Mathematical
Inequalities
& Applications
Volume 23, Number 4 (2020), 1391–1423 doi:10.7153/mia-2020-23-101

EXTENSIONS OF QUADRATIC TRANSFORMATION IDENTITIES FOR HYPERGEOMETRIC FUNCTIONS

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(*Communicated by S. Varošanec*)

Abstract. In the article, we extend the identities $F_0(x) = (1+r)F_0(r)$, $2F_0(\sqrt{1-x}) = (1+r)F_0(1-r^2)$, $2\overline{F}_0(y) = \sqrt{1+3r}\,\overline{F}_0(1-r^2)$ and $\overline{F}_0(1-y) = \sqrt{1+3r}\,\overline{F}_0(r^2)$ for hypergeometric functions $F_0(r) = {}_2F_1(1/2,1;3/2;r)$ and $\overline{F}_0(r) = {}_2F_1(1/4,3/4;1;r)$, performed by the quadratic transformations $r \mapsto x = 4r/(1+r)^2$, $r \mapsto \sqrt{1-x}$, $r \mapsto y = (1-r)^2/(1+3r)^2$ and $r \mapsto 1-y$, to the zero-balanced hypergeometric function ${}_2F_1(a,b;a+b;r)$, by showing new properties of ${}_2F_1(a,b;a+b;r)$ and the Ramanujan type constant, and the monotonicity properties of certain combinations in terms of hypergeometric and elementary functions. These extensions give complete solutions of the problem of extending the transformation identities abovementioned to ${}_2F_1(a,b;a+b;r)$, and perfect all the known related results. By these results, sharp transformation inequalities are obtained for the generalized Grötzsch ring function appearing in Ramanujan's modular equations.

1. Introduction

For real numbers a,b and c with $c \neq 0,-1,-2,\cdots$, the Gaussian hypergeometric function $F(a,b;c;x$ [29, 44, 45, 50, 52, 54, 55, 73] is defined by

$$F(a,b;c;x) = {}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)n!}x^n,\ |x|<1, \tag{1}$$

where $(a,0)=1$ for $a \neq 0$, and $(a,n) = a(a+1)(a+2)\cdots(a+n-1)$ for $n \in \mathbb{N} = \{n|\ n$ is a positive integer$\}$ is the shifted factorial function. The function $F(a,b;c;x)$ is said to be zero-balanced [16, 49, 63, 64] if $c = a+b$. It is well known that $F(a,b;c;x)$ has wide applications in mathematics, physics, as well as in some fields of engineering [19, 21, 28, 41, 46, 51, 56, 57, 62, 66, 67], and many other special functions in mathematical physics and even some elementary functions are particular or limiting cases of $F(a,b;c;x)$ [1, 3, 4, 5, 6, 7, 8, 9, 10, 13, 15, 25, 35]. For example, $\mathscr{K}_a$ and $\mathscr{K}_a'$ ($\mathscr{E}_a$ and $\mathscr{E}_a'$) [17, 18, 20, 23, 27, 38, 47, 53, 58, 60, 71, 72, 74], defined by

$$\mathscr{K}_a(r) = \frac{\pi}{2}F\left(a,1-a;1;r^2\right),\ \mathscr{K}_a'(r) = \mathscr{K}_a\left(\sqrt{1-r^2}\right), \tag{2}$$

Mathematics subject classification (2010): 33C05, 33F05, 30C62, 11F03.
Keywords and phrases: Zero-balanced hypergeometric function, quadratic transformation, generalized Grötzsch ring function, monotonicity, extension.
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$$\mathscr{E}_a(r) = \frac{\pi}{2} F\left(a-1, 1-a; 1; r^2\right), \ \mathscr{E}_a'(r) = \mathscr{E}_a\left(\sqrt{1-r^2}\right) \tag{3}$$

for $a \in (0, 1/2]$ and $r \in (0,1)$, are the well-known generalized elliptic integrals of the first kind (the second kind, respectively), while $\mathscr{K}(r) = \mathscr{K}_{1/2}(r)$ and $\mathscr{K}'(r) = \mathscr{K}_{1/2}'(r)$ ($\mathscr{E}(r) = \mathscr{E}_{1/2}(r)$ and $\mathscr{E}'(r) = \mathscr{E}_{1/2}'(r)$) are the complete elliptic integrals of the first kind (the second kind, respectively).

For $x, y \in (0, \infty)$, the classical gamma, psi (digamma) and beta functions are defined as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \ \psi(x) = \frac{d}{dx} \log \Gamma(x), \ B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \tag{4}$$

respectively [1, 3, 4, 35, 36, 65, 68, 70].

Let $\gamma = \lim_{n\to\infty} [\sum_{k=1}^n (1/k) - \log n] = 0.577215664\cdots$ be the Euler-Mascheroni constant [22]. Then it is well known that (see [1, 6.1.15, 6.3.2, 6.3.3, 6.3.5, 6.3.8, 6.3.16 & 6.4.10] and [11, p.4232])

$$x\Gamma(x) = \Gamma(x+1), \ \psi^{(n)}(x+1) = \psi^{(n)}(x) + (-1)^n n! x^{-n-1}, \ n \in \mathbb{N}_0, \tag{5}$$

$$\psi(x) = -\gamma - \frac{1}{x} + \sum_{k=1}^\infty \frac{x}{k(k+x)}, \ \psi^{(n)}(x) = \sum_{k=1}^\infty \frac{(-1)^{n+1} n!}{(k+x)^{n+1}}, \ n \in \mathbb{N}, \tag{6}$$

$$2\psi(2x) = \psi(x) + \psi\left(x + \frac{1}{2}\right) + \log 4, \ \psi(1) = -\gamma, \ \psi\left(\frac{1}{2}\right) = -\gamma - \log 4. \tag{7}$$

$$\psi(1/4) + \psi(3/4) = -2\gamma - \log 64. \tag{8}$$

For $a, b \in (0, \infty)$, we denote

$$R(a,b) = -2\gamma - \psi(a) - \psi(b), \tag{9}$$

$$R_c(a) = R(a, c-a) \equiv -2\gamma - \psi(a) - \psi(c-a), \tag{10}$$

$$R(a) = R(a, 1-a) = -2\gamma - \psi(a) - \psi(1-a), \tag{11}$$

$$B(a) = B(a, 1-a) = \Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin(\pi a)}. \tag{12}$$

$R(a,b)$ and $R(a)$ are called the Ramanujan type constants in literature [31]. It follows from (4) and (7)–(12) that

$$\begin{cases} B(1/2) = \pi, \ B(1/2, 1) = 2, B(1/4) = \sqrt{2}\pi, \\ R(1/2) = \log 16, \ R(1/2, 1) = \log 4, \ R(1/4) = \log 64, \end{cases} \tag{13}$$

and by the symmetry, we may assume that $a \in (0, c/2]$ in (10), and $a \in (0, 1/2]$ in (11) and (12).

Throughout this paper, we denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $r' = \sqrt{1-r^2}$ for each $r \in [0,1]$. For $a, b, a_1, b_1 \in (0, \infty)$ with $c = a+b$ and $c_1 = a_1 + b_1$, and for $r \in (0,1)$, let $\alpha = ab/c$, $\overline{\alpha} = ab/(c+1)$, $\overline{\alpha}_1 = a_1 b_1/(c_1+1)$, $\alpha_1 = a_1 b_1/c_1$, $B = B(a,b)$, $B_1 =$

$B(a_1,b_1)$, $\overline{B}=B(a+1,b+1)$, $\overline{B}_1=B(a_1+1,b_1+1)$, $R=R(a,b)$, $R_1=R(a_1,b_1)$, $\overline{R}=R(a+1,b+1)$, $\overline{R}_1=R(a_1+1,b_1+1)$,

$$\begin{cases} F(r)=F(a,b;c;r),\ G(r)=F(a,b;c+1;r),\\ F_1(r)=F(a_1,b_1;c_1;r),\ G_1(r)=F(a_1,b_1;c_1+1;r),\\ F_0(r)=F(1/2,1;3/2;r),\ G_0(r)=F(1/2,1;5/2;r),\\ \overline{F}_0(r)=F(1/4,3/4;1;r),\ \overline{G}_0(r)=F(1/4,3/4;2;r),\\ F_+(r)=F(a+1,b+1;c+2;r),\ F_{1+}(r)=F(a_1+1,b_1+1;c_1+2;r). \end{cases} \tag{14}$$

It follows from (4)–(5) and (9) that

$$\overline{B}=\alpha B/(c+1)=\overline{\alpha}B/c \ \text{ and } \ \overline{R}=R-1/\alpha \tag{15}$$

if $a,b\in(0,\infty)$ with $c=a+b$.

In addition, by the symmetry of the parameters a and b in the function $F(a,b;a+b;x)$, without loss of generality, we assume that $a\leqslant b$. Observe that for $a,b\in(0,\infty)$ with $c=a+b$,

$$a\leqslant b\Rightarrow a\leqslant c/2\leqslant b \ \text{ and } \ ab=a(c-a)\leqslant c^2/4. \tag{16}$$

The following formulas are well-known

$$F(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},\ c>a+b, \tag{17}$$

$$\frac{d}{dx}F(a,b;c;x)=\frac{ab}{c}F(a+1,b+1;c+1;x), \tag{18}$$

$$F(a,b;c;x)=(1-x)^{c-a-b}F(c-a,c-b;c;x), \tag{19}$$

$$BF(a,b;a+b;r)=\log\frac{e^R}{1-r}+O((1-r)\log(1-r))\ (r\to 1) \tag{20}$$

(see [1, 15.1.20, 15.2.1,15.3.3, & 15.3.10] and [4, 5]). From [1, 15.3.10], (5) and (9), we obtain the following refinement of (20)

$$BF(r)=[1+ab(1-r)]\log\frac{e^R}{1-r}+(2ab-a-b)(1-r)+O\left((1-r)^2\log(1-r)\right). \tag{21}$$

It follows from [1, 15.1.4], (17)–(19) and the third equality in (12) that

$$F'(r)=\frac{\alpha G(r)}{1-r},\ F_0(r)=\frac{\operatorname{arth}(\sqrt{r})}{\sqrt{r}},\ F_0'(r)=\frac{G_0(r)}{3(1-r)}=\frac{\sqrt{r}-(1-r)\operatorname{arth}(\sqrt{r})}{2r^{3/2}(1-r)}, \tag{22}$$

$$\overline{F}_0'(r)=\frac{3\overline{G}_0(r)}{16(1-r)},\ G_0(1)=\frac{3}{2},\ G(1)=\frac{1}{\alpha B},\ \overline{G}_0(1)=\frac{8\sqrt{2}}{3\pi}. \tag{23}$$

One kind of the important properties of the zero-balanced hypergeometric functions are their transformation identities. In addition to the well-known Landen transformation identities [1, 8, 11]

$$F\left(\frac{1}{2},\frac{1}{2},1;\frac{4r}{(1+r)^2}\right)=(1+r)F\left(\frac{1}{2},\frac{1}{2},1;r^2\right), \tag{24}$$

$$F\left(\frac{1}{2},\frac{1}{2},1;\left(\frac{1-r}{1+r}\right)^2\right)=\frac{1+r}{2}F\left(\frac{1}{2},\frac{1}{2},1;r'^2\right), \tag{25}$$

many other beautiful transformation identities can be found in [1, 8, 11]. For instance, for $r\in(0,1)$, the following quadratic transformation identities hold

$$F\left(\frac{1}{2},1;\frac{3}{2};\frac{4r}{(1+r)^2}\right)=(1+r)F\left(\frac{1}{2},1;\frac{3}{2};r\right), \tag{26}$$

$$F\left(\frac{1}{2},1;\frac{3}{2};\frac{1-r}{1+r}\right)=\frac{1+r}{2}F\left(\frac{1}{2},1;\frac{3}{2};r'^2\right), \tag{27}$$

$$F\left(\frac{1}{4},\frac{3}{4};1;\left(\frac{1-r}{1+3r}\right)^2\right)=\frac{\sqrt{1+3r}}{2}F\left(\frac{1}{4},\frac{3}{4};1;r'^2\right), \tag{28}$$

$$F\left(\frac{1}{4},\frac{3}{4};1;1-\left(\frac{1-r}{1+3r}\right)^2\right)=\sqrt{1+3r}F\left(\frac{1}{4},\frac{3}{4};1;r^2\right), \tag{29}$$

where (26) and (27) are the special cases of [1, 15.3.19] (see also [8, 25]), while (28) and (29) were proved in [11, Theorem 9.4] (see also [8, 25]). It is natural to raise the following Problem 1.1.

PROBLEM 1.1. Can we extend the transformation identities above-mentioned such as (26)–(29) to zero-balanced hypergeometric function $F(a,b;a+b;r)$ for $a,b\in(0,\infty)$ and $r\in(0,1)$?

During the past few years, several authors studied this problem, and many results have been obtained in the literature [32, 37, 39, 40, 42, 43, 48, 61]. For example, Simić and Vuorinen proved several extensions of (24) and (25) to zero-balanced hypergeometric functions in [37], while Wang and Chu [39] studied the problem of the generalizations of (26)–(29) and obtained several results, some of which are the following two theorems (with some simplifications here for their formulations).

THEOREM 1.2. ([39, Theorem 3.5]) *For $a,b\in(0,\infty)$ with $c=a+b$ and $r\in(0,1)$, if $ab\leqslant\min\{1/2,c/3\}$, then*

$$0\leqslant(1+r)F(a,b;c;r)-F\left(a,b;c;\frac{4r}{(1+r)^2}\right)\leqslant\frac{R-\log 4}{B}, \tag{30}$$

and if $ab\geqslant\max\{1/2,c/3\}$, then each inequality in (30) is reversed.

THEOREM 1.3. ([39, Theorem 4.5]) *For* $a,b \in (0,\infty)$, $c = a+b$ *and for* $r \in (0,1)$, *if* $ab \leqslant \min\{3/16, c/16\}$, *then*

$$0 \leqslant \sqrt{1+3r}F\left(a,b;c;r^2\right) - F\left(a,b;c;1-\left(\frac{1-r}{1+3r}\right)^2\right) \leqslant \frac{R-\log 64}{B}, \tag{31}$$

and if $ab \geqslant \max\{3/16, c/16\}$, *then each inequality in* (31) *is reversed.*

However, the known results concerning the extensions of (26)–(29) are neither sharp nor complete. This may be due to the lack of the known properties of $R(a,b)$ and the innovation in methodology.

