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REMARKS ON A LIMITING CASE OF HARDY TYPE INEQUALITIES

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Abstract. The classical Hardy inequality holds in Sobolev spaces $W_0^{1,p}$ when $1 \leqslant p < N$. In the limiting case where $p = N$, it is known that by introducing a logarithmic weight function in the Hardy potential, some inequality which is called the critical Hardy inequality holds in $W_0^{1,N}$. In this note, in order to give an explanation of the appearance of the logarithmic function in the potential, we derive the logarithmic function from the classical Hardy inequality with best constant via some limiting procedure as $p \nearrow N$. We show that our limiting procedure is also available for the classical Rellich inequality in second order Sobolev spaces $W_0^{2,p}$ with $p \in (1, \frac{N}{2})$ and the Poincaré inequality.

1. Introduction

Let $B_1 \subset \mathbb{R}^N$ be the unit ball, $1 < p < N$ and $N \geqslant 2$. The classical Hardy inequality

$$\left(\frac{N-p}{p}\right)^p \int_{B_1} \frac{|u|^p}{|x|^p} dx \leqslant \int_{B_1} |\nabla u|^p dx, \tag{1}$$

holds for all $u \in W_0^{1,p}(B_1)$, where $W_0^{1,p}(B_1)$ is a completion of $C_c^\infty(B_1)$ with respect to the norm $\|\nabla(\cdot)\|_{L^p(B_1)}$. Note that the Hardy inequality (1) expresses the embedding $W_0^{1,p}(B_1) \hookrightarrow L^p(B_1;|x|^{-p}dx)$ which is equivalent to $W_0^{1,p}(B_1) \hookrightarrow L^{p^*,p}(B_1)$ thanks to rearrangement techniques, where $p^* = \frac{Np}{N-p}$ and $L^{p,q}$ are Lorentz spaces. Therefore by a property of Lorentz spaces, we see that for any $q > p$

$$W_0^{1,p} \hookrightarrow L^{p^*,p} \hookrightarrow L^{p^*,q} \hookrightarrow L^{p^*,\infty}.$$

For the above inclusion relations, we can observe that the Hardy inequality (1) derives the Sobolev inequality, without best constant, which expresses the Sobolev embedding $W_0^{1,p} \hookrightarrow L^{p^*} = L^{p^*,p^*}$. On the contrary, the Sobolev inequality with best constant derives the Hardy inequality with best constant as an infinite-dimensional form of the Sobolev inequality, see [32]. Besides, Hardy's best constant $\left(\frac{N-p}{p}\right)^p$ plays an important role to investigate several properties of solution to elliptic and parabolic partial differential

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equations, for example, stability of solution, instantaneous blow-up solution, global-in-time solution, see [8, 4], to name a few.

On the other hand, in the limiting case where $p = N$ the Hardy inequality (1) degenerates as the best constant vanishes. However, by introducing a logarithmic weight function in the Hardy potential, the following inequality holds, so-called the critical Hardy inequality:

$$\left(\frac{N-1}{N}\right)^N \int_{B_1} \frac{|u|^N}{|x|^N (\log \frac{a}{|x|})^N} dx \leqslant \int_{B_1} |\nabla u|^N dx \tag{2}$$

for all $u \in W_0^{1,N}(B_1)$ and $a \geqslant 1$, see [25, 24]. The inequality (2) expresses the embedding $W_0^{1,N}(B_1) \hookrightarrow L^N(B_1; |x|^{-N}(\log \frac{a}{|x|})^{-N} dx)$. Since for large a the weight functions $|x|^{-N}(\log \frac{a}{|x|})^{-N}$ are radially decreasing with respect to $|x|$, the embedding with $a > 1$ is equivalent to $W_0^{1,N} \hookrightarrow L^{\infty,N}(\log L)^{-1}$ thanks to rearrangement techniques, see Proposition 1 in §2. Here $L^{p,q}(\log L)^r$ are Lorentz-Zygmund spaces which are given by

$$L^{p,q}(\log L)^r = \left\{ u : B_1 \to \mathbb{R} \text{ measurable} \;\middle|\; \|u\|_{L^{p,q}(\log L)^r} < \infty \right\}$$

