^+])\\ &+\mathrm{Var}(M_{\alpha,\beta}f_j;[b_l^+,a_{l+1}^+])+\mathrm{Var}(M_{\alpha,\beta}f_j;[a_{l+1}^+,\infty))\leqslant C\varepsilon,\end{aligned}$$

which proves (26) in this case.

(e) There exists the string of local maxima $[a_0^-,a_0^+]$ in $[\Lambda,\infty)$, but the string of local maxima $[a_{l+1}^-,a_{l+1}^+]$ does not exist. Then $M_{\alpha,\beta}f_j$ is monotonically non-decreasing on $[\Lambda,a_0^-]$ and is monotonically non-increasing on $[a_0^+,b_0^-]$. By (30) and (34), we have

$$\begin{aligned}&\mathrm{Var}(M_{\alpha,\beta}f_j;[\Lambda,\infty))\\ =&\mathrm{Var}(M_{\alpha,\beta}f_j;[\Lambda,a_0^-])+\mathrm{Var}(M_{\alpha,\beta}f_j;[a_0^+,b_0^-])+\mathrm{Var}(M_{\alpha,\beta}f_j;[b_0^-,b_l^+])\\ &+\mathrm{Var}(M_{\alpha,\beta}f_j;[b_l^+,\infty))\leqslant C\varepsilon,\end{aligned}$$

which gives (26) in this case.

(g) The sequence (13) exists the string of local maxima $[a_{l+1}^-,a_{l+1}^+]$ in $[\Lambda,\infty)$, but the string of local maxima $[a_0^-,a_0^+]$ does not exist. Then $M_{\alpha,\beta}f_j$ is monotonically non-increasing on $[\Lambda,b_0^-]$ and $[a_{l+1}^+,\infty)$ and is monotonically non-decreasing on $[b_l^+,a_{l+1}^+]$. By (30) and (34) again, we get

$$\begin{aligned}&\mathrm{Var}(M_{\alpha,\beta}f_j;[\Lambda,\infty))\\ =&\mathrm{Var}(M_{\alpha,\beta}f_j;[\Lambda,b_0^-])+\mathrm{Var}(M_{\alpha,\beta}f_j;[b_0^-,b_l^+])+\mathrm{Var}(M_{\alpha,\beta}f_j;[b_l^+,a_{l+1}^+])\\ &+\mathrm{Var}(M_{\alpha,\beta}f_j;[a_{l+1}^+,\infty))\leqslant C\varepsilon,\end{aligned}$$

which gives (26) in this case.

We now proceed with the rest of the proof. By assumption (iii) we have that $(M_{\alpha,\beta}f_j)'(n)\to(M_{\alpha,\beta}f)'(n)$ uniformly for $n\in\mathbb{Z}$. There exists $N_3=N_3(\Lambda,\varepsilon)>0$ such that

$$|(M_{\alpha,\beta}f_j-M_{\alpha,\beta}f)'(n)|<\frac{\varepsilon}{2\Lambda+1} \tag{35}$$

for any $|n| \leqslant \Lambda$ and $j \geqslant N_3$. It follows from (35) that

$$\mathrm{Var}(M_{\alpha,\beta} f_j - M_{\alpha,\beta} f; [-\Lambda,\Lambda]) \leqslant \|(M_{\alpha,\beta} f_j - M_{\alpha,\beta} f)' \chi_{\{|n|\leqslant\Lambda\}}\|_{\ell^1(\mathbb{Z})} < \varepsilon \tag{36}$$