The main purpose of the article is to study the problem of extending (26)–(29) to zero-balanced hypergeometric functions, give complete solutions to Problem 1.1 in this case, and substantially improve all the known related results such as Theorems 1.2 and 1.3. (See the results proved in Sections 4–5.) In addition, the authors will obtain several new properties of the Ramanujan type constant $R(a,b)$ and the hypergeometric functions in Sections 2–3, including the relations between two Ramanujan constants $R(a,b)$ and $R(a_1,b_1)$ and between two hypergeometric functions with distinct parameters (a,b) and (a_1,b_1), monotonicity properties and sharp functional inequalities, which play a key role in the proofs of our results obtained in Sections 4–5 and yield some properties of $\mathscr{K}(r)$, $\mathscr{E}(r)$, $\mathscr{K}_a(r)$ and $\mathscr{E}_a(r)$ (See Section 7). As examples of applications of these results, several quadratic transformation properties of the generalized Grötzsch ring function, which appears in Ramanujan's modular equations, are obtained in Section 6.

2. Preliminaries

In this section, we shall give several lemmas showing some properties of $R(a,b)$ and hypergeometric functions. First, we show some properties of $R(a,b)$.

LEMMA 2.1. *(1) For each* $c \in (0,\infty)$, *as functions of* a, $g_1(a) \equiv R_c(a) = -2\gamma - \psi(a) - \psi(c-a)$ *and* $g_2(a) \equiv B(a,c-a)$ *are both strictly decreasing and convex on* $(0,c/2]$.

(2) For $a,b \in (0,\infty)$, $R(a,b)$ *can be expressed by the following function of* $x = ab$ *and* $c = a+b$ *or a function of* $\alpha = ab/c$ *and* c

$$R(a,b) = g_3(x,c) \equiv \frac{c}{x} - \sum_{k=1}^{\infty} \frac{ck+2x}{k(k^2+ck+x)} \tag{32}$$

$$= g_4(\alpha,c) \equiv \frac{1}{\alpha} - \sum_{k=1}^{\infty} \frac{k+2\alpha}{k[(k^2/c)+k+\alpha]}. \tag{33}$$

Moreover, g_3 *is strictly decreasing and convex both in* $x \in (0,c^2/4]$ *and in* $c \in (0,\infty)$, *with* $g_3(0^+,c) = \infty$, $g_5(c) \equiv g_3\left(c^2/4, c\right)$ *is strictly decreasing and convex from* $(0,\infty)$ *onto* $(-\infty,\infty)$ *with* $g_5(1) = R(1/2) = \log 16$ *and* $g_5(2) = R(1,1) = 0$.

(3) For each $c \in (0,\infty)$, *the function* $g_6(x) \equiv x g_3(x,c)$ *is strictly decreasing in* x *from* $(0,\infty)$ *onto* $(-\infty,c)$.

(4) g_4 *is strictly decreasing and convex in* $\alpha \in (0,\infty)$, *and in* $c \in (0,\infty)$.

Proof. Parts (1)–(3) except for (33) were proved in [30, Lemma 2.1], while (33) is clear.

Part (4) follows from the partial derivative

$$\frac{\partial g_4}{\partial \alpha} = -\left\{\frac{1}{\alpha^2} + \sum_{k=1}^{\infty} \frac{(2k/c)+1}{[(k^2/c)+k+\alpha]^2}\right\}, \quad \frac{\partial g_4}{\partial c} = -\sum_{k=1}^{\infty} \frac{k(k+2\alpha)}{(k^2+ck+c\alpha)^2}. \quad \square$$

THEOREM 2.2. *Let* $a,b,a_1,b_1 \in (0,\infty)$, $c = a+b$ *and* $c_1 = a_1 + b_1$. *Then the following statements are true:*

(1) If $ab \leqslant a_1 b_1$ *and* $c \leqslant c_1$, *then*

$$R(a,b) \geqslant R(a_1,b_1), \tag{34}$$

with equality if and only if $(a,b) = (a_1,b_1)$.

(2) If $ab \geqslant \max\{a_1b_1, c\alpha_1\} = c\alpha_1$, *then the inequality (34) is reversed.*

(3) In other case not stated in parts (1)–(2), that is, $a_1b_1 < ab < c\alpha_1$, *then* $R(a,b)$ *and* $R(a_1,b_1)$ *are not directly comparable, namely neither (34) nor its reversed inequality holds for all* $a,b,a_1,b_1 \in (0,\infty)$.

Proof. (1) It follows from Lemma 2.1(2) that

$$R(a_1,b_1) - R(a,b) = \sum_{k=0}^{\infty} \frac{(c-c_1)k^2 + 2(ab-a_1b_1)k + c_1(ab - c\alpha_1)}{(k^2+ck+ab)(k^2+c_1k+a_1b_1)}, \tag{35}$$

and if $ab = a_1b_1$, then

$$R(a,b) = g_3(ab,c) = g_3(a_1b_1,c) \begin{cases} > g_3(a_1b_1,c_1) = R_1, & \text{if } c < c_1, \\ = g_3(a_1b_1,c_1) = R_1, & \text{if } c = c_1, \\ < g_3(a_1b_1,c_1) = R_1, & \text{if } c > c_1. \end{cases} \tag{36}$$

Clearly, the conditions $ab \leqslant a_1b_1$ and $c \leqslant c_1$ imply that one of the following three conditions is fulfilled:
(i) $ab = a_1b_1$ and $c \leqslant c_1$,
(ii) $ab \leqslant \min\{a_1b_1, c\alpha_1\} = c\alpha_1$,
(iii) $c\alpha_1 < ab < a_1b_1$.
If $ab = a_1b_1$ and $c \leqslant c_1$, or if $ab \leqslant \min\{a_1b_1, c\alpha_1\}$, then (34) follows from (35) and (36).

Next, let g_3 be given as in Lemma 2.1. Note that if $c\alpha_1 < ab < a_1b_1$, then $c < c_1$, and hence by Lemma 2.1(2),

$$R(a,b) = g_3(ab,c) > g_3(a_1b_1,c_1) = R(a_1,b_1),$$

showing that (34) holds.

From the above discussion, we can easily see that the equality in (34) holds if and only if $(a,b)=(a_1,b_1)$.

(2) Since $\max\{a_1b_1,c\alpha_1\}=c\alpha_1$ implies that $c\geqslant c_1$, part (2) follows from (35) and (36).

(3) It is easy to see that the remaining case not stated in parts (1)–(2) is that a,b,a_1,b_1 satisfy the condition $a_1b_1<ab<c\alpha_1$, which implies that $c>c_1$. By (32),

$$\lim_{c\to\infty}\lim_{ab\to(a_1b_1)^+}R(a,b)=\lim_{c\to\infty}c\left[\frac{1}{a_1b_1}-\sum_{k=1}^{\infty}\frac{k+2a_1b_1/c}{k\,(k^2+ck+a_1b_1)}\right]=\infty,$$
$$\lim_{c\to\infty}\lim_{ab\to(c\alpha_1)^-}R(a,b)=\lim_{c\to\infty}\left[\frac{1}{\alpha_1}-\sum_{k=1}^{\infty}\frac{c(k+2\alpha_1)}{k\,(k^2+ck+c\alpha_1)}\right]=-\infty.$$

This shows that $R(a,b)>R(a_1,b_1)$ ($R(a,b)<R(a_1,b_1)$) when ab is close to a_1b_1 and c is sufficiently large (ab is close to $c\alpha_1$ and c is sufficiently large, respectively). Hence part (3) follows.

Given the values of a_1 and b_1 in Theorem 2.2, one can obtain the corresponding comparison of $R(a,b)$ and the value $R(a_1b_1)$. For example, one can easily obtain the following

COROLLARY 2.3. *(1) For $a,b,a_1,b_1\in(0,\infty)$ with $c=a+b$ and $c_1=a_1+b_1$, if $4ab\leqslant 1$, then*

$$R(a,b)\geqslant\log 16, \tag{37}$$

with the equality if and only if $a=b=1/2$. If $4ab\geqslant c$, then the inequality (37) is reversed.

(2) For $a,b,a_1,b_1\in(0,\infty)$ with $c=a+b$ and $c_1=a_1+b_1$, if $1<4ab<c$, then $R(a,b)$ and $\log 16$ are not directly comparable, that is, neither (37) nor its inverse inequality holds for all $a,b>0$ with $1<4ab<c$.

As we know, [26, Lemma 2.1] gives an effective tool for us to show the monotonicity properties of a ratio of two power series. In [61, Theorem 2.1] (see also [39, Lemma 1.1]), Yang, Chu and Wang proved a good criterion for the monotonicity of the quotient $\varphi(x)\equiv A(x)/B(x)$, where $A=A(x)=\sum_{n=0}^{\infty}a_nx^n$ and $B=B(x)=\sum_{n=0}^{\infty}b_nx^n$ have a common radius r of convergence. They use the sign of $H_{A,B}(r^-)$ of the function $H_{A,B}=(A'B/B')-A$ to determine the monotonicity properties of φ. Since $H_{A,B}(x)=B(x)^2\varphi'(x)/B'(x)$, it is easy to see that [61, Theorem 2.1] can be changed to the following more natural and convenient conclusions.

LEMMA 2.4. *Suppose that the real power series $A(x)=\sum_{n=0}^{\infty}a_nx^n$ and $B(x)=\sum_{n=0}^{\infty}b_nx^n$ with $b_n>0$ are of a common radius $r\in(0,\infty)$ of convergence, and $\{a_n/b_n\}$ is a non-constant sequence. Let $\varphi(x)=A(x)/B(x)$.*

(1) If there is an $n_0\in\mathbb{N}$ such that the sequence $\{a_n/b_n\}$ is increasing (decreasing) for $0\leqslant n\leqslant n_0$, and decreasing (increasing) for $n\geqslant n_0$, then φ is increasing (decreasing) on $(0,r)$ if and only if $\varphi'(r^-)\geqslant 0$ ($\varphi'(r^-)\leqslant 0$, respectively).

(2) If there is an $n_0 \in \mathbb{N}$ such that the sequence $\{a_n/b_n\}$ is increasing (decreasing) for $0 \leqslant n \leqslant n_0$, and decreasing (increasing) for $n \geqslant n_0$, and if $\varphi'(r^-) < 0$ ($\varphi'(r^-) > 0$), then there exists a number $x_0 \in (0,r)$ such that φ is strictly increasing (decreasing) on $(0,x_0]$ and decreasing (increasing, respectively) on $[x_0,r)$.

LEMMA 2.5. *For $a,b \in (0,\infty)$ with $c = a+b$ and $d \in (-\infty,\infty)$, the function $g_7(r) \equiv (1-r)^d F(a,b;c;r)$ is increasing (decreasing) on $(0,1)$ if and only if $d \leqslant 0$ ($d \geqslant \alpha$).*

Proof. Let $d_n = (a,n)(b,n)/[(c+1,n)n!]$ and $\overline{d}_n = (a,n)(b,n)/[(c,n)n!]$. By differentiation and (22),

$$\frac{(1-r)^{1-d}}{F(r)} g_7'(r) = \alpha g_8(r) - d, \; g_8(r) = \frac{G(r)}{F(r)} = \frac{\sum_{n=0}^{\infty} d_n r^n}{\sum_{n=0}^{\infty} \overline{d}_n r^n}. \tag{38}$$

Clearly, $g_8(0) = 1$ and $g_8(1^-) = 0$. Since $d_n/\overline{d}_n = c/(n+c)$ is strictly decreasing in $n \in \mathbb{N}_0$, g_8 is strictly decreasing on $(0,1)$ by [26, Lemma 2.1]. Hence by (38),

$$g_7'(r) > 0 \Leftrightarrow d \leqslant \alpha \inf_{0<r<1}\{g_8(r)\} = 0, \quad g_7'(r) < 0 \Leftrightarrow d \geqslant \alpha \sup_{0<r<1}\{g_8(r)\} = \alpha.$$

In [39, Lemma 2.2], some monotonicity properties of $f_1(r) \equiv F(r)/F_1(r)$ and $f_2(r) \equiv G(r)/G_1(r)$, for $r \in (0,1)$, were obtained. However, the formulation of the conditions in [39, Lemma 2.2] is not simple and clear enough, the results for f_2 are not complete, and the proof of [39, Lemma 2.2] is not natural, because of lack of the help of Lemma 2.1 and Theorem 2.2. For this reason, we prove the following results.

THEOREM 2.6. *For $a,b,a_1,b_1 \in (0,\infty)$ with $c = a+b$ and $c_1 = a_1+b_1$, and for $r \in (0,1)$, let $f_1(r) = F(r)/F_1(r)$ and $f_2(r) = G(r)/G_1(r)$. Then we have the following conclusions:*

(1) If $ab \leqslant \min\{a_1b_1, c\alpha_1\}$, or if $a_1b_1 < ab < c\alpha_1$ with $R \geqslant R_1$, then f_1 is decreasing from $[0,1)$ onto $(B_1/B,1]$. Moreover, if $(a,b) \neq (a_1,b_1)$, then the monotonicity of f_1 is strict.

(2) If $ab \geqslant \max\{a_1b_1, c\alpha_1\}$, then f_1 is increasing from $[0,1)$ onto $[1,B_1/B)$. Moreover, if $(a,b) \neq (a_1,b_1)$, then the monotonicity of f_1 is strict.

(3) In other cases not stated in parts (1)–(2), namely $a_1b_1 < ab < c\alpha_1$ with $R < R_1$ (or $c\alpha_1 < ab < a_1b_1$), there exists a number $r_1 = r_1(a,b,a_1,b_1) \in (0,1)$ ($r_2 = r_2(a,b,a_1,b_1) \in (0,1)$) such that f_1 is decreasing (increasing) on $(0,r_1]$ ($(0,r_2]$), and increasing (decreasing) on $[r_1,1)$ ($[r_2,1)$, respectively). If $c \leqslant 4\alpha_1$, then the case "$c\alpha_1 < ab < a_1b_1$" does not appear, and in particular, if $c \leqslant 1$ and $\alpha_1 \geqslant 1/4$, then the case "$c\alpha_1 < ab < a_1b_1$" does not appear.

(4) If $ab \leqslant \min\{a_1b_1+c_1-c, (c+1)\overline{\alpha}_1\}$ or $a_1b_1+c_1-c < ab \leqslant a_1b_1$, then f_2 is decreasing from $[0,1)$ onto $(\alpha_1B_1/(\alpha B),1]$. Moreover, if $(a,b) \neq (a_1,b_1)$, then the monotonicity of f_2 is strict.

(5) If $ab \geqslant \max\{a_1b_1+c_1-c, (c+1)\overline{\alpha}_1\}$ or $a_1b_1 \leqslant ab < a_1b_1+c_1-c$, then f_2 is increasing from $[0,1)$ onto $[1,\alpha_1B_1/(\alpha B))$. Moreover, if $(a,b) \neq (a_1,b_1)$, then the monotonicity of f_2 is strict.