$$\|u\|_{L^{p,q}(\log L)^r} = \begin{cases} \left(\int_0^{|B_1|} s^{\frac{q}{p}-1} \left(1 + \log \frac{|B_1|}{s}\right)^{rq} (u^*(s))^q \, ds \right)^{\frac{1}{q}} & \text{if } 1 \leqslant q < \infty, \\ \sup_{0<s<|B_1|} s^{\frac{1}{p}} \left(1 + \log \frac{|B_1|}{s}\right)^r u^*(s) & \text{if } q = \infty, \end{cases}$$

where u^* be the decreasing rearrangement of u. Note that $L^{p,q}(\log L)^0$ become Lorentz spaces $L^{p,q}$ and $L^{\infty,\infty}(\log L)^r$ become Zygmund spaces Z^{-r} which can be equivalent to Orlicz space $L_{e^{|u|^{-1/r}}} = \mathrm{ExpL}^{-\frac{1}{r}}$ in some sense, see [6, 7, 14]. By a property of Lorentz-Zygmund spaces, see e.g. [6] Theorem 9.5., we see that for any $q > N$

$$W_0^{1,N} \hookrightarrow L^{\infty,N}(\log L)^{-1} \hookrightarrow L^{\infty,q}(\log L)^{-1+\frac{1}{N}-\frac{1}{q}} \hookrightarrow L^{\infty,\infty}(\log L)^{-1+\frac{1}{N}} = \mathrm{ExpL}^{\frac{N}{N-1}}.$$

Variational problems related to best constants in embedding inequalities are intensively studied, see [3, 10, 1, 20, 14, 22, 31].

In this note, in order to give an explanation of the appearance of the logarithmic function in the limiting case $p = N$ of the classical Hardy inequality, we shall derive the logarithmic function in (2) from the classical Hardy inequality with best constant by some limiting procedure as $p \nearrow N$ based on extrapolation. This will give a reason *why Lorentz-Zygmund spaces* $L^{p,q}(\log L)^r$ *appear in embedding of critical Sobolev spaces* $W_0^{1,N}$. Our main result is the following.

THEOREM 1. *The following critical Hardy type inequality (3) can be derived by a limiting procedure for the classical Hardy inequality (1) as* $p \nearrow N$.

$$C \int_{B_1} \frac{|u|^N}{|x|^N \left(\log \frac{a}{|x|}\right)^\beta} dx \leqslant \int_{B_1} \left| \nabla u \cdot \frac{x}{|x|} \right|^N dx \quad (u \in C_c^1(B_1)). \tag{3}$$

Here $\beta > 2N, a > 1$*, and the constant* $C = C(\beta, a, N) > 0$ *is independent of* u.

Note that in the inequality (3) the exponent β as well as the constant C are not optimal and that the inequality itself follows from the critical Hardy inequality (2) and the following obvious fact:

$$\int_{B_1} \frac{|u|^N}{|x|^N \left(\log \frac{a}{|x|}\right)^\beta} dx \leqslant (\log a)^{N-\beta} \int_{B_1} \frac{|u|^N}{|x|^N \left(\log \frac{a}{|x|}\right)^N} dx$$

However, our limiting procedure for the classical Hardy inequality is new and gives some explanation of the appearance of the logarithmic function in the Hardy potential in the limiting case $p = N$. Our limiting procedure can be regarded as an analogue of Trudinger's argument in [34] for the Sobolev inequality as $p \nearrow N$, see also [9] Theorem 1.7. For several limiting procedures, we refer to [5] (The Sobolev inequality as $N \nearrow \infty$), [35], [36]XII 4.41. (L^p boundedness of the Hilbert transformation as $p \searrow 1$ or $p \nearrow \infty$), [29] Corollary 3.2.4 (The Sobolev inequality is derived from the Nash inequality), see also [33] which is a survey concerning two kinds of limiting procedures for the Hardy and the Sobolev inequalities as $p \nearrow N$ and $N \nearrow \infty$.