for any $j \geqslant N_3$. (36) together with (24), (26) and (27) yields that

$$\begin{aligned} &\mathrm{Var}(M_{\alpha,\beta} f_j - M_{\alpha,\beta} f) \\ &= \mathrm{Var}(M_{\alpha,\beta} f_j - M_{\alpha,\beta} f; [-\Lambda,\Lambda]) + \mathrm{Var}(M_{\alpha,\beta} f_j - M_{\alpha,\beta} f; (-\infty,-\Lambda]) \\ &\quad + \mathrm{Var}(M_{\alpha,\beta} f_j - M_{\alpha,\beta} f; [\Lambda,\infty)) \\ &\leqslant \varepsilon + \mathrm{Var}(M_{\alpha,\beta} f; (-\infty,-\Lambda]) + \mathrm{Var}(M_{\alpha,\beta} f; [\Lambda,\infty)) \\ &\quad + \mathrm{Var}(M_{\alpha,\beta} f_j; (-\infty,-\Lambda]) + \mathrm{Var}(M_{\alpha,\beta} f_j; [\Lambda,\infty)) \leqslant C\varepsilon \end{aligned} \tag{37}$$

for all $j \geqslant \max\{N_1,N_2,N_3\}$. Then (22) follows from (37). This proves Lemma 2.

REMARK 2. It should be pointed out that the condition (i) of Lemma 2 is the crux of the BV-continuity. In order to illustrate this point, two examples will be listed as follows:

(i) Let $\{f_j\}_{j\geqslant 1} \subset \mathrm{BV}(\mathbb{Z})$ and $f \in \mathrm{BV}(\mathbb{Z})$ be such that $\|f_j - f\|_{\mathrm{BV}(\mathbb{Z})} \to 0$ as $j \to \infty$. In order to establish the continuity of $\widetilde{M} : \mathrm{BV}(\mathbb{Z}) \to \mathrm{BV}(\mathbb{Z})$. Carneiro et al. [6] proved that for fixed $\varepsilon > 0$, there exist $N_1 = N_1(\varepsilon) > 0$ and $\Lambda_1 = \Lambda_1(\varepsilon) > 0$ such that

$$\max\{\mathrm{Var}(\widetilde{M} f_j; (-\infty,-\Lambda_1]), \mathrm{Var}(\widetilde{M} f_j; [\Lambda_1,\infty))\} \leqslant C\varepsilon, \quad \forall j \geqslant N_1,$$

where $C > 0$ is independent of ε (see (2.14) and (2.15) in [6]).

(ii) Let $\{f_j\}_{j\geqslant 1} \subset \mathrm{BV}(\mathbb{Z})$ and $f \in \mathrm{BV}(\mathbb{Z})$ be such that $\|f_j - f\|_{\mathrm{BV}(\mathbb{Z})} \to 0$ as $j \to \infty$. In order to establish the continuity for $M_{0,0} : \mathrm{BV}(\mathbb{Z}) \to \mathrm{BV}(\mathbb{Z})$. Madrid [29] proved that for fixed $\varepsilon > 0$, there exist $N_2 = N_2(\varepsilon) > 0$ and $\Lambda_2 = \Lambda_2(\varepsilon) > 0$ such that

$$\max\{\mathrm{Var}(M_{0,0} f_j; (-\infty,-\Lambda_2]), \mathrm{Var}(M_{0,0} f_j; [\Lambda_2,\infty))\} \leqslant C\varepsilon, \quad \forall j \geqslant N_2,$$

where $C > 0$ is independent of ε (see (4.24) and (4.25) in [29]).

3. Proof of Theorem 1

Let us first prove (i) of Theorem 1. Let $f \in \mathrm{BV}(\mathbb{Z})$. By (8), we have $M_{0,\beta} f \not\equiv \infty$. It was shown in [32, Theorem 1] that

$$\mathrm{Var}(M_{0,0} f) \leqslant C\mathrm{Var}(f).$$

This together with (7) and Proposition 1 yields that

$$\mathrm{Var}(M_{0,\beta} f) \leqslant C\mathrm{Var}(f), \quad \forall \beta \geqslant 0; \tag{38}$$

$$\mathrm{Var}(M_{0,\beta} f) \leqslant \mathrm{Var}(f), \quad \forall \beta \geqslant 1. \tag{39}$$