(6) In other cases not stated in parts (4)–(5), namely $a_1b_1 < ab < (c+1)\overline{\alpha}_1$ (or $(c+1)\overline{\alpha}_1 < ab < a_1b_1$), there exists a number $r_3 = r_3(a,b,a_1,b_1) \in (0,1)$ ($r_4 = r_4(a,b,a_1,b_1) \in (0,1)$) such that f_2 is decreasing (increasing) on $(0,r_3]$ ($(0,r_4]$) and increasing (decreasing) on $[r_3,1)$ ($[r_4,1)$, respectively). If $c^2 \leqslant 4(c+1)\overline{\alpha}_1$, then the case "$(c+1)\overline{\alpha}_1 < ab < a_1b_1$" does not appear. In particular, if $c \leqslant 1$ and $\overline{\alpha}_1 \geqslant 1/8$, then the case "$(c+1)\overline{\alpha}_1 < ab < a_1b_1$" does not appear.

Proof. Clearly, $f_1(0) = f_2(0) = 1$. By (20) and (23), $f_1(1^-) = B_1/B$ and $f_2(1) = \alpha_1 B_1/(\alpha B)$. Differentiation gives

$$f_1'(r) = \frac{\alpha G(r)F_1(r) - \alpha_1 F(r)G_1(r)}{(1-r)F_1(r)^2}, \tag{39}$$

$$f_2'(r) = \frac{\overline{\alpha}F_+(r)G_1(r) - \overline{\alpha}_1 F_{1+}(r)G(r)}{G_1(r)^2}. \tag{40}$$

By (23), $\alpha BR_1G(1) - \alpha_1 B_1 RG_1(1) = R_1 - R$. By (15), (20) and (22), and by l'Hôpital's rule,

$$\begin{aligned}
\lim_{r\to 1} \frac{\alpha BG(r) - \alpha_1 B_1 G_1(r)}{(1-r)F_1(r)} &= \lim_{r\to 1} \frac{\alpha_1\overline{\alpha}_1 B_1 F_{1+}(r) - \alpha\overline{\alpha} BF_+(r)}{F_1(r) - \alpha_1 G_1(r)} \\
&= \lim_{r\to 1} \left[\alpha_1\overline{\alpha}_1 B_1 \frac{F_{1+}(r)}{F_1(r)} - \alpha\overline{\alpha}B\frac{F_+(r)}{F_1(r)}\right] \\
&= (a_1b_1 - ab)B_1, \\
\lim_{r\to 1} \frac{\alpha BR_1G(r) - \alpha_1 B_1 RG_1(r)}{(1-r)F_1(r)^2} &= 0 \text{ if } R = R_1.
\end{aligned}$$

Hence by (15), (20), (22)–(23) and (39), we obtain the limiting value

$$\begin{aligned}
f_1'(1^-) =& \frac{1}{BB_1} \lim_{r\to 1} \frac{\alpha BG(r)\log\left(e^{R_1}/(1-r)\right) - \alpha_1 B_1 G_1(r)\log\left(e^{R}/(1-r)\right)}{(1-r)F_1(r)^2} \\
=& \frac{1}{B} \lim_{r\to 1} \left[\frac{\alpha BR_1G(r) - \alpha_1 B_1 RG_1(r)}{B_1(1-r)F_1(r)^2} + \frac{\alpha BG(r) - \alpha_1 B_1 G_1(r)}{(1-r)F_1(r)} \cdot \frac{\log[1/(1-r)]}{\log\left[e^{R_1}/(1-r)\right]}\right] \\
=& \frac{B_1}{B}(a_1b_1 - ab) + \frac{1}{BB_1} \lim_{r\to 1} \frac{\alpha BR_1G(r) - \alpha_1 B_1 RG_1(r)}{(1-r)F_1(r)^2} \\
=& \begin{cases} -\infty, & R > R_1, \\ B_1(a_1b_1 - ab)/B, & R = R_1, \\ \infty, & R < R_1. \end{cases}
\end{aligned} \tag{41}$$

Put $D_1 = \alpha_1 B_1(abR - a_1b_1R_1 + c_1 - c)/(\alpha B)$. Then (23) leads to

$$\frac{(abR - c)G_1(1)}{\alpha B} - \frac{(a_1b_1R_1 - c_1)G(1)}{\alpha_1 B_1} = \frac{D_1}{(\alpha_1 B_1)^2},$$

$$cB_1G_1(1) - c_1BG(1) = \frac{ab - a_1b_1}{\alpha\alpha_1},$$

and by (20) and l'Hôpital's rule,

$$\lim_{r\to 1}\frac{cB_1G_1(r)-c_1BG(r)}{r'}=0 \text{ if } ab=a_1b_1.$$

Hence by (15), (20), (22) and (40), we obtain the limiting value

$$\begin{aligned} f_2'(1^-)=&\frac{1}{G_1(1)^2}\lim_{r\to 1}\left[\frac{\overline{\alpha}G_1(r)}{\overline{B}}\log\frac{e^{\overline{R}}}{1-r}-\frac{\overline{\alpha}_1G(r)}{\overline{B}_1}\log\frac{e^{\overline{R}_1}}{1-r}\right]\\ =&D_1+\frac{\alpha_1^2B_1}{B}\lim_{r\to 1}[cB_1G_1(r)-c_1BG(r)]\log\frac{1}{1-r}\\ =&\begin{cases}-\infty, & ab<a_1b_1,\\ D_1, & ab=a_1b_1,\\ \infty, & ab>a_1b_1.\end{cases} \end{aligned} \tag{42}$$

By Theorem 2.2(1)–(2), if $ab=a_1b_1$, then

$$D_1=\frac{cB_1}{c_1B}[ab(R-R_1)+c_1-c]\begin{cases}<0, & \text{if } c_1<c,\\ >0, & \text{if } c_1>c.\end{cases} \tag{43}$$

Next, for $a,b,a_1,b_1\in(0,\infty)$ with $c=a+b$ and $c_1=a_1+b_1$, and for $n\in\mathbb{N}_0$, let

$$\begin{aligned} &\widetilde{a}_n=\frac{(a,n)(b,n)}{(c,n)n!},\quad \widetilde{b}_n=\frac{(a_1,n)(b_1,n)}{(c_1,n)n!},\quad \widetilde{c}_n=\frac{\widetilde{a}_n}{\widetilde{b}_n},\\ &\overline{a}_n=\frac{(a,n)(b,n)}{(c+1,n)n!},\quad \overline{b}_n=\frac{(a_1,n)(b_1,n)}{(c_1+1,n)n!},\quad \overline{c}_n=\frac{\overline{a}_n}{\overline{b}_n},\\ &\Delta_1=\Delta_1(n,a,b,a_1,b_1)=(ab-a_1b_1)n+c_1(ab-c\alpha_1),\\ &\Delta_2=\Delta_2(n,a,b,a_1,b_1)=(ab+c-a_1b_1-c_1)n+ab(c_1+1)-a_1b_1(c+1)\\ &\quad=(ab+c-a_1b_1-c_1)n+(c_1+1)[ab-(c+1)\overline{\alpha}_1]. \end{aligned}$$

Then by (1),

$$f_1(r)=\frac{\sum_{n=0}^{\infty}\widetilde{a}_nr^n}{\sum_{n=0}^{\infty}\widetilde{b}_nr^n},\quad \frac{\widetilde{c}_{n+1}}{\widetilde{c}_n}=1+\frac{\Delta_1(n,a,b,a_1,b_1)}{(n+c)(n^2+c_1n+a_1b_1)}, \tag{44}$$

$$f_2(r)=\frac{\sum_{n=0}^{\infty}\overline{a}_nr^n}{\sum_{n=0}^{\infty}\overline{b}_nr^n},\quad \frac{\overline{c}_{n+1}}{\overline{c}_n}=1+\frac{\Delta_2(n,a,b,a_1,b_1)}{(n+c+1)(n^2+c_1n+a_1b_1)}. \tag{45}$$

(1) If $ab\leqslant\min\{a_1b_1,c\alpha_1\}$, then $\Delta_1(n,a,b,a_1,b_1)\leqslant 0$, so that $\widetilde{c}_n$ is decreasing in $n\in\mathbb{N}_0$ by (44). Hence f_1 is decreasing on $[0,1)$ by [26, Lemma 2.1].

If $a_1b_1<ab<c\alpha_1$ and $R\geqslant R_1$, then $\widetilde{c}_n$ is decreasing and then increasing in $n\in\mathbb{N}_0$ by (44), and $f_1'(1^-)<0$ by (41). Hence by Lemma 2.4(1), f_1 is decreasing on $[0,1)$.

(2) If $ab\geqslant\max\{a_1b_1,c\alpha_1\}$, then $\Delta_1(n,a,b,a_1,b_1)\geqslant 0$, and $\widetilde{c}_n$ is increasing in $n\in\mathbb{N}_0$ by (44). Hence f_1 is increasing on $[0,1)$ by [26, Lemma 2.1].

(3) If $a_1b_1 < ab < c\alpha_1$ and $R < R_1$, then $\widetilde{c}_n$ is decreasing and then increasing in $n \in \mathbb{N}_0$ by (44), and $f_1'(1^-) = \infty$ by (41). Hence the piecewise monotonicity of f_1 follows from Lemma 2.4(2).

If $c\alpha_1 < ab < a_1b_1$, then $c < c_1$, c_n is increasing and then decreasing in $n \in \mathbb{N}_0$ by (44), $R > R_1$ by Theorem 2.2(1), $f_1'(1^-) = -\infty$ by (41), and the piecewise monotonicity of f_1 follows from Lemma 2.4(2).

If $c \leqslant 4\alpha_1$, then $ab \leqslant c^2/4 \leqslant c\alpha_1$ by (16), so that the case "$c\alpha_1 < ab < a_1b_1$" does not appear. In particular, if $c \leqslant 1$ and $\alpha_1 \geqslant 1/4$, then $ab \leqslant c^2/4 \leqslant c^2\alpha_1 \leqslant c\alpha_1$ by (16), so that the case "$c\alpha_1 < ab < a_1b_1$" does not appear.

(4) If $ab \leqslant \min\{a_1b_1 + c_1 - c, (c+1)\overline{\alpha}_1\}$, then $\Delta_2(n,a,b,a_1,b_1) \leqslant 0$, so that $\overline{c}_n$ is decreasing in $n \in \mathbb{N}_0$ by (45). Hence f_2 is decreasing on $[0,1)$ by [26, Lemma 2.1].

If $a_1b_1 + c_1 - c < ab \leqslant a_1b_1$, then $c_1 < c$ and $ab(c_1+1) < a_1b_1(c+1)$, so that $\overline{c}_n$ is decreasing and then increasing in $n \in \mathbb{N}_0$ by (45). By (42) and (43), $f_2'(1^-) = -\infty$ if $ab < a_1b_1$, and $f_2'(1^-) = D_1 < 0$ if $ab = a_1b_1$. Hence f_2 is decreasing on $[0,1)$ by Lemma 2.4(1).

(5) If $ab \geqslant \max\{a_1b_1 + c_1 - c, (c+1)\overline{\alpha}_1\}$, then $\Delta_2(n,a,b,a_1,b_1) \geqslant 0$, so that $\overline{c}_n$ is increasing in $n \in \mathbb{N}_0$ by (45). Hence f_2 is increasing on $[0,1)$ by [26, Lemma 2.1].

If $a_1b_1 \leqslant ab < a_1b_1 + c_1 - c$, then $c_1 > c$, $ab(c_1+1) > a_1b_1(c+1)$, so that $\overline{c}_n$ is increasing and then decreasing in $n \in \mathbb{N}_0$ by (45). By (42) and (43), $f_2'(1^-) = \infty$ if $ab > a_1b_1$, and $f_2'(1^-) = D_1 > 0$ if $ab = a_1b_1$. Hence f_2 is increasing on $[0,1)$ by Lemma 2.4(1).

(6) If $a_1b_1 < ab < (c+1)\overline{\alpha}_1$, then $c > c_1$, $ab + c > a_1b_1 + c_1$ and $ab(c_1+1) < (c_1+1)(c+1)\overline{\alpha}_1 = a_1b_1(c+1)$, so that $\overline{c}_n$ in decreasing and then increasing in $n \in \mathbb{N}_0$ by (45). By (42), $f_2'(1^-) = \infty$, and hence the piecewise monotonicity of f_2 follows from Lemma 2.4(2).

If $(c+1)\overline{\alpha}_1 < ab < a_1b_1$, then $c < c_1$, $ab + c < a_1b_1 + c_1$ and $ab(c_1+1) > (c_1+1)(c+1)\overline{\alpha}_1 = a_1b_1(c+1)$, so that $\overline{c}_n$ in increasing and then decreasing in $n \in \mathbb{N}_0$ by (45), and $f_2'(1^-) = -\infty$ by (42). Hence the assertion on the piecewise monotonicity of f_2 follows from Lemma 2.4(2).

If $c^2 \leqslant 4(c+1)\overline{\alpha}_1$, then by (16), $ab \leqslant c^2/4 \leqslant (c+1)\overline{\alpha}_1$. Hence the situation "$(c+1)\overline{\alpha}_1 < ab < a_1b_1$" does not exist. In particular, if $c \leqslant 1$ and $\overline{\alpha}_1 \geqslant 1/8$, then $ab \leqslant c^2/4 \leqslant (c+1)/8 \leqslant (c+1)\overline{\alpha}_1$ by (16), so that the case "$(c+1)\overline{\alpha}_1 < ab < a_1b_1$" does not appear. $\square$

LEMMA 2.7. *For $a,b,a_1,b_1 \in (0,\infty)$, $c = a+b$, $c_1 = a_1+b_1$, and for $r \in (0,1)$, let $f_3(r) = [F(r)-1]/[F_1(r)-1]$.*

(1) If $ab \leqslant \min\{a_1b_1 + c_1 - c, (c+1)\overline{\alpha}_1\}$ or $a_1b_1 + c_1 - c < ab \leqslant a_1b_1$, then f_3 is decreasing from $(0,1)$ onto $(B_1/B, \alpha/\alpha_1)$. Moreover, if $(a,b) \neq (a_1,b_1)$, then the monotonicity of f_3 is strict. In particular, for $r \in (0,1)$,

$$1 - \frac{B_1}{B} + \frac{B_1}{B} F(a_1,b_1;c_1;r) \leqslant F(a,b;c;r) \leqslant 1 - \frac{\alpha}{\alpha_1} + \frac{\alpha}{\alpha_1} F(a_1,b_1;c_1 r), \tag{46}$$

with equality in each instance if and only if $(a,b) = (a_1,b_1)$.