A few comments on Theorem 1 are in order. Very recently, Ioku [21] showed an improved Hardy inequality on B_1 which is equivalent to the classical Hardy inequality on $\mathbb{R}^N$ via a transformation. The improved Hardy inequality yields the critical Hardy inequality (2) with best constant as the limiting form of the improved Hardy inequality as $p \nearrow N$. However, in the higher order or fractional order case, these beautiful and simple structures and transformations might be useless. Indeed, we can see a different strucure in the second order case, see [30] §2. In this note, we also treat the second order case. For interesting equivalence in functional inequalities underlie to function space embeddings, see [16, 26, 13, 15].

Our limiting procedure also gives an interesting relationship between the Hardy inequality and the Poincaré inequality. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. The Poincaré inequality follows from the Hardy inequality since

$$\int_\Omega |u|^p \, dx \leqslant \left(\max_{x\in\Omega} |x|^p\right) \int_\Omega \frac{|u|^p}{|x|^p} dx \leqslant \left(\max_{x\in\Omega} |x|^p\right) \left(\frac{N-p}{p}\right)^{-p} \int_\Omega |\nabla u|^p \, dx.$$

Therefore, we can say that the Hardy inequality $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega; |x|^{-p}dx)$ is stronger than the Poincaré inequality $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ if we ignore the optimality of Poincaré's best constant $\lambda(\Omega)$. The converse is not straightforward, however, by using the information of Poincaré's best constant $\lambda(\Omega)$ as $|\Omega| \searrow 0$ and our limiting procedure, we obtain the reverse inequality though we miss the best constant. For the details, see §4.

This paper is organized as follows: In §2, necessary preliminary facts are presented. In §3, we give the limiting procedure as $p \nearrow N$ for the Hardy inequality in Theorem 1. We also apply our limiting procedure for the Rellich inequality. In §4, we consider *a limit* as $|\Omega| \searrow 0$ for the Poincaré inequality via our limiting procedure.

Let us fix some notation. B_R denote a N-dimensional ball centered 0 with radius R and ω_{N-1} denotes an area of the unit sphere in $\mathbb{R}^N$. $|A|$ denotes the Lebesgue measure of a set $A \subset \mathbb{R}^N$ and $X_{\mathrm{rad}} = \{u \in X \,|\, u \text{ is radial}\}$. Throughout the paper, if a radial function u is written as $u(x) = \tilde{u}(|x|)$ by some function $\tilde{u} = \tilde{u}(r)$, we write $u(x) = u(|x|)$ for simplicity.

2. Preliminaries

We recall some basic facts which will be used in the sequel.

LEMMA 1. *For any radial functions* $u \in C^1(B_R) \cap C(\overline{B_R})$ *satisfying* $u|_{\partial B_R} = 0$, *for any* $r \in (0,R)$ *the following estimate holds.*

$$|u(r)| \leqslant \begin{cases} \omega_{N-1}^{-1} \|\nabla u\|_{L^1(B_R)} r^{-(N-1)} & \textit{if } p = 1, \\ \left(\frac{p-1}{|N-p|}\right)^{\frac{p-1}{p}} \omega_{N-1}^{-\frac{1}{p}} \|\nabla u\|_{L^p(B_R)} \left| r^{-\frac{N-p}{p-1}} - R^{-\frac{N-p}{p-1}} \right|^{\frac{p-1}{p}} & \textit{if } 1 < p \neq N, \\ \omega_{N-1}^{-\frac{1}{N}} \|\nabla u\|_{L^N(B_R)} \left(\log \frac{R}{r}\right)^{\frac{N-1}{N}} & \textit{if } p = N. \end{cases}$$

The pointwise estimate in Lemma 1 is well-known and follows form the fundamental theorem of calculus and the Hölder inequality immediately. Here we give the proof for reader's convenience.

Proof. When $p = 1$, we have

$$u(r) = -\int_r^R u'(s)\,ds = -\int_r^R u'(s)\,s^{N-1} s^{-(N-1)}\,ds \leqslant \omega_{N-1}^{-1} \|\nabla u\|_{L^1(B_R)} r^{-(N-1)}.$$

When $1 < p \leqslant N$, we have

$$u(r) = -\int_r^R u'(s)\,s^{\frac{N-1}{p}} s^{-\frac{N-1}{p}}\,ds \leqslant \left(\int_r^R |u'(s)|^p s^{N-1}\,ds\right)^{\frac{1}{p}} \left(\int_r^R s^{-\frac{N-1}{p-1}}\,ds\right)^{\frac{p-1}{p}}$$

which implies the desired estimate. □

When the potential function is not radially decreasing, it is difficult to apply rearrangement techniques. The next lemma will enable us to reduce the problem to the radial setting.