(38), (39) and (8) imply the boundedness of $M_{0,\beta} : \mathrm{BV}(\mathbb{Z}) \to \mathrm{BV}(\mathbb{Z})$. Let $f_j \to f$ in $\mathrm{BV}(\mathbb{Z})$ as $j \to \infty$. By the sublinearity of $M_{0,\beta}$ and (8), it holds that

$$\|M_{0,\beta} f_j - M_{0,\beta} f\|_{\ell^\infty(\mathbb{Z})} \leqslant \|M_{0,\beta}(f_j - f)\|_{\ell^\infty(\mathbb{Z})} \leqslant \|f_j - f\|_{\ell^\infty(\mathbb{Z})} \leqslant \|f_j - f\|_{\mathrm{BV}(\mathbb{Z})},$$

which yields that $M_{0,\beta}f_j \to M_{0,\beta}f$ in $\ell^\infty(\mathbb{Z})$ as $j \to \infty$. This together with (38), (39), (ii) of Remark 2 and Lemma 2 yields the continuity of $M_{0,\beta} : \mathrm{BV}(\mathbb{Z}) \to \mathrm{BV}(\mathbb{Z})$.

We now prove (ii). Let $0 \leqslant \alpha < 1$, $\beta \geqslant 0$ and $f \in \ell^1(\mathbb{Z})$. It is easy to see that

$$\|M_{\alpha,\beta}f\|_{\ell^\infty(\mathbb{Z})} \leqslant \|f\|_{\ell^1(\mathbb{Z})}. \tag{40}$$

It was shown in [15, Theorem 1.2] that

$$\mathrm{Var}(M_{\alpha,0}f) \leqslant 2\|f\|_{\ell^1(\mathbb{Z})}. \tag{41}$$

(41) together with (40) and Proposition 1 yields the boundedness for $M_{\alpha,\beta} : \ell^1(\mathbb{Z}) \to \mathrm{BV}(\mathbb{Z})$ and

$$\mathrm{Var}(M_{\alpha,\beta}f) \leqslant 2\|f\|_{\ell^1(\mathbb{Z})}, \quad \forall \beta \geqslant 0.$$

Let $f_j \to f$ in $\ell^1(\mathbb{Z})$ as $j \to \infty$. It follows from (4) that $f_j \to f$ in $\mathrm{BV}(\mathbb{Z})$ as $j \to \infty$. By the sublinearity of $M_{\alpha,\beta}$ and (40), we have

$$\|M_{\alpha,\beta}f_j - M_{\alpha,\beta}f\|_{\ell^\infty(\mathbb{Z})} \leqslant \|M_{\alpha,\beta}(f_j - f)\|_{\ell^\infty(\mathbb{Z})} \leqslant \|f_j - f\|_{\ell^1(\mathbb{Z})},$$

which yields that $M_{\alpha,\beta}f_j \to M_{\alpha,\beta}f$ in $\ell^\infty(\mathbb{Z})$ as $j \to \infty$. On the other hand, we get from the proof of [15, Theorem 1.4] (see [15, p.116] by taking $\Phi(t) = t^{\alpha-1}$) that for fixed $\varepsilon > 0$, there exist $N = N(\varepsilon) > 0$ and $\Lambda = \Lambda(\varepsilon) > 0$ such that

$$\max\{\mathrm{Var}(M_{\alpha,0}f_j; (-\infty, -\Lambda]), \mathrm{Var}(M_{\alpha,0}f_j; [\Lambda, \infty))\} \leqslant C\varepsilon, \quad \forall j > N,$$

where $C > 0$ is independent of ε. The above facts together with Lemma 2 yield the continuity of $M_{\alpha,\beta} : \ell^1(\mathbb{Z}) \to \mathrm{BV}(\mathbb{Z})$. Hence Theorem 1 is proved. □

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