(2) If $ab \geqslant \max\{a_1b_1+c_1-c,(c+1)\overline{\alpha}_1\}$ *or* $a_1b_1 \leqslant ab < a_1b_1+c_1-c$ *, then* f_3 *is increasing from* $[0,1)$ *onto* $(\alpha/\alpha_1, B_1/B)$*, and each inequality in (46) is reversed. Moreover, the monotonicity of* f_3 *is strict if* $(a,b) \neq (a_1,b_1)$*.*

Proof. Let f_2 be as in Theorem 2.6, $g_9(r)=F(r)-1$ and $g_{10}(r)=F_1(r)-1$. Then $g_9(0)=g_{10}(0)=0$ and

$$\frac{g_9'(r)}{g_{10}'(r)}=\frac{\alpha G(r)}{\alpha_1 G_1(r)}=\frac{\alpha}{\alpha_1}f_2(r). \tag{47}$$

Hence the monotonicity properties of f_3 follow from Theorem 2.6(4)–(5) and [3, Theorem 1.25].

By (47), $f_3(0^+)=\alpha/\alpha_1$, and by (20), $f_3(1^-)=B_1/B$. The remaining conclusions are clear. □

3. Some properties of hypergeometric functions

In this section, we mainly show several properties of hypergeometric functions, including their sharp bounds given in terms of elementary functions, and the relations between $F(r)$ and $F_0(r)$, $G(r)$ and $G_0(r)$, $F(r)$ and $\overline{F}_0(r)$, and between $G(r)$ and $\overline{G}_0(r)$. Some of these relations embody the stabilities of the hypergeometric functions $F_0(r)$, $G_0(r)$, $\overline{F}_0(r)$ and $\overline{G}_0(r)$ with respect to the parameters, in some extent. These results are needed in the proofs of our results in Sections 4 and 5.

First, taking $a_1=1/2$ and $b_1=1$, we can immediately obtain the following Corollaries 3.1–3.2 (Corollary 3.3) from Theorem 2.6 (Lemma 2.7, respectively) and (22)–(23).

COROLLARY 3.1. *For* $a,b\in(0,\infty)$ *with* $c=a+b$*, and for* $r\in(0,1)$*, let* $f_4(r)=F(r)/F_0(r)=\sqrt{r}F(r)/\mathrm{arth}(\sqrt{r})$*.*

(1) If $ab \leqslant \min\{1/2,c/3\}$*, or if* $1/2<ab<c/3$ *with* $R\geqslant \log 4$*, then* f_4 *is decreasing from* $[0,1)$ *onto* $(2/B,1]$*. The monotonicity of* f_4 *is strict if* $(a,b)\neq(1/2,1)$*. In particular, for* $r\in(0,1)$*,*

$$2\frac{\mathrm{arth}(r)}{Br}=\frac{2}{B}F\left(\frac{1}{2},1;\frac{3}{2};r^2\right)\leqslant F\left(a,b;c;r^2\right)\leqslant F\left(\frac{1}{2},1;\frac{3}{2};r^2\right)=\frac{\mathrm{arth}(r)}{r}, \tag{48}$$

with equality in each inequality if and only if $(a,b)=(1/2,1)$*.*

(2) If $ab\geqslant\max\{1/2,c/3\}$*, then* f_4 *is increasing from* $[0,1)$ *onto* $[1,2/B)$*, and each inequality in (48) is reversed. Moreover, the monotonicity of* f_4 *is strict if* $(a,b)\neq(1/2,1)$*.*

(3) In other cases not stated in parts (1)–(2), namely $1/2<ab<c/3$ *with* $R<\log 4$ *(or* $c/3<ab<1/2$*), there exists* $r_5=r_5(a,b)\in(0,1)$ *(*$r_6=r_6(a,b)\in(0,1)$*) such that* f_4 *is decreasing (increasing) on* $[0,r_5]$ *(*$[0,r_6]$*), and increasing (decreasing) on* $[r_5,1)$ *(*$[r_6,1)$*, respectively), with* $f_4(0)=1$ *and* $f_4(1^-)=2/B$*. If* $c\leqslant 4/3$*, then the case "*$c/3<ab<1/2$*" does not appear.*

COROLLARY 3.2. *For* $a,b\in(0,\infty)$ *with* $c=a+b$, *and for* $r\in(0,1)$, *let* $f_5(r)=G(r)/G_0(r)$.

(1) If $ab\leqslant\min\{2-c,(c+1)/5\}$ *or* $2-c<ab\leqslant 1/2$, *then* f_5 *is decreasing from* $[0,1)$ *onto* $(2/(3\alpha B),1]$. *Moreover, if* $(a,b)\neq(1/2,1)$, *then the monotonicity of* f_5 *is strict. In particular, for* $r\in(0,1)$,

$$P_0(r)/(\alpha B)\leqslant F\left(a,b;c+1;r^2\right)\leqslant 3P_0(r)/2, \tag{49}$$

with equality in each instance if and only if $a=1/2$ *and* $b=1$, *where* $P_0(r)=r^{-3}\left[r-r'^2 arth(r)\right]$.

(2) If $ab\geqslant\max\{2-c,(c+1)/5\}$ *or* $1/2\leqslant ab<2-c$, *then* f_5 *is increasing from* $[0,1)$ *onto* $[1,2/(3\alpha B))$, *and each inequality in (49) is reversed. Moreover, if* $(a,b)\neq(1/2,1)$, *then the monotonicity of* f_5 *is strict.*

(3) In other cases not stated in parts (1)–(2), that is, $1/2<ab<(c+1)/5$ *(or* $(c+1)/5<ab<1/2$*), there exists a number* $r_7=r_7(a,b)\in(0,1)$ *(*$r_8=r_8(a,b)\in(0,1)$*) such that* f_5 *is decreasing (increasing) on* $[0,r_7]$ *(*$[0,r_8]$*), and increasing (decreasing) on* $[r_7,1)$ *(*$[r_8,1)$*, respectively), with* $f_5(0)=1$ *and* $f_5(1^-)=2/(3\alpha B)$. *If* $c^2\leqslant 4(c+1)/5$, *then the case "*$(c+1)/5<ab<1/2$*" does not appear.*

COROLLARY 3.3. *For* $a,b\in(0,\infty)$ *with* $c=a+b$, *and for* $r\in(0,1)$, *let* $f_6(r)=[F(r)-1]/[F_0(r)-1]$.

(1) If $ab\leqslant\min\{2-c,(c+1)/5\}$ *or* $2-c<ab\leqslant 1/2$, *then* f_6 *is decreasing from* $(0,1)$ *onto* $(2/B,3\alpha)$. *Moreover, if* $(a,b)\neq(1/2,1)$, *then the monotonicity of* f_6 *is strict. In particular, for* $r\in(0,1)$,

$$1+\frac{2}{B}\left[\frac{arth(r)}{r}-1\right]\leqslant F\left(a,b;c;r^2\right)\leqslant 1+3\alpha\left[\frac{arth(r)}{r}-1\right], \tag{50}$$

with equality in each instance if and only if $a=1/2$ *and* $b=1$, *and*

$$\frac{\pi}{2}-1+\frac{arth(r)}{r}<\mathscr{K}(r)<\frac{\pi}{8}+3\pi\frac{arth(r)}{8r}. \tag{51}$$

(2) If $ab\geqslant\max\{2-c,(c+1)/5\}$ *or* $1/2\leqslant ab<2-c$, *then* f_6 *is increasing from* $[0,1)$ *onto* $(3\alpha,2/B)$, *and each inequality in (50) is reversed. Moreover, if* $(a,b)\neq(1/2,1)$, *then the monotonicity of* f_6 *is strict.*

COROLLARY 3.4. *For* $r\in(0,1)$, $D_2=\sqrt{2}/\pi=0.45015\cdots$ *and* $D_3=8\sqrt{2}/3\pi=1.20042\cdots$,

$$D_2\frac{arth(r)}{r}<F\left(\frac{1}{4},\frac{3}{4};1;r^2\right)<\frac{arth(r)}{r}, \tag{52}$$

$$D_3\frac{r-r'^2 arth(r)}{r^3}<F\left(\frac{1}{4},\frac{3}{4};2;r^2\right)<3\frac{r-r'^2 arth(r)}{2r^3}. \tag{53}$$

The coefficients of the lower and upper bounds in (52) and (53) are all best possible.

Proof. Take $a=1/4$ and $b=3/4$ in Corollary 3.1(1). Then $c=1$, $B(1/4,3/4)=\sqrt{2}\pi$ by (13), $ab=\alpha=3/16<1/3$, and hence (52) follows from Corollary 3.1(1). The coefficients of the lower and upper bounds in (52) are both best possible, since $\lim_{r\to 0}\overline{F}_0(r)/F_0(r)=1$ and $\lim_{r\to 1}\overline{F}_0(r)/F_0(r)=\sqrt{2}/\pi$ by (20).

Similarly, the remaining conclusions follow from Corollary 3.2(1). □

COROLLARY 3.5. *For $a,b\in(0,\infty)$ with $c=a+b$, and for $r\in(0,1)$, let $f_7(r)=F(r)/\overline{F}_0(r)$.*

(1) If $16ab/3\leqslant\min\{1,c\}$, or if $1<16ab/3<c$ with $R\geqslant\log 64$, then f_7 is decreasing from $[0,1)$ onto $(\sqrt{2}\pi/B,1]$, so that for $r\in(0,1)$,

$$\frac{\sqrt{2}\pi}{B}F\left(\frac{1}{4},\frac{3}{4};1;r^2\right)\leqslant F\left(a,b;c;r^2\right)\leqslant F\left(\frac{1}{4},\frac{3}{4};1;r^2\right), \tag{54}$$

with equality in each instance if and only if $a=1/4=b/3$. The monotonicity of f_7 is strict if $(a,b)\neq(1/4,3/4)$.

(2) If $16ab/3\geqslant\max\{1,c\}$, then f_7 is increasing from $[0,1)$ onto $[1,\sqrt{2}\pi/B)$, so that each inequality in (54) is reversed. Moreover, if $(a,b)\neq(1/4,3/4)$, then the monotonicity of f_7 is strict.

(3) In other cases not stated in parts (1)–(2), namely $1<16ab/3<c$ with $R<\log 64$ (or $c<16ab/3<1$), there exists $r_9=r_9(a,b)\in(0,1)$ ($r_{10}=r_{10}(a,b)\in(0,1)$) such that f_7 is decreasing (increasing) on $[0,r_9]$ ($[0,r_{10}]$), and increasing (decreasing) on $[r_9,1)$ ($[r_{10},1)$, respectively), with $f_7(0)=1$ and $f_7(1^-)=\sqrt{2}\pi/B$. If $c\leqslant 3/4$, then the case "$c<16ab/3<1$" does not appear.

Proof. The results follow from Theorem 2.6(1)–(3) with $a_1=1/4$ and $b_1=3/4$ and (13). □

Similarly, Theorem 2.6(4)–(6) with $a_1=1/4$ and $b_1=3/4$, (13) and (53) yield the following corollary.

COROLLARY 3.6. *For $a,b\in(0,\infty)$ with $c=a+b$, and for $r\in(0,1)$, let $f_8(r)=G(r)/\overline{G}_0(r)$.*

(1) If $ab\leqslant\min\{(19/16)-c,3(c+1)/32\}$ or $(19/16)-c<ab\leqslant 3/16$, then f_8 is decreasing from $[0,1)$ onto $(3\sqrt{2}\pi/(16\alpha B),1]$. The monotonicity of f_8 is strict if $(a,b)\neq(1/4,3/4)$. In particular, for $r\in(0,1)$,

$$\frac{3\sqrt{2}\pi}{16\alpha B}F\left(\frac{1}{4},\frac{3}{4};2;r^2\right)\leqslant F\left(a,b;c+1;r^2\right)\leqslant F\left(\frac{1}{4},\frac{3}{4};2;r^2\right), \tag{55}$$

with equality in each instance if and only if $(a,b)=(1/4,3/4)$. Moreover, the coefficients $1/(\alpha B)$ and 3/2 of the lower and upper bounds in (55) are both best possible.

(2) If $ab\geqslant\max\{(19/16)-c,3(c+1)/32\}$ or $3/16\leqslant ab<(19/16)-c$, then f_8 is increasing from $[0,1)$ onto $[1,3\sqrt{2}\pi/(16\alpha B))$, so that each inequality in (55) is reversed. Moreover, the monotonicity of f_8 is strict if $(a,b)\neq(1/4,3/4)$.

(3) In other cases not stated in parts (1)–(2), that is, $3/16 < ab < 3(c+1)/32$ *(or* $3(c+1)/32 < ab < 3/16$*), there exists a number* $r_{11} = r_{11}(a,b) \in (0,1)$ *(*$r_{12} = r_{12}(a,b) \in (0,1)$*) such that* f_8 *is decreasing (increasing) on* $[0, r_{11}]$ *(*$[0, r_{12}]$*), and increasing (decreasing) on* $[r_{11}, 1)$ *(*$[r_{12}, 1)$*, respectively) with* $f_8(0) = 1$ *and* $f_8(1^-) = 3\sqrt{2}\pi/(16\alpha B)$*. If* $c^2 \leqslant 3(c+1)/8$*, then the case "*$3(c+1)/32 < ab < 3/16$*" does not appear.*

Taking $a_1 = 1/4$ and $b_1 = 3/4$ in Lemma 2.7, and applying (13), we obtain the following corollary.

COROLLARY 3.7. *For* $a, b \in (0, \infty)$ *with* $c = a + b$*, and for* $r \in (0,1)$*, let* $f_9(r) = [F(r) - 1]/[\overline{F}_0(r) - 1]$*.*

(1) If $ab \leqslant \min\{(19/16) - c, 3(c+1)/32\}$ *or* $(19/16) - c < ab \leqslant 3/16$*, then* f_9 *is decreasing from* $(0,1)$ *onto* $(\sqrt{2}\pi/B, 16\alpha/3)$*. Moreover, the monotonicity of* f_9 *is strict if* $(a,b) \neq (1/4, 3/4)$*. In particular, for* $r \in (0,1)$*,*

$$1 + \frac{\sqrt{2}\pi}{B}\left[F\left(\frac{1}{4}, \frac{3}{4}; 1; r^2\right) - 1\right] \leqslant F\left(a, b; c; r^2\right) \leqslant 1 + \frac{16\alpha}{3}\left[F\left(\frac{1}{4}, \frac{3}{4}; 1; r^2\right) - 1\right], \tag{56}$$

with equality in each instance if and only if $a = 1/4$ *and* $b = 3/4$*.*

(2) If $ab \geqslant \max\{(19/16) - c, 3(c+1)/32\}$ *or* $3/16 \leqslant ab < (19/16) - c$*, then* f_9 *is increasing from* $(0,1)$ *onto* $(16\alpha/3, \sqrt{2}\pi/B)$*, so that each inequality in (56) is reversed. Moreover, the monotonicity of* f_9 *is strict if* $(a,b) \neq (1/4, 3/4)$*.*

Next, we present some more properties of zero-balanced hypergeometric functions. The following theorem and its corollaries 3.9–3.10 play a key role in the proofs of our results obtained in Section 4.