LEMMA 2. *Let* $1 < q < \infty$, $V = V(x)$ *be a radial function on* B_R. *If there exists* $C > 0$ *such that for any radial functions* $u \in C_c^1(B_R)$ *the inequality*

$$C \int_{B_R} |u|^q V(x)\,dx \leqslant \int_{B_R} |\nabla u|^q\,dx < \infty \tag{4}$$

holds, then for any functions $w \in C_c^1(B_R)$ *the inequality*

$$C \int_{B_R} |w|^q V(x)\,dx \leqslant \int_{B_R} \left|\nabla w \cdot \frac{x}{|x|}\right|^q dx < \infty \tag{5}$$

holds.

Proof. We refer to [31]. For any $w \in C_c^1(B_R)$, define the radial function W as follows.

$$W(r) = \left(\omega_{N-1}^{-1} \int_{\omega \in \partial B_1} |w(r\omega)|^q \, dS_\omega \right)^{\frac{1}{q}} \quad (0 \leqslant r \leqslant R).$$

Then we have

$$\begin{aligned} |W'(r)| &= \omega_{N-1}^{-\frac{1}{q}} \left(\int_{\partial B_1} |w(r\omega)|^q \, dS_\omega \right)^{\frac{1}{q}-1} \int_{\partial B_1} |w|^{q-1} \left| \frac{\partial w}{\partial r} \right| dS_\omega \\ &\leqslant \omega_{N-1}^{-\frac{1}{q}} \left(\int_{\partial B_1} \left| \frac{\partial w}{\partial r}(r\omega) \right|^q dS_\omega \right)^{\frac{1}{q}}. \end{aligned}$$

Therefore, we have

$$\int_{B_R} |\nabla W|^q \, dx \leqslant \int_{B_R} \left| \nabla w \cdot \frac{x}{|x|} \right|^q dx, \tag{6}$$

$$\int_{B_R} |W|^q V(x) \, dx = \int_{B_R} |w|^q V(x) \, dx. \tag{7}$$

From (4) for W, (6), and (7), we obtain (5) for any w. □

We next establish the pointwise estimates for radial functions and their derivatives in $W_0^{2,p}(B_R)$ proved in [11, 12]. For much higher order case, see Proposition 2 in §3.

LEMMA 3. *Let $N \geqslant 3$ and $u \in C^2(B_R) \cup C(\overline{B_R})$ be a radial function satisfying $u|_{\partial B_R} = 0$. Then the following pointwise estimates hold for any $r \in (0,R)$.*

$$|u(r)| \leqslant \begin{cases} \frac{p}{|N-2p|} \omega_{N-1}^{-\frac{1}{p}} N^{\frac{1}{p}-1} \|\Delta u\|_{L^p(B_R)} \left| r^{-\frac{N-2p}{p}} - R^{-\frac{N-2p}{p}} \right| & \textit{if } 1 \leqslant p \neq \frac{N}{2}, \\ \omega_{N-1}^{-\frac{2}{N}} N^{\frac{2}{N}-1} \|\Delta u\|_{L^{N/2}(B_R)} \log \frac{R}{r} & \textit{if } p = \frac{N}{2}. \end{cases} \tag{8}$$

$$|u'(r)| \leqslant \frac{\|\Delta u\|_{L^p(B_R)}}{\omega_{N-1}^{\frac{1}{p}} N^{1-\frac{1}{p}}} r^{-\frac{N-p}{p}} \quad \textit{for any } p \geqslant 1. \tag{9}$$

Proof. We refer to [12]. Consider the following transformation:

$$w(t) = Au(r), \text{ where } r = R(t+1)^{-\frac{1}{N-2}} \text{ and } A^p = \omega_{N-1} R^{N-2p} (N-2)^{2p-1} \tag{10}$$