THEOREM 3.8. *For* $a, b, a_1, b_1 \in (0, \infty)$ *with* $c = a + b$ *and* $c_1 = a_1 + b_1$*, let* r_3 *and* r_4 *be as in Theorem 2.6,* $\xi = 1 - \alpha/\alpha_1$*,* $\delta = (R - R_1)/B$*, and for* $r \in (0,1)$*, let* $f_{10}(r) = F(r) - F_1(r)F'(r)/F_1'(r)$*.*

(1) If $ab \leqslant \min\{a_1b_1 + c_1 - c, (c+1)\overline{\alpha}_1\}$ *or* $a_1b_1 + c_1 - c < ab \leqslant a_1b_1$*, then* f_{10} *is increasing from* $(0,1)$ *onto* (ξ, δ)*. Moreover, if* $(a,b) \neq (a_1, b_1)$*, then the monotonicity of* f_{10} *is strict. In particular, for* $r \in (0,1)$*,*

$$\xi + \frac{B_1}{B}F(a_1, b_1; c_1; r) \leqslant F(a, b; c; r) \leqslant \delta + \frac{\alpha}{\alpha_1}F(a_1, b_1; c_1; r), \tag{57}$$

with equality in each instance if and only if $a = a_1$ *and* $b = b_1$*.*

(2) If $ab \geqslant \max\{a_1b_1 + c_1 - c, (c+1)\overline{\alpha}_1\}$ *or* $a_1b_1 \leqslant ab < a_1b_1 + c_1 - c$*, then* f_{10} *is decreasing from* $(0,1)$ *onto* (δ, ξ)*, and each inequality in (57) is reversed. Moreover, if* $(a,b) \neq (a_1, b_1)$*, then the monotonicity of* f_{10} *is strict.*

(3) In other cases not stated in parts (1)–(2), that is, $a_1b_1 < ab < (c+1)\overline{\alpha}_1$ *(or* $(c+1)\overline{\alpha}_1 < ab < a_1b_1$*),* f_{10} *is increasing (decreasing) on* $[0, r_3]$ *(*$[0, r_4]$*), and decreasing (increasing) on* $[r_3, 1)$ *(*$[r_4, 1)$*, respectively). If* $c^2 \leqslant 4(c+1)\overline{\alpha}_1$*, then the case "*$(c+1)\overline{\alpha}_1 < ab < a_1b_1$*" does not appear.*

Proof. Let f_2 be as in Theorem 2.6. Then by (22), $F'(r)/F_1'(r) = \alpha G(r)/[\alpha_1 G_1(r)] = \alpha f_2(r)/\alpha_1$, so that

$$f_{10}(r) = F(r) - F_1(r)\frac{\alpha G(r)}{\alpha_1 G_1(r)} = F(r) - \frac{\alpha}{\alpha_1}F_1(r)f_2(r), \tag{58}$$

$$f_{10}'(r) = -F_1(r)\frac{d}{dr}\left[\frac{F'(r)}{F_1'(r)}\right] = -\frac{\alpha}{\alpha_1}F_1(r)f_2'(r). \tag{59}$$

Hence the monotonicity properties of f_{10}, given in parts (1)–(3), follow from (59) and Theorem 2.6(4)–(6).

By (58), we see that $f_{10}(0) = 1-\alpha/\alpha_1$. Since $\lim_{r\to 1}[\alpha_1 B_1 G_1(r) - \alpha B G(r)]/r' = 0$ by l'Hôpital's rule and (20), it follows from (58), (20) and (23) that

$$\begin{aligned} f_{10}(1^-) &= B_1 \lim_{r\to 1}\left[\frac{\alpha_1 G_1(r)}{B}\log\frac{e^R}{1-r} - \frac{\alpha G(r)}{B_1}\log\frac{e^{R_1}}{1-r}\right] \\ &= \delta + \lim_{r\to 1}\frac{\alpha_1 B_1 G_1(r) - \alpha B G(r)}{Br'}\cdot\left(r'\log\frac{1}{1-r}\right) = \delta. \end{aligned}$$

It follows from (58) and the monotonicity of f_{10} given in part (1) that

$$\xi + \frac{\alpha}{\alpha_1}F_1(r)f_2(r) \leqslant F(a,b;c;r) \leqslant \delta + \frac{\alpha}{\alpha_1}F_1(r)f_2(r),$$

and hence (57) follows from Theorem 2.6(4). The remaining conclusions are clear. □

Taking $a_1 = 1/2$ and $b_1 = 1$ ($a_1 = 1/4$ and $b_1 = 3/4$) in Theorem 3.8, we immediately obtain the following Corollary 3.9 (Corollary 3.10, respectively).

COROLLARY 3.9. *For $a,b\in(0,\infty)$ with $c=a+b$, let r_7 and r_8 be as in Corollary 3.2, $\delta_1 = (R-\log 4)/B$, and for $r\in(0,1)$, let $f_{11}(r) = F(r) - F_0(r)F'(r)/F_0'(r)$ and $Q_0(r) = [\text{arth}(r)]/r$.*

(1) If $ab \leqslant \min\{2-c,(c+1)/5\}$ or $2-c<ab<1/2$, then f_{11} is increasing from $(0,1)$ onto $(1-3\alpha,\delta_1)$. Moreover, if $(a,b)\neq(1/2,1)$, then the monotonicity of f_{11} is strict. In particular, for $r\in(0,1)$,

$$1-3\alpha+2Q_0(r)/B \leqslant F\left(a,b;c;r^2\right) \leqslant \delta_1 + 3\alpha Q_0(r), \tag{60}$$

with equality in each instance if and only if $a=1/2$ and $b=1$.

(2) If $ab \geqslant \max\{2-c,(c+1)/5\}$ or $1/2<ab<2-c$, then f_{11} is decreasing from $(0,1)$ onto $(\delta_1, 1-3\alpha)$, and each inequality in (60) is reversed. Furthermore, if $(a,b)\neq(1/2,1)$, then the monotonicity of f_{11} is strict.

(3) In other cases not stated in parts (1)–(2), that is, $1/2<ab<(c+1)/5$ (or $(c+1)/5<ab<1/2$), f_{11} is increasing (decreasing) on $[0,r_7]$ ($[0,r_8]$), and decreasing (increasing) on $[r_7,1)$ ($[r_8,1)$, respectively). If $c^2 \leqslant 4(c+1)/5$, then the case "$(c+1)/5<ab<1/2$" does not appear.

COROLLARY 3.10. *For $a,b\in(0,\infty)$ with $c=a+b$, and for $r\in(0,1)$, let r_{11} and r_{12} be as in Corollary 3.6, $\delta_2=(R-\log 64)/B$, $\beta_1=1-16\alpha/3$, and $f_{12}(r)=F(r)-\overline{F}_0(r)F'(r)/\overline{F}'_0(r)$.*

(1) If $ab\leqslant\min\{(19/16)-c,3(c+1)/32\}$ or $(19/16)-c<ab<3/16$, then f_{12} is increasing from $(0,1)$ onto (η,δ_2). Moreover, if $(a,b)\neq(1/4,3/4)$, then the monotonicity of f_{12} is strict. In particular, for $r\in(0,1)$,

$$1-\frac{16\alpha}{3}+\frac{\sqrt{2}\pi}{B}F\left(\frac{1}{4},\frac{3}{4};1;r^2\right)\leqslant F\left(a,b;c;r^2\right)\leqslant\delta_2+\frac{16\alpha}{3}F\left(\frac{1}{4},\frac{3}{4};1;r^2\right),\quad(61)$$

with equality in each instance if and only if $a=1/4$ and $b=3/4$.

(2) If $ab\geqslant\max\{(19/16)-c,3(c+1)/32\}$ or $3/16<ab<(19/16)-c$, then f_{12} is decreasing from $(0,1)$ onto (δ_2,η), and for $r\in(0,1)$, each inequality in (61) is reversed. Furthermore, if $(a,b)\neq(1/4,3/4)$, then the monotonicity of f_{12} is strict.

(3) In other cases not stated in parts (1)–(2), that is, $3/16<ab<3(c+1)/32$ (or $3(c+1)/32<ab<3/16$), f_{12} is increasing (decreasing) on $[0,r_{11}]$ ($[0,r_{12}]$), and decreasing (increasing) on $[r_{11},1)$ ($[r_{12},1)$, respectively). If $c^2\leqslant 3(c+1)/8$, then the case "$3(c+1)/32<ab<3/16$" does not appear.

COROLLARY 3.11. *(1) For $a,b,a_1,b_1\in(0,\infty)$ with $c=a+b$ and $c_1=a_1+b_1$, let f_1 be as in Theorem 2.6. If $\alpha_1\leqslant 1/2$ and $ab\leqslant\min\{a_1b_1+c_1-c,(c+1)\overline{\alpha}_1\}$, or if $\alpha_1\leqslant 1/2$ and $a_1b_1+c_1-c<ab\leqslant a_1b_1$, then f_1 is concave on $(0,1)$.*

(2) If $ab\leqslant\min\{2-c,(c+1)/5\}$ or $2-c<ab\leqslant 1/2$, then f_4 defined in Corollary 3.1 is concave on $(0,1)$.

(3) If $ab\leqslant\min\{(19/16)-c,3(c+1)/32\}$ or $(19/16)-c<ab\leqslant 3/16$, then f_7 defined in Corollary 3.5 is concave on $(0,1)$.

Proof. (1) Let f_{10} be as in Theorem 3.8. By differentiation,

$$\begin{aligned}-f_1'(r)&=\frac{F_1'(r)}{F_1(r)^2}\left[F(r)-\frac{F'(r)}{F_1'(r)}F_1(r)\right]\\&=\frac{F_1'(r)}{F_1(r)^2}f_{10}(r)\\&=\frac{\alpha_1G_1(r)}{(1-r)F_1(r)^2}\cdot f_{10}(r),\end{aligned}$$

which is product of two positive and increasing functions on $(0,1)$ by Theorem 3.8(1) and Lemma 2.5. This yields part (1).

(2) If $a_1=1/2$ and $b_1=1$ in part (1), then $\alpha_1=1/3<1/2$, $a_1b_1+c_1=2$, $a_1b_1/(c_1+1)=1/5$, and hence the concavity of f_4 follows from part (1).

(3) Similarly, part (3) follows from part (1) with $a_1=1/4$ and $b_1=3/4$. □

4. Extensions of transformation identities (26) and (27)

In this section, we extend the identities (26) and (27) to zero-balanced hypergeometric functions by proving the following Theorems 4.1 and 4.2. These results substantially improve all the known related results such as Theorem 1.2.

THEOREM 4.1. *For $a,b\in(0,\infty)$ with $c=a+b$, let $\beta=1-3\alpha$ and $\delta_1=(R-\log 4)/B$, and define the function f on $(0,1)$ by*

$$f(r)=(1+r)F(a,b;c;r)-F\left(a,b;c;\frac{4r}{(1+r)^2}\right)-\beta r.$$

(1) If $ab\leqslant\min\{2-c,(c+1)/5\}$ or $2-c<ab<1/2$, then f is increasing from $[0,1)$ onto $[0,\delta_1-\beta)$. Moreover, if $(a,b)\neq(1/2,1)$, then the monotonicity of f is strict. In particular, for $r\in(0,1)$,

$$\beta r\leqslant(1+r)F(a,b;c;r)-F\left(a,b;c;\frac{4r}{(1+r)^2}\right)\leqslant\beta r+\delta_1-\beta, \tag{62}$$

with equality in each instance if and only if $a=1/2$ and $b=1$.

(2) If $ab\geqslant\max\{2-c,(c+1)/5\}$ or $1/2<ab<2-c$, then f is decreasing from $[0,1)$ onto $(\delta_1-\beta,0]$. Moreover, if $(a,b)\neq(1/2,1)$, then the monotonicity of f is strict. In particular, for $r\in(0,1)$,

$$\beta r+\delta_1-\beta\leqslant(1+r)F(a,b;c;r)-F\left(a,b;c;\frac{4r}{(1+r)^2}\right)\leqslant\beta r, \tag{63}$$

with equality in each instance if and only if $a=1/2$ and $b=1$.

(3) In other cases not stated in parts (1)–(2), that is, $1/2<ab<(c+1)/5$ or $(c+1)/5<ab<1/2$, f is not monotone on $(0,1)$, and neither the double inequality (62) nor (63) holds for all $r\in(0,1)$ and for all $a,b\in(0,\infty)$ with $1/2<ab<(c+1)/5$ or $(c+1)/5<ab<1/2$. If $c^2\leqslant4(c+1)/5$, then the case "$(c+1)/5<ab<1/2$" does not appear.