Then we have

$$w''(t) = \frac{AR^2}{(N-2)^2}(t+1)^{-2\frac{N-1}{N-2}} \left(u''(r) + \frac{N-1}{r} u'(r) \right) = \frac{AR^2}{(N-2)^2}(t+1)^{-2\frac{N-1}{N-2}} \Delta u \tag{11}$$

which yields that

$$\int_{B_R} |\Delta u|^p \, dx = \int_0^\infty |w''(t)|^p (t+1)^{2\frac{(N-1)(p-1)}{N-2}} \, dt.$$

Since $w(0) = w'(\infty) = 0$, we have

$$\begin{aligned}
w(t) &= -\int_0^t \int_s^\infty w''(u)\, du\, ds \\
&\leqslant \int_0^t \left(\int_0^\infty |w''(u)|^p (u+1)^{2\frac{(N-1)(p-1)}{N-2}} \, du \right)^{\frac{1}{p}} \left(\int_s^\infty (u+1)^{-2\frac{N-1}{N-2}} \, du \right)^{\frac{p-1}{p}} ds \\
&= \left(\frac{N-2}{N} \right)^{\frac{p-1}{p}} \|\Delta u\|_{L^p(B_R)} \int_0^t (s+1)^{-\frac{N(p-1)}{(N-2)p}} \, ds \\
&= \begin{cases} \left(\frac{N-2}{N}\right)^{\frac{p-1}{p}} \frac{(N-2)p}{N-2p} \|\Delta u\|_{L^p(B_R)} \left((t+1)^{\frac{N-2p}{(N-2)p}} - 1 \right) & \text{if } p \neq \frac{N}{2}, \\ \left(\frac{N-2}{N}\right)^{1-\frac{2}{N}} \|\Delta u\|_{L^{N/2}(B_R)} \log(t+1) & \text{if } p = \frac{N}{2}. \end{cases}
\end{aligned}$$

Therefore we obtain (8). On the other hand, since

$$w'(t) = -\int_t^\infty w''(u)\, du \leqslant \left(\frac{N-2}{N} \right)^{\frac{p-1}{p}} \|\Delta u\|_{L^p(B_R)} (t+1)^{\frac{N(p-1)}{(N-2)p}}$$

and $w'(t) = -Au'(r)\frac{R}{N-2}(t+1)^{-\frac{N-1}{N-2}}$, we also obtain (9). □

Finally, we show the equivalence in two embeddings of critical Sobolev spaces $W_0^{1,N}(B_1)$. Let us recall the Schwarz symmetrization $u^\#\colon \mathbb{R}^N \to [0,\infty]$ of u which is given by

$$u^\#(x) = u^\#(|x|) = \inf\left\{ \tau > 0 : |\{ y \in \mathbb{R}^N : |u(y)| > \tau \}| \leqslant |B_{|x|}| \right\} = u^*\left(\frac{\omega_{N-1}}{N} |x|^N \right).$$

PROPOSITION 1. *Let* $a > 1$.
Then the embedding $W_0^{1,N}(B_1) \hookrightarrow L^N(B_1; |x|^{-N}(\log \frac{a}{|x|})^{-N} dx)$ *is equivalent to the embedding* $W_0^{1,N}(B_1) \hookrightarrow L^{\infty,N}(\log L)^{-1}$.

Proof. Since there exists a constant $C > 0$ such that for any $x \in B_1$

$$C^{-1} \log \frac{e}{|x|} \leqslant \log \frac{a}{|x|} \leqslant C \log \frac{e}{|x|},$$

it is enough to show the equivalence when $a = e$. We see that

$$\begin{aligned}\|u\|^N_{L^{\infty,N}(\log L)^{-1}} &= \int_0^{|B_1|} s^{-1}\left(1+\log\frac{|B_1|}{s}\right)^{-N}(u^*(s))^N\,ds\\ &= N\int_0^1 r^{-1}\left(N\log\frac{e^{\frac{1}{N}}}{r}\right)^{-N}|u^\#(r)|^N\,dr\\ &\geqslant N^{1-N}\omega_{N-1}^{-1}\int_{B_1}\frac{|u^\#|^N}{|x|^N\left(\log\frac{e}{|x|}\right)^N}\,dx\\ &\geqslant N^{1-N}\omega_{N-1}^{-1}\int_{B_1}\frac{|u|^N}{|x|^N\left(\log\frac{e}{|x|}\right)^N}\,dx,\end{aligned}$$

where the last inequality comes from the Hardy-Littlewood inequality. Therefore we can derive the embedding $W_0^{1,N}(B_1)\hookrightarrow L^N(B_1;|x|^{-N}(\log\frac{e}{|x|})^{-N}dx)$ from the embedding $W_0^{1,N}(B_1)\hookrightarrow L^{\infty,N}(\log L)^{-1}$. On the other hand, we assume that the embedding $W_0^{1,N}(B_1)\hookrightarrow L^N(B_1;|x|^{-N}(\log\frac{e}{|x|})^{-N}dx)$ holds. Since $\frac{1}{N}\log\frac{e}{|x|}\leqslant\log\frac{e^{\frac{1}{N}}}{|x|}$ for any $x\in B_1$, we see that

$$\|u\|^N_{L^{\infty,N}(\log L)^{-1}}\leqslant N\omega_{N-1}^{-1}\int_{B_1}\frac{|u^\#|^N}{|x|^N\left(\log\frac{e}{|x|}\right)^N}\,dx\leqslant\int_{B_1}|\nabla u^\#|^N\,dx\leqslant\int_{B_1}|\nabla u|^N\,dx$$

where the last inequality comes from the Polyá-Szegö inequality. Therefore, we can obtain the desired equivalence.

3. A limiting procedure for the Hardy type inequalities

3.1. Proof of Theorem 1: The Hardy inequality

First, we prepare for making the optimal constant $(\frac{N-p}{p})^p$ which goes to zero as $p\nearrow N$ compete with $\int_{B_1}\frac{|u|^p}{|x|^p}\,dx$ which goes to infinity, in general, as $p\nearrow N$.

Let $p_k = N-\frac{1}{k}$ for $k\in\mathbb{N}$, $f\in C^1(-\infty,\infty)$ be a monotone-decreasing function which satisfies $\lim_{t\to+\infty}f(t)=0$, and $\{\phi_k\}_{k\in\mathbb{Z}}\subset C_c^\infty(\mathbb{R}^N\setminus\{0\})$ be radial functions which satisfy

$$\begin{aligned}&(i)\ \sum_{k=-\infty}^{+\infty}\phi_k(x)^N = 1,\ 0\leqslant\phi_k(x)\leqslant 1\ \left(\forall x\in\mathbb{R}^N\setminus\{0\}\right),\\ &(ii)\ \operatorname{supp}\phi_k\subset B_{f(k)}\setminus B_{f(k+2)}.\end{aligned}$$

For any radial functions $u\in C_c^1(B_1)$, set $u_k = u\,\phi_k$ and

$$A_k = \operatorname{supp} u_k\subset B_1\cap\left(B_{f(k)}\setminus B_{f(k+2)}\right).$$

In order to obtain *a limit* for the classical Hardy inequality (1) as $p \nearrow N$, the left-hand side of (1) for u_k and p_k must not be vanishing as $k \to \infty$. We shall determine such f. Note that if $x \in A_k$, then $f(k+2) \leqslant |x| \leqslant f(k)$ and $k \leqslant f^{-1}(|x|) \leqslant k+2$. By Lemma 1 we have

$$\begin{aligned}
&\left(\frac{N-p_k}{p_k}\right)^{p_k} \int_{A_k} \frac{|u_k|^{p_k}}{|x|^{p_k}} dx = p_k^{-p_k} \int_{A_k} \left(\frac{|u_k(x)|}{|x|\,k}\right)^{N-\frac{1}{k}} dx \\
&\geqslant C \int_{A_k} \frac{|u_k(x)|^N}{|x|^N \left(f^{-1}(|x|)\right)^N} \left(\frac{|x|\,k}{|u_k(x)|}\right)^{\frac{1}{k}} dx \\
&\geqslant C \|\nabla u_k\|_{L^N(A_k)}^{-\frac{1}{k}} \int_{A_k} \frac{|u_k(x)|^N}{|x|^N \left(f^{-1}(|x|)\right)^N} \left(f(k+2)\left(\log \frac{f(k)}{f(k+2)}\right)^{-\frac{N-1}{N}}\right)^{\frac{1}{k}} dx. \qquad (12)
\end{aligned}$$