Proof. Put $x=4r/(1+r)^2$. Then $x>r$ and

$$1-x=\left(\frac{1-r}{1+r}\right)^2,\ \frac{dx}{dr}=\frac{4(1-r)}{(1+r)^3},\ \frac{1}{1-x}\frac{dx}{dr}=\frac{4}{r'^2}. \tag{64}$$

Clearly, $f(0)=0$. By (20) and (64), we obtain

$$\begin{aligned} f(1^-)&=\frac{1}{B}\lim_{r\to1}\left[(1+r)\log\frac{e^R}{1-r}-\log\frac{e^R}{1-x}\right]-\beta\\ &=\frac{1}{B}\lim_{r\to1}\left[rR+(1+r)\log\frac{1}{1-r}-2\log\frac{1+r}{1-r}\right]-\beta=\delta_1-\beta. \end{aligned} \tag{65}$$

Let $f_{13}(r)=f_{14}(r)/f_{15}(r)$, $f_{14}(r)=r'^2F(r)+\alpha(1+r)^2G(r)-4\alpha G(x)$ and $f_{15}(r)=r'^2$. Then by (22) and (64), and by differentiation, we have

$$f'(r)=F(r)+\alpha\frac{1+r}{1-r}G(r)-\frac{4\alpha}{1-r^2}G(x)-\beta=f_{13}(r)-\beta, \tag{66}$$

$$f_{14}'(r)=3\alpha(1+r)G(r)-2rF(r)+\alpha\overline{\alpha}(1+r)^2F_+(r)-\frac{16\alpha\overline{\alpha}(1-r)}{(1+r)^3}F_+(x). \tag{67}$$

Clearly,

$$f_{14}(0)-\beta=f_{14}(1^-)=0,\ f_{14}'(0)=\frac{15\alpha}{c+1}\left(\frac{c+1}{5}-ab\right). \tag{68}$$

Next, by (26), $F_0(x)=(1+r)F_0(r)$. Differentiating both sides of this identity with respect to r, and using (18), (22) and (64), we obtain the following relation

$$G_0(x)=\frac{3r'^2}{4}\left[F_0(r)+\frac{1+r}{3(1-r)}G_0(r)\right]. \tag{69}$$

(1) If $ab\leqslant\min\{2-c,(c+1)/5\}$ or $2-c<ab<1/2$, then by Corollary 3.2(1),

$$G(x)\leqslant G(r)G_0(x)/G_0(r). \tag{70}$$

Let f_{11} be as in Corollary 3.9. Then it follows from (66), (69), (70) and Corollary 3.9(1) that

$$\begin{aligned}f'(r)&\geqslant F(r)+\alpha\frac{1+r}{1-r}G(r)-\frac{4\alpha G(r)}{r'^2G_0(r)}G_0(x)-\beta\\&=F(r)-3\alpha F_0(r)\frac{G(r)}{G_0(r)}-\beta=f_{11}(r)-\beta\geqslant 0.\end{aligned} \tag{71}$$

This yields the monotonicity of f. The remaining conclusions in part (1) are clear.

(2) If $ab\geqslant\max\{2-c,(c+1)/5\}$ or $1/2<ab<2-c$, then by Corollaries 3.2(2) and 3.9(2), each inequality in (70) –(71) is reversed, and hence part (2) follows.

(3) By (66), we see that $f'(0)=0$. By l'Hôpital's rule and (67)–(68), we obtain

$$\lim_{r\to0}\frac{r'^2}{r}f'(r)=\lim_{r\to0}\left[\frac{f_{14}(r)-\beta}{r}+\beta r\right]=f_{14}'(0^+)=\frac{15\alpha}{c+1}\left(\frac{c+1}{5}-ab\right). \tag{72}$$

By (15), (20), (22)–(23) and (66)–(67), and by l'Hôpital's rule, we obtain

$$\begin{aligned}f'(1^-)&=f_{13}(1^-)-\beta=\lim_{r\to1}\frac{f_{14}'(r)}{f_{15}'(r)}-\beta=-\frac{1}{2}f_{14}'(1^-)-\beta\\&=\frac{1}{2}\lim_{r\to1}\left[2rF(r)+\frac{16\alpha\overline{\alpha}(1-r)}{(1+r)^3}F_+(x)-3\alpha(1+r)G(r)-\alpha\overline{\alpha}(1+r)^2F_+(r)\right]-\beta\\&=-\frac{3}{B}+\lim_{r\to1}\left[rF(r)-\frac{\alpha\overline{\alpha}}{2}(1+r)^2F_+(r)\right]-\beta\end{aligned}$$

$$
\begin{aligned}
&= -\frac{3}{B} + \lim_{r\to 1}\left[\frac{r}{B}\log\frac{e^R}{1-r} - \frac{\alpha\overline{\alpha}(1+r)^2}{2\overline{B}}\log\frac{e^{\overline{R}}}{1-r}\right] - \beta \\
&= \frac{2c+(1-2ab)R-3}{B} + \frac{1}{B}\lim_{r\to 1}\left[r - \frac{ab}{2}(1+r)^2\right]\log\frac{1}{1-r} - \beta \\
&= \begin{cases} \infty, & \text{if } ab < 1/2, \\ -\infty, & \text{if } ab > 1/2. \end{cases}
\end{aligned} \tag{73}
$$

If $1/2 < ab < (c+1)/5$, then by (72) and (73), there exist numbers $r_{13}, r_{14} \in (0,1)$ with $r_{13} < r_{14}$ such that $f'(r) > 0$ for $r \in (0, r_{13})$, and $f'(r) < 0$ for $r \in (r_{14}, 1)$. Hence f is not monotone on $(0,1)$, and the second inequality in (62) (the second inequality in (63)) is reversed for $r \in (r_{14}, 1)$ ($r \in (0, r_{13}]$, respectively).

Similarly, if $(c+1)/5 < ab < 1/2$, then f is not monotone on $(0,1)$, and neither the double inequality (62) nor (63) holds for all $r \in (0,1)$. The remaining conclusion is clear. $\square$

COROLLARY 4.2. *For $a, b \in (0,\infty)$ with $c = a+b$, let $\beta = 1-3\alpha$ and $\delta_1 = (R-\log 4)/B$, and define the function g on $(0,1)$ by*

$$
g(r) = \frac{2}{1+r}F\left(a,b;c;\frac{1-r}{1+r}\right) - F\left(a,b;c;r'^2\right) - \beta\frac{1-r}{1+r}.
$$

(1) If $ab \leqslant \min\{2-c,(c+1)/5\}$ or $2-c < ab < 1/2$, then g is decreasing from $(0,1]$ onto $[0, \delta_1 - \beta)$. Moreover, if $(a,b) \neq (1/2,1)$, then the monotonicity of g is strict. In particular, for $r \in (0,1)$,

$$
\beta(1-r) \leqslant 2F\left(a,b;c;\frac{1-r}{1+r}\right) - (1+r)F\left(a,b;c;r'^2\right) \leqslant \beta(1-r) + (\delta_1-\beta)(1+r), \tag{74}
$$

with equality in each instance if and only if $a = 1/2$ and $b = 1$.

(2) If $ab \geqslant \max\{2-c,(c+1)/5\}$ or $1/2 < ab < 2-c$, then g is increasing from $(0,1]$ onto $(\delta_1 - \beta, 0]$. Moreover, if $(a,b) \neq (1/2,1)$, then the monotonicity of g is strict. In particular, for $r \in (0,1)$,

$$
\beta(1-r) + (\delta_1-\beta)(1+r) \leqslant 2F\left(a,b;c;\frac{1-r}{1+r}\right) - (1+r)F\left(a,b;c;r'^2\right) \leqslant \beta(1-r), \tag{75}
$$

with equality in each instance if and only if $a = 1/2$ and $b = 1$.

(3) In other cases not stated in parts (1)–(2), that is, $1/2 < ab < (c+1)/5$ or $(c+1)/5 < ab < 1/2$, g is not monotone on $(0,1)$, and neither (74) nor (75) holds for all $r \in (0,1)$ and for all $a,b \in (0,\infty)$ with $1/2 < ab < (c+1)/5$ or $(c+1)/5 < ab < 1/2$. If $c^2 \leqslant 4(c+1)/5$, then the case "$(c+1)/5 < ab < 1/2$" does not appear.

Proof. Let f be as in Theorem 4.1, and $t = (1-r)/(1+r)$. Then $2/(1+r) = 1+t$, $r = (1-t)/(1+t)$, $r'^2 = 4t/(1+t)^2$, and $g(r) = (1+t)F(t) - F\left(4t/(1+t)^2\right) - \beta t = f(t)$. Hence the results in Corollary 4.2 follow from Theorem 4.1. $\square$

5. Extensions of identities (28) and (29)

In this section, we apply the results proved in Section 3 to extend the transformation identities (28) and (29) to zero-balanced hypergeometric functions by proving the following theorem and its corollary.

THEOREM 5.1. *For* $a,b\in(0,\infty)$ *with* $c=a+b$, *and for* $r\in(0,1)$, *let* $\eta=1-16\alpha/3$, $\delta_2=(R-\log 64)/B$,

$$P_1(r)=\eta\left(\sqrt{1+3r}-1\right)=\frac{3\eta}{1+\sqrt{1+3r}},$$
$$h(r)=\sqrt{1+3r}F\left(a,b;c;r^2\right)-F\left(a,b;c;1-\left(\frac{1-r}{1+3r}\right)^2\right)-P_1(r).$$

(1) If $ab\leqslant\min\{(19/16)-c,3(c+1)/32\}$ *or* $(19/16)-c<ab<3/16$, *then* h *is increasing from* $[0,1)$ *onto* $[0,\delta_2-\eta)$. *Moreover, if* $(a,b)\neq(1/4,3/4)$, *then the monotonicity of* h *is strict. In particular, for* $r\in(0,1)$,

$$P_1(r)\leqslant\sqrt{1+3r}F\left(a,b;c;r^2\right)-F\left(a,b;c;1-\left(\frac{1-r}{1+3r}\right)^2\right)\leqslant P_1(r)+\delta_2-\eta, \quad (76)$$

with equality in each instance if and only if $a=1/4$ *and* $b=3/4$.

(2) If $ab\geqslant\max\{(19/16)-c,3(c+1)/32\}$ *or* $3/16<ab<(19/16)-c$, *then* h *is decreasing from* $[0,1)$ *onto* $(\delta_2-\eta,0]$. *Moreover, if* $(a,b)\neq(1/4,3/4)$, *then the monotonicity of* h *is strict. In particular, for* $r\in(0,1)$,

$$P_1(r)+\delta_2-\eta\leqslant\sqrt{1+3r}F\left(a,b;c;r^2\right)-F\left(a,b;c;1-\left(\frac{1-r}{1+3r}\right)^2\right)\leqslant P_1(r), \quad (77)$$

with equality in each instance if and only if $a=1/4$ *and* $b=3/4$.

(3) In other cases not stated in parts (1)–(2), that is, $3/16<ab<3(c+1)/32$ *or* $3(c+1)/32<ab<3/16$, h *is not monotone on* $(0,1)$, *and neither the double inequality (76) nor (77) holds for all* $r\in(0,1)$ *and all* $a,b\in(0,\infty)$ *with* $3/16<ab<3(c+1)/32$ *or* $3(c+1)/32<ab<3/16$. *If* $c^2\leqslant3(c+1)/8$, *then the case "*$3(c+1)/32<ab<3/16$*" does not appear.*

Proof. Set $y=1-[(1-r)/(1+3r)]^2=8r(1+r)/(1+3r)^2$. Then $y>r>r^2$ for $r\in(0,1)$, and

$$\frac{dy}{dr}=\frac{8(1-r)}{(1+3r)^3},\ \frac{1}{1-y}\frac{dy}{dr}=\frac{8}{(1-r)(1+3r)}. \quad (78)$$

Clearly, $h(0)=0$. By (20) and (78), we obtain

$$\begin{aligned} h(1^-) &= \frac{1}{B}\lim_{r\to 1}\left[\sqrt{1+3r}\log\frac{e^R}{1-r^2}-\log\frac{e^R}{1-y}\right]-\eta \\ &= \frac{1}{B}\lim_{r\to 1}\left[\left(\sqrt{1+3r}-1\right)R+\sqrt{1+3r}\log\frac{1}{1-r^2}-2\log\frac{1+3r}{1-r}\right]-\eta \\ &= (R-\log 64)/B-\eta=\delta_2-\eta. \end{aligned} \tag{79}$$

For $a,b\in(0,\infty)$ with $c=a+b$, and for $r\in(0,1)$, let $h_1(r)=h_2(r)/h_3(r)$, where

$$\begin{aligned} h_2(r) &= 3r'^2\sqrt{1+3r}F\left(r^2\right)+4\alpha r(1+3r)^{3/2}G\left(r^2\right)-16\alpha(1+r)G(y), \\ h_3(r) &= r'^2\sqrt{1+3r}. \end{aligned}$$

Then by (22) and (78), and by differentiation, we obtain

$$\begin{aligned} 2\sqrt{1+3r}\,h'(r) =& 3F\left(r^2\right)+\frac{4\alpha r(1+3r)}{1-r^2}G\left(r^2\right)-\frac{16\alpha G(y)}{(1-r)\sqrt{1+3r}}-3\eta \\ =& h_1(r)-3\eta, \end{aligned} \tag{80}$$

$$\begin{aligned} h_2'(r) =& \frac{3\left(3-4r-15r^2\right)}{2\sqrt{1+3r}}F\left(r^2\right)+4\alpha(1+9r)\sqrt{1+3r}G\left(r^2\right) \\ &+8\alpha\overline{\alpha}r^2(1+3r)^{3/2}F_+\left(r^2\right)-16\alpha G(y)-\frac{128\alpha\overline{\alpha}r'^2}{(1+3r)^3}F_+(y). \end{aligned} \tag{81}$$

Set $D_4=64[1-c+(ab-3/16)R]/B$. Since

$$\lim_{r\to 1}\left[16abr^2(1+3r)^2-3\left(15r^2+4r-3\right)\right]=256(ab-3/16),$$

it follows from (81), (15), (20) and (22)–(23) that

$$h_2(0)=3\eta,\ h_2(1^-)=0,\ h_2'(0)=\frac{9}{2}-4\alpha\left(3+\frac{32ab}{c+1}\right), \tag{82}$$

$$\begin{aligned} h_2'(1^-) =& \frac{64}{B}+\lim_{r\to 1}\left[8\alpha\overline{\alpha}r^2(1+3r)^{3/2}F_+\left(r^2\right)-\frac{3\left(15r^2+4r-3\right)}{2\sqrt{1+3r}}F\left(r^2\right)\right] \\ =& \frac{64}{B}+\frac{1}{4B}\lim_{r\to 1}\left[16abr^2(1+3r)^2\log\frac{e^{\overline{R}}}{1-r^2}-3\left(15r^2+4r-3\right)\log\frac{e^R}{1-r^2}\right] \\ =& D_4+\lim_{r\to 1}\frac{16abr^2(1+3r)^2-3\left(15r^2+4r-3\right)}{4B}\log\frac{1}{1-r^2} \\ =& \begin{cases} -\infty, & \text{if } ab<3/16, \\ \infty, & \text{if } ab>3/16. \end{cases} \end{aligned} \tag{83}$$

Next, by (29), $\overline{F}_0(y)=\sqrt{1+3r}\,\overline{F}_0\left(r^2\right)$. Differentiating both sides of this identity with respect to r, and applying (23), we obtain

$$\overline{G}_0(y)=(1-r)\sqrt{1+3r}\left[\overline{F}_0\left(r^2\right)+\frac{r(1+3r)}{4\left(1-r^2\right)}\overline{G}_0\left(r^2\right)\right]. \tag{84}$$

(1) If $ab \leqslant \min\{(19/16)-c, 3(c+1)/32\}$ or $(19/16)-c < ab < 3/16$, then by Corollary 3.6(1), we obtain

$$G(y) \leqslant G\left(r^2\right)\overline{G}_0(y)/\overline{G}_0\left(r^2\right), \tag{85}$$

with equality if and only if $a=1/4$ and $b=3/4$.