On the other hand, by the classical Hardy inequality (1) we have

$$\left(\frac{N-p_k}{p_k}\right)^{p_k} \int_{A_k} \frac{|u_k|^{p_k}}{|x|^{p_k}} dx \leqslant \int_{A_k} |\nabla u_k|^{p_k} dx \leqslant C \|\nabla u_k\|_{L^N(A_k)}^{-\frac{1}{k}} \int_{A_k} |\nabla u_k|^N dx.$$

Therefore, if for any $k \in \mathbb{N}$ the function f satisfies

$$\left(f(k+2)\left(\log \frac{f(k)}{f(k+2)}\right)^{-\frac{N-1}{N}}\right)^{\frac{1}{k}} \geqslant C > 0, \qquad (13)$$

then the information of the classical Hardy inequality (1) for u_k and p_k will be remaining even if we sum up (1) for u_k and p_k with respect to $k \in \mathbb{Z}$. From (13) and l'Hôpital's rule, we have an ordinary differential inequality for f as follows:

$$\frac{d}{dt} f(t) \geqslant -C f(t)$$

whose solution satisfies $f(t) \geqslant e^{-Ct}$. Thus $f^{-1}(t) \geqslant \frac{1}{C} \log \frac{1}{t}$. We believe that the above calculation and consideration give some explanation of the appearance of the logarithmic function in the Hardy potential in the limiting case $p = N$.

Hereinafter we set $f(t) = e^{-t}$.

Proof of Theorem 1. From Lemma 2, it is enough to show the inequality (3) for any radial functions $u \in C_c^1(B_1)$. Applying the classical Hardy inequality (1) for u_k and p_k for $k \geqslant 1$, we have

$$\left(\frac{N-p_k}{p_k}\right)^{p_k} \int_{A_k} \frac{|u_k|^{p_k}}{|x|^{p_k}} dx \leqslant \int_{A_k} |\nabla u_k|^{p_k} dx \leqslant |A_k|^{1-\frac{p_k}{N}} \|\nabla u_k\|_N^{N-\frac{1}{k}}.$$

By (12) and (13), for $k \geqslant 1$

$$C \int_{A_k} \frac{|u_k|^N}{|x|^N \left(\log \frac{1}{|x|}\right)^N} dx \leqslant \int_{A_k} |\nabla u_k|^N dx.$$

Since $k \leqslant \log \frac{1}{|x|}$ for $x \in A_k$,

$$C \int_{A_k} \frac{|u_k|^N}{|x|^N \left(\log \frac{a}{|x|}\right)^{\beta}} dx \leqslant b_k \int_{A_k} |\nabla u_k|^N dx \tag{14}$$

for $k \geqslant 1, a > 1$, and $\beta > 2N$, where b_k is given by

$$b_k = \begin{cases} k^{N-\beta} & \text{if } k \geqslant 1, \\ 1 & \text{if } k = 0, -1, \\ 0 & \text{if } k \leqslant -2. \end{cases}$$

Here, note that the inequalities (14) with $k = 0, -1$ come form the Poincaré inequality and the boundedness of the function $|x|^{-N}(\log \frac{a}{|x|})^{-\beta}$ on $A_0 \cup A_{-1} \subset B_1 \setminus B_{e^{-2}}$. Summing both sides on (14), we have

$$C \sum_{k \in \mathbb{Z}} \int_{B_1} \frac{|u\phi_k|^N}{|x|^N \left(\log \frac{a}{|x|}\right)^{\beta}} dx \leqslant \sum_{k \in \mathbb{Z}} b_k \int_{A_k} |\nabla(u\phi_k)|^N dx$$

which yields that

$$\begin{aligned} C \int_{B_1} \frac{|u|^N}{|x|^N \left(\log \frac{a}{|x|}\right)^{\beta}} dx &\leqslant 2^{N-1} \sum_{k \in \mathbb{Z}} b_k \int_{A_k} \phi_k^N |\nabla u|^N + |u|^N |\nabla \phi_k|^N dx \\ &\leqslant 2^{N-1} \int_{B_1} |\nabla u|^N dx + C \sum_{k=1}^{+\infty} b_k e^{kN} \int_{A_k} |u|^N dx. \end{aligned} \tag{15}$$