Let f_{12} be as in Corollary 3.10. Then it follows from (22)–(23), (80) and (84)–(85) that

$$\begin{aligned} h_1(r) &\geqslant 3F\left(r^2\right)+\frac{4\alpha r(1+3r)}{1-r^2}G\left(r^2\right)-\frac{16\alpha}{(1-r)\sqrt{1+3r}}\frac{G\left(r^2\right)}{\overline{G}_0\left(r^2\right)}\overline{G}_0(y) \\ &= 3\left[F\left(r^2\right)-\frac{16\alpha G\left(r^2\right)}{3\overline{G}_0\left(r^2\right)}\overline{F}_0\left(r^2\right)\right]=3f_{12}\left(r^2\right). \end{aligned} \tag{86}$$

The first equality in (86) holds if and only if $a=1/4$ and $b=3/4$. Hence by Corollary 3.10(1) and (80),

$$2h'(r)\sqrt{1+3r}=h_1(r)-3\eta \geqslant 3\left[f_{12}\left(r^2\right)-\eta\right] \geqslant 0,\ r\in(0,1), \tag{87}$$

with equality in each instance if and only if $a=1/4$ and $b=3/4$. This yields the monotonicity of h.

Since $\sqrt{1+3r}-1=3r/\left(1+\sqrt{1+3r}\right)$, (76) follows from the monotonicity of h. The remaining conclusions in part (1) are clear.

(2) With the conditions in part (2), each inequality in (85)–(87) is reversed by Corollaries 3.6(2) and 3.10(2). Hence part (2) follows.

(3) It follows from (80) that

$$\begin{aligned} 2\frac{\sqrt{1+3r}}{r}h'(r) &= \frac{h_2(r)-3\eta+3\eta\left[\left(1-\sqrt{1+3r}\right)+r^2\sqrt{1+3r}\right]}{rr'^2\sqrt{1+3r}} \\ &= \frac{h_2(r)-3\eta}{rr'^2\sqrt{1+3r}}-\frac{9\eta}{r'^2\left(1+\sqrt{1+3r}\right)\sqrt{1+3r}}+3\eta\frac{r}{r'^2}. \end{aligned}$$

Hence by (82) and l'Hôpital's rule,

$$\begin{aligned} 2\lim_{r\to 0}\frac{h'(r)}{r} &= 2\lim_{r\to 0}\frac{\sqrt{1+3r}}{r}h'(r)=\lim_{r\to 0}\frac{h_2(r)-3\eta}{r}-\frac{9\eta}{2} \\ &= h_2'(0^+)-\frac{9\eta}{2}=\frac{128\alpha}{c+1}\left[\frac{3(c+1)}{32}-ab\right]. \end{aligned} \tag{88}$$

On the other hand, by (80) and (82)–(83), and by l'Hôpital's rule, we obtain

$$\begin{aligned} 4h'(1^-) &= h_1(1^-)-3\eta=\frac{1}{4}\lim_{r\to 1}\frac{h_2(r)}{1-r}-3\eta \\ &= -\frac{1}{4}h_2'(1^-)-3\eta=\begin{cases}\infty, & \text{if } ab<3/16,\\ -\infty, & \text{if } ab>3/16.\end{cases} \end{aligned} \tag{89}$$

If $3/16 < ab < 3(c+1)/32$, then by (88) and (89), there exist $r_{15}, r_{16} \in (0,1)$ with $r_{15} < r_{16}$ such that $h'(r) > 0$ for $r \in (0, r_{15})$, and $h'(r) < 0$ for $r \in (r_{16}, 1)$. Hence h is not monotone on $(0,1)$, and the second inequality in (76) (the second inequality in (77)) is reversed for $r \in [r_{16}, 1)$ ($r \in (0, r_{15}]$, respectively).

Similarly, if $3(c+1)/32 < ab < 3/16$, then h is not monotone on $(0,1)$, and neither the double inequality (76) nor (77) holds for all $r \in (0,1)$. The remaining conclusion is clear. □

COROLLARY 5.2. *For $a,b \in (0,\infty)$ with $c = a+b$, let $\eta = 1 - 16\alpha/3$, $\delta_2 = (R - \log 64)/B$ and*

$$P(r) = \frac{3\eta(1-r)}{2+\sqrt{1+3r}}, \; Q(r) = P(r) + (\delta_2 - \eta)\sqrt{1+3r},$$

and define the function H on $(0,1)$ by

$$H(r) = \frac{2}{\sqrt{1+3r}} F\left(a,b;c;\left(\frac{1-r}{1+3r}\right)^2\right) - F\left(a,b;c;1-r^2\right) - \eta\left(\frac{2}{\sqrt{1+3r}} - 1\right).$$

(1) If $ab \leqslant \min\{(19/16) - c, 3(c+1)/32\}$ or $(19/16) - c < ab < 3/16$, then H is decreasing from $(0,1]$ onto $[0, \delta_2 - \eta)$. Moreover, if $(a,b) \neq (1/4, 3/4)$, then the monotonicity of H is strict. In particular, for $r \in (0,1)$,

$$P(r) \leqslant 2F\left(a,b;c;\left(\frac{1-r}{1+3r}\right)^2\right) - \sqrt{1+3r}F\left(a,b;c;1-r^2\right) \leqslant Q(r), \tag{90}$$

with equality in each instance if and only if $a = 1/4$ and $b = 3/4$.

(2) If $ab \geqslant \max\{(19/16) - c, 3(c+1)/32\}$ or $3/16 < ab < (19/16) - c$, then H is increasing from $(0,1]$ onto $(\delta_2 - \eta, 0]$. Moreover, if $(a,b) \neq (1/4, 3/4)$, then the monotonicity of H is strict.

$$Q(r) \leqslant 2F\left(a,b;c;\left(\frac{1-r}{1+3r}\right)^2\right) - \sqrt{1+3r}F\left(a,b;c;1-r^2\right) \leqslant P(r), \tag{91}$$

with equality in each instance if and only if $a = 1/4$ and $b = 3/4$.

(3) In other cases not stated in parts (1)–(2), that is, $3/16 < ab < 3(c+1)/32$ or $3(c+1)/32 < ab < 3/16$, H is not monotone on $(0,1)$, and neither the double inequality (90) nor (91) holds for all $r \in (0,1)$ and $a,b \in (0,\infty)$ with $3/16 < ab < 3(c+1)/32$ or $3(c+1)/32 < ab < 3/16$. If $c^2 \leqslant 3(c+1)/8$, then the case "$3(c+1)/32 < ab < 3/16$" does not appear.

Proof. Put $t = (1-r)/(1+3r)$. Then $r = (1-t)/(1+3t)$, $1+3r = 4/(1+3t)$ and

$$H(r) = \sqrt{1+3t}F\left(a,b;c;t^2\right) - F\left(a,b;c;1-\left(\frac{1-t}{1+3t}\right)^2\right) - \eta\left(\sqrt{1+3t} - 1\right) = h(t),$$

where h is as in Theorem 5.1. Hence the results in Corollary 5.2 follow from Theorem 5.1. □

6. Transformation inequalities for the generalized Grötzsch ring function

For $a,b\in(0,\infty)$ with $c=a+b$ and $r\in(0,1)$, the generalized Grötzsch ring function is defined by

$$\mu_{a,b}(r)=\frac{B(a,b)}{2}\frac{F\left(a,b;c;1-r^2\right)}{F\left(a,b;c;r^2\right)}. \tag{92}$$

For $0<a\leqslant 1/2$, the function $\mu_a\equiv\mu_{a,1-a}$ is also said to be the generalized Grötzsch ring function, while $\mu\equiv\mu_{1/2}$ is exactly the well-known Grötzsch ring function in quasiconformal theory. The function $\mu_{a,b}$ has applications in several fields of mathematics such as the theories of quasiconformal mappings and Ramanujan's modular equations. (Cf. [2, 3, 12, 24, 33, 34, 40, 59, 69]). Many properties of the functions μ and μ_a have been revealed. However, only a few results have been obtained for the function $\mu_{a,b}$. In this section, we apply the results proved in Sections 2–3 to show several properties of $\mu_{a,b}$.

It is well known that the function μ satisfies the following Landen transformation identity (cf. [3, 24])

$$\mu(r)=2\mu\left(\frac{2\sqrt{r}}{1+r}\right),\ \mu(r)\mu\left(\frac{1-r}{1+r}\right)\equiv\frac{\pi^2}{2},\ r\in(0,1), \tag{93}$$

and it is clear that for $r\in(0,1)$,

$$\mu_{a,b}(r)\mu_{a,b}(r')=B^2/4. \tag{94}$$

Let $t=[(1-r)/(1+3r)]^2$. Since $\sqrt{1-t}=2\sqrt{2r(1+r)}/(1+3r)$, it follows from (13), (28)–(29), (92) and (94) that

$$\mu_{1/4}(r)=\frac{\sqrt{2}\pi}{2}\frac{\overline{F}_0(r'^2)}{\overline{F}_0(r^2)}=\frac{\sqrt{2}\pi\overline{F}_0(t)}{\overline{F}_0(1-t)}=2\mu_{1/4}\left(\frac{2\sqrt{2r(1+r)}}{1+3r}\right)=\frac{\pi^2}{\mu_{1/4}(\sqrt{t})},$$

which yields

$$\mu_{1/4}(r)=2\mu_{1/4}\left(\frac{2\sqrt{2r(1+r)}}{1+3r}\right),\ \mu_{1/4}(r)\mu_{1/4}\left(\frac{1-r}{1+3r}\right)\equiv\pi^2. \tag{95}$$

Now we show several properties of $\mu_{a,b}$, and extend (93) and (95) to $\mu_{a,b}(r)$. First, we prove the following theorem, which gives the relations between $\mu_{a,b}(r)$ and $\mu_{a_1,b_1}(r)$.

THEOREM 6.1. *For $a,b,a_1,b_1\in(0,\infty)$ with $c=a+b$ and $c_1=a_1+b_1$, and for $r\in(0,1)$, let $f_{16}(r)=\mu_{a,b}(r)/\mu_{a_1,b_1}(r)$.*

(1) If $ab\leqslant\min\{a_1b_1,c\alpha_1\}$ ($ab\geqslant\max\{a_1b_1,c\alpha_1\}$), or if $a_1b_1<ab<c\alpha_1$ with $R\geqslant R_1$, then f_{16} is increasing (decreasing) from $(0,1)$ onto $(1,B^2/B_1^2)$ ($(B^2/B_1^2,1)$, respectively). Moreover, if $(a,b)\neq(a_1,b_1)$, then the monotonicity properties of f_{16} are

strict. In particular, if $ab \leqslant \min\{a_1b_1, c\alpha_1\}$*, or if* $a_1b_1 < ab < c\alpha_1$ *with* $R \geqslant R_1$*, then for* $r \in (0,1)$*,*

$$B_1^2\mu_{a_1,b_1}(r) \leqslant B_1^2\mu_{a,b}(r) \leqslant B^2\mu_{a_1,b_1}(r), \tag{96}$$

with equality in each instance if and only if $(a,b) = (a_1,b_1)$*. If* $ab \geqslant \max\{a_1b_1, c\alpha_1\}$*, then each inequality in (96) is reversed.*

(2) For $a_1 \in (0,1/2]$*, let* $b_1 = 1-a_1$*. If* $ab \leqslant a_1(1-a_1)\min\{1,c\}$*, or if* $a_1(1-a_1) < ab < ca_1(1-a_1)$ *with* $R \geqslant R(a_1)$*, then for* $r \in (0,1)$*,*

$$\mu_{a_1}(r) \leqslant \mu_{a,b}(r) \leqslant B^2[\pi\sin(a_1\pi)]^{-2}\mu_{a_1}(r), \tag{97}$$

with equality in each instance if and only if $(a,b) = (a_1,1-a_1)$*. If* $ab \geqslant a_1(1-a_1)\max\{1,c\}$*, then each inequality in (97) is reversed.*

Proof. (1) Let f_1 be as in Theorem 2.6. Then by (92), $f_{16}(r)$ can be written as $f_{16}(r) = Bf_1(r'^2)/[B_1f_1(r^2)]$. Hence the monotonicity properties of f_{16} follow from Theorem 2.6(1)–(2).

The double inequality (96) and its equality case, and the remaining conclusion are clear.

(2) Part (2) follows from part (1). $\square$

COROLLARY 6.2. *(1) For* $a,b \in (0,\infty)$*, if* $4ab \leqslant \min\{1,c\}$*, or if* $1 < 4ab < c$ *with* $R \geqslant \log 16$*, then for* $r \in (0,1)$*,*

$$\pi^2\mu(r) \leqslant \pi^2\mu_{a,b}(r) \leqslant B^2\mu(r), \tag{98}$$

with equality in each instance if and only if $a = b = 1/2$*. Each inequality in (98) is reversed if* $4ab \geqslant \max\{1,c\}$*.*

(2) For all $a \in (0,1/2]$ *and* $r \in (0,1)$*,*

$$\mu(r) \leqslant \mu_a(r) \leqslant \mu(r)\sin^{-2}(a\pi), \tag{99}$$

with equality in each instance if and only if $a = 1/2$*.*

Proof. Taking $a_1 = b_1 = 1/2$, we obtain part (1) from Theorem 6.1 and (13). Part (2) is the special case of part (1) when $a \in (0,1/2]$ and $b = 1-a$, in which case $c = 1$ and $4ab = 4a(1-a) \leqslant 1 = \min\{1,c\}$. $\square$

COROLLARY 6.3. *For* $a,b \in (0,\infty)$*, if* $4ab \leqslant \min\{1,c\}$*, or if* $1 < 4ab < c$ *with* $R \geqslant \log 16$*, then for* $r \in (0,1)$*,*

$$\mu_{a,b}(r) \leqslant 2\mu_{a,b}\left(\frac{2\sqrt{r}}{1+r}\right) \leqslant \left(\frac{B}{\pi}\right)^2\mu_{a,b}(r), \tag{100}$$

$$\frac{\pi^2}{2} \leqslant \mu_{a,b}(r)\mu_{a,b}\left(\frac{1-r}{1+r}\right) \leqslant \frac{B^2}{2}, \tag{101}$$

with equality in each instance if and only if $a = b = 1/2$*. If* $4ab \geqslant \max\{1,c\}$*, then each inequality in (100) and (101) is reversed.*

Proof. Let f_1 be as in Theorem 2.6, $f_{17}(r) = f_1(r)|_{a_1=b_1=1/2}$, $t = 2\sqrt{r}/(1+r)$, $F_{1/2}(r) = F(1/2,1/2;1;r)$, $f_{18}(r) = f_{17}(r'^2)/f_{17}(r^2)$, $f_{19}(r) = 2\mu_{a,b}(t)/\mu_{a,b}(r)$ and $f_{20}(r) = \mu_{a,b}(r)\mu_{a,b}((1-r)/(1+r))$. Then by (92)–(94),

$$f_{19}(r) = \frac{F(t'^2)}{F_{1/2}(t'^2)} \cdot \frac{F_{1/2}(t^2)}{F(t^2)} \cdot \frac{F(r^2)}{F_{1/2}(r^2)} \cdot \frac{F_{1/2}(r'^2)}{F(r'^2)} \cdot \frac{2\mu(t)}{\mu(r)}$$

$$= \frac{f_{17}(1-t^2)}{f_{17}(t^2)} \cdot \frac{f_{17}(r^2)}{f_{17}(1-r^2)} = \frac{f_{18}(t)}{f_{18}(r)}, \tag{102}$$

$$f_{20}(r) = \frac{B^2\mu_{a,b}(r)}{4\mu_{a,b}(t)} = \frac{B^2}{2f_{19}(r)}. \tag{103}$$

If $4ab \leqslant \min\{1,c\}$, or if $1 < 4ab < c$ with $R \geqslant \log 16$, then by Theorem 2.6(1) and (13), f_{18} is increasing from $(0,1)$ onto $(\pi/B, B/\pi)$. If $(a,b) \neq (1/2,1/2)$, then the monotonicity of f_{18} is strict. Hence it follows from (102)–(103) that

$$1 = \frac{f_{18}(r)}{f_{18}(r)} \leqslant \frac{f_{18}(t)}{f_{18}(r)} = f_{19}(r) \leqslant \frac{f_{18}(1)}{f_{18}(0)} = \left(\frac{B}{\pi}\right)^2, \tag{104}$$

$$\pi^2/2 \leqslant f_{20}(r) \leqslant B^2/2, \tag{105}$$

with equality in each inequality if and only if $a = b = 1/2$. This yields (100)–(101) and their equality case.

If $4ab \geqslant \max\{1,c\}$, then f_{18} is decreasing from $(0,1)$ onto $(B/\pi, \pi/B)$, and the monotonicity of f_{18} is strict if $(a,b) \neq (1/2,1/2)$. Hence each inequality in (100)–(101) is reversed. $\square$

COROLLARY 6.4. *For $a,b \in (0,\infty)$, if $16ab/3 \leqslant \min\{1,c\}$, or if $1 < 16ab/3 < c$ with $R \geqslant \log 64$, then*

$$\mu_{a,b}(r) \leqslant 2\mu_{a,b}\left(\frac{2\sqrt{2r(1+r)}}{1+3r}\right) \leqslant \frac{1}{2}\left(\frac{B}{\pi}\right)^2 \mu_{a,b}(r), \tag{106}$$

$$\pi^2 \leqslant \mu_{a,b}(r)\mu_{a,b}\left(\frac{1-r}{1+3r}\right) \leqslant \frac{B^2}{2} \tag{107}$$

for $r \in (0,1)$, with equality in each instance if and only if $(a,b) = (1/4,3/4)$. If $16ab/3 \geqslant \max\{1,c\}$, then each inequality in (106)–(107) is reversed.

Proof. For $a,b \in (0,\infty)$ and $r \in (0,1)$, let f_7 be as in Corollary 3.5, $y = 1 - [(1-r)/(1+3r)]^2 = 8r(1+r)/(1+3r)^2$,

$$H_1(r) = \frac{\mu_{a,b}(r)}{\mu_{a,b}(\sqrt{y})},\ H_2(r) = \mu_{a,b}(r)\mu_{a,b}\left(\frac{1-r}{1+3r}\right),\ H_3(r) = \frac{f_7(1-r)}{f_7(r)}.$$

Then $\sqrt{y}=2\sqrt{2r(1+r)}/(1+3r)$, and by (92) and (28)–(29),

$$\begin{aligned} H_1(r)=&\frac{F(r'^2)}{F(r^2)}\frac{F(y)}{F(1-y)}=\frac{F(y)}{\overline{F}_0(y)}\frac{\overline{F}_0(1-y)}{F(1-y)}\frac{F(r'^2)}{\overline{F}_0(r'^2)}\frac{\overline{F}_0(r^2)}{F(r^2)}\frac{\overline{F}_0(y)\overline{F}_0(r'^2)}{\overline{F}_0(1-y)\overline{F}_0(r^2)}\\ =&2\frac{f_7(y)}{f_7(1-y)}\frac{f_7(1-r^2)}{f_7(r^2)}=2\frac{H_3(r^2)}{H_3(y)}, \qquad (108)\\ H_2(r)=&\frac{\mu_{a,b}(r)}{\mu_{a,b}(\sqrt{y})}\mu_{a,b}(\sqrt{y})\mu_{a,b}(\sqrt{1-y})=\frac{B^2}{4}H_1(r). \qquad (109)\end{aligned}$$

If $16ab/3\leqslant\min\{1,c\}$, or if $1<16ab/3<c$ with $R\geqslant\log 64$, then by Corollary 3.5(1), H_3 is increasing from $(0,1)$ onto $(\sqrt{2}\pi/B, B/(\sqrt{2}\pi))$, and the monotonicity of H_3 is strict if $(a,b)\neq(1/4,3/4)$. Since $y>r>r^2$ for $r\in(0,1)$, it follows from (108) that

$$4\pi^2/B^2=2H_3(0)/H_3(1)\leqslant H_1(r)\leqslant 2H_3(y)/H_3(y)=2, \qquad (110)$$

with equality in each inequality in (110) if and only if $(a,b)=(1/4,3/4)$. This, together with (109), yields (106) –(107) and their equality case.

The remaining conclusion follows from Corollary 3.5(2) and (108)–(109). □

REMARK 6.5. The double inequalities (100) and (101) extend the identities in (93) to $\mu_{a,b}(r)$, and Corollary 6.4 extends (95) to $\mu_{a,b}(r)$. By Theorem 6.1, one can apply the known identities and bounds of $\mu(r)$ and $\mu_{1/4}(r)$ to obtain inequalities for $\mu_{a,b}(r)$, although such kind of inequalities may be not sharp enough. For example,

$$2\left(\frac{\pi}{B}\right)^2\mu_{a,b}(r)\leqslant\mu_{1/4}(r)=2\mu_{1/4}\left(\frac{2\sqrt{2(1+r)}}{1+3r}\right)\leqslant 2\mu_{a,b}\left(\frac{2\sqrt{2(1+r)}}{1+3r}\right) \qquad (111)$$

for $16ab/3\leqslant\min\{1,c\}$, by (95) and (96). However, by Corollary 3.5(1), $B\geqslant\sqrt{2}\pi$ in this case, and hence the lower bound in (111) is less than that in (106) if $(a,b)\neq(1/4,3/4)$.

7. Concluding remarks

(i) We can derive some properties of $\mathscr{K}(r)$, $\mathscr{E}(r)$, $\mathscr{K}_a(r)$ and $\mathscr{E}_a(r)$ from the results obtained in Section 3, which are even probably new. We only give several examples below.

Letting f_1 be as in Theorem 2.6 with $a_1=b_1=1/2$, we obtain the following conclusions: For $a,b\in(0,\infty)$ with $c=a+b$, if $4ab\leqslant\min\{1,c\}$ ($4ab\geqslant\max\{1,c\}$), or if $1<4ab<c$ with $R\geqslant\log 16$, then the function $f_{21}(r)\equiv F(r^2)/\mathscr{K}(r)=2f_1(r^2)/\pi$ is decreasing (increasing) from $[0,1)$ onto $(2/B,2/\pi]$ ($[2/\pi,2/B)$, respectively). In particular, for $r\in(0,1)$, if $4ab\leqslant\min\{1,c\}$, or if $1<4ab<c$ with $R\geqslant\log 16$, then

$$\frac{\pi}{B}\mathscr{K}(r)\leqslant\frac{\pi}{2}F\left(a,b;a+b;r^2\right)\leqslant\mathscr{K}(r),\ r\in(0,1), \qquad (112)$$

with equality in each instance if and only if $a=b=1/2$. Each inequality in (112) is reversed if $4ab \geqslant \max\{1,c\}$.

Taking $a_1=b_1=1/2$ in Lemma 2.7, we obtain the following conclusions: For $a,b\in(0,\infty)$ with $c=a+b$, and for $r\in(0,1)$, if $ab\leqslant\min\{(5/4)-c,(c+1)/8\}$ or $(5/4)-c<ab<1/4$, then for $r\in(0,1)$,

$$1-\frac{\pi}{B}+\frac{2}{B}\mathscr{K}(r)\leqslant F\left(a,b;a+b;r^2\right)\leqslant 1-4\alpha+\frac{8\alpha}{\pi}\mathscr{K}(r), \tag{113}$$

with equality in each instance if and only if $a=b=1/2$. If $ab\geqslant\max\{(5/4)-c,(c+1)/8\}$ or $1/4<ab<(5/4)-c$, then each inequality in (113) is reversed.

Taking $c=1$ in (50), we obtain

$$\frac{\pi}{2}+\left[\frac{\operatorname{arth}(r)}{r}-1\right]\sin(\pi a)<\mathscr{K}_a(r)<\frac{\pi}{2}+\frac{3\pi a(1-a)}{2}\left[\frac{\operatorname{arth}(r)}{r}-1\right],\ r\in(0,1). \tag{114}$$

As we know, many good results for $\mathscr{K}(r)$ have been obtained, including sharp lower and upper bounds expressed in terms of elementary functions. Applying (112)–(113) (or their reversed double inequalities) and the known functional inequalities satisfied by $\mathscr{K}(r)$, one can obtain lower and upper bounds given in terms of elementary functions for the function $F(a,b;a+b;r)$.

Let $c=a+b$ for $a,b\in(0,\infty)$. Then it follows from (2), (18) and [2, Theorem 4.1] that

$$\frac{d\mathscr{K}_a}{dr}=\pi a(1-a)\frac{r}{1-r^2}F\left(a,1-a;2;r^2\right)=2(1-a)\frac{\mathscr{E}_a(r)-r'^2\mathscr{K}_a(r)}{rr'^2},$$

which yields

$$\mathscr{E}_a(r)-r'^2\mathscr{K}_a(r)=\frac{\pi a}{2}r^2F\left(a,1-a;2;r^2\right).$$

Hence it follows from (49) that for $r\in(0,1)$,

$$\frac{\sin(\pi a)}{2(1-a)}\frac{r-r'^2\operatorname{arth}(r)}{r}<\mathscr{E}_a(r)-r'^2\mathscr{K}_a(r)<\frac{3\pi a}{4}\frac{r-r'^2\operatorname{arth}(r)}{r}, \tag{115}$$

$$\frac{r-r'^2\operatorname{arth}(r)}{r}<\mathscr{E}(r)-r'^2\mathscr{K}(r)<3\pi\frac{r-r'^2\operatorname{arth}(r)}{8r}. \tag{116}$$

(ii) Applying the related results in Sections 2–5, one can obtain the bounds for the quotients

$$\frac{(1+r)F(r)}{F\left(\frac{4r}{(1+r)^2}\right)},\ \frac{(1+r)F\left(r'^2\right)}{F\left(\frac{1-r}{1+r}\right)},\ \frac{\sqrt{1+3r}F\left(r^2\right)}{F\left(1-\left(\frac{1-r}{1+3r}\right)^2\right)}\ \text{and}\ \frac{\sqrt{1+3r}F\left(r'^2\right)}{F\left(\left(\frac{1-r}{1+3r}\right)^2\right)}.$$

As an example, we only give the following inequalities: For $a,b\in(0,\infty)$ with $c=a+b$ and for $r\in(0,1)$,

$$1\leqslant\frac{(1+r)F(r)}{F\left(4r/(1+r)^2\right)}\leqslant\frac{B}{2}\ \text{ if }\ ab\leqslant\min\left\{\frac{1}{2},\frac{c}{3}\right\}, \tag{117}$$

$$\frac{B}{2} \leqslant \frac{(1+r)F(r)}{F\left(4r/(1+r)^2\right)} \leqslant 1 \quad \text{if } ab \geqslant \max\left\{\frac{1}{2}, \frac{c}{3}\right\}, \tag{118}$$

with equality in each instance if and only if $a=1/2$ and $b=1$. As a matter of fact, if $ab \leqslant \min\{1/2, c/3\}$ and $x = 4r/(1+r)^2$, then it follows from Corollary 3.1 and (26) that

$$1 = \frac{f_4(r)}{f_4(r)} \leqslant \frac{(1+r)F(r)}{F(x)} = \frac{F_0(x)}{F(x)} \cdot \frac{F(r)}{F_0(r)} = \frac{f_4(r)}{f_4(x)} \leqslant \frac{f_4(0)}{f_4(1)} = \frac{B}{2},$$

where f_4 is as in Corollary 3.1. The second and fifth equalities hold if and only if $a=1/2$ and $b=1$. This yields the double inequality (117) and its equality case. Similarly, we can prove (118) and its equality case.

(iii) By applying the results in Section 6, one can obtain some properties of the so-called generalized Hersch-Pluger distortion function $\varphi_K(a,b,r) \equiv \mu_{a,b}^{-1}(\mu_{a,b}(r)/K)$, which can express the solutions of Ramanujan's modular equations (cf. [2, 3, 10]). These results will be presented in a separate paper.

(iv) *Conjecture.* Let f and h be as in Theorem 4.1 and in Theorem 5.1, respectively. Our computation seems to show that the following conjecture is true.

CONJECTURE 7.1. If $ab \leqslant \min\{2-c, (c+1)/5\}$ or $2-c < ab < 1/2$ ($ab \geqslant \max\{2-c, (c+1)/5\}$ or $1/2 < ab < 2-c$), then f is convex (concave, respectively) on $(0,1)$. If $ab \leqslant \min\{(19/16)-c, 3(c+1)/32\}$ or $(19/16)-c < ab < 3/16$ ($ab \geqslant \max\{(19/16)-c, 3(c+1)/32\}$ or $3/16 < ab < (19/16)-c$), then h is convex (concave, respectively) on $(0,1)$. *If this conjecture is true, then the double inequalities in Theorems 4.1 and 5.1 can be improved.*

Acknowledgements. This work is supported by the Natural Science Foundation of China (Grant Nos. 11771400, 11701176 and 61673169) and Zhejiang Provincial Natural Science Foundation of China (Grant No. LQ17A010010).

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(Received December 1, 2019)

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