By Lemma 1 we have

$$\begin{aligned} b_k e^{kN} \int_{A_k} |u|^N dx &\leqslant C b_k e^{kN} \|\nabla u\|_N^N \int_{A_k} \left(\log \frac{1}{|x|}\right)^{N-1} dx \\ &\leqslant C b_k e^{kN} \|\nabla u\|_N^N \int_k^{k+2} s^{N-1} e^{-sN} ds \leqslant C b_k k^{N-1} \|\nabla u\|_N^N. \end{aligned}$$

From (15) we have

$$\begin{aligned} C \int_{B_1} \frac{|u|^N}{|x|^N \left(\log \frac{a}{|x|}\right)^{\beta}} dx &\leqslant C \int_{B_1} |\nabla u|^N dx + C \left(\sum_{k=1}^{+\infty} k^{-1-(\beta-2N)} \right) \int_{B_1} |\nabla u|^N dx \\ &\leqslant C \int_{B_1} |\nabla u|^N dx. \quad \square \end{aligned}$$

3.2. The Rellich inequality

Let $1 < p < \frac{N}{2}$. The classical Rellich inequality

$$\left(\frac{N(p-1)(N-2p)}{p}\right)^p \int_{B_1} \frac{|u|^p}{|x|^{2p}} dx \leqslant \int_{B_1} |\Delta u|^p dx \tag{16}$$

holds for all $u \in W_0^{2,p}(B_1)$, where $W_0^{2,p}(B_1)$ is a completion of $C_c^\infty(B_1)$ with respect to the norm $\|\Delta(\cdot)\|_{L^p(B_1)}$, see [28], [17], [27]. In this subsection, we apply our limiting procedure in §3.1 to the Rellich inequality (16) as $p \nearrow \frac{N}{2}$.

THEOREM 2. *The following critical Rellich type inequality (17) can be derived by a limiting procedure for the classical Rellich inequality (16) as $p \nearrow \frac{N}{2}$.*

$$C \int_{B_1} \frac{|u|^{\frac{N}{2}}}{|x|^N \left(\log \frac{a}{|x|}\right)^\beta} dx \leqslant \int_{B_1} |\Delta u|^{\frac{N}{2}} dx \quad (u \in C^2_{c,rad}(B_1)). \tag{17}$$

Here $\beta > N+2, a > 1$, and the constant $C = C(\beta, a, N) > 0$ is independent of u.

REMARK 1. Like Theorem 1, in the inequality (17) the exponent β as well as the constant C are not optimal. For the optimal exponent and the best constant of the critical Rellich inequality, see e.g. [18], [2].

Proof. We shall show (17) for any $u \in C^2_{c,\mathrm{rad}}(B_1)$. The strategy of the proof is the same as it in §3.1.

Let $p_k = \frac{N}{2} - \frac{1}{2k}$ for $k \geqslant 2$ and only condition (i) of ϕ_k in §3.1 is changed to $\sum_{k=-\infty}^{+\infty} \phi_k(x)^{\frac{N}{2}} = 1$ for any $x \in \mathbb{R}^N \setminus \{0\}$. Applying the classical Rellich inequality (16) for $u_k = u\phi_k$ for $u \in C^2_{c,\mathrm{rad}}(B_1)$ and p_k for $k \geqslant 2$, we have

$$\left(\frac{(N-2p_k)(p_k-1)N}{p_k}\right)^{p_k} \int_{A_k} \frac{|u_k|^{p_k}}{|x|^{2p_k}} dx \leqslant \int_{A_k} |\Delta u_k|^{p_k} dx. \tag{18}$$

On the left-hand side of (18), by (8) in Lemma 3 we have

$$\begin{aligned}
&\left(\frac{(N-2p_k)(p_k-1)N}{p_k}\right)^{p_k} \int_{A_k} \frac{|u_k|^{p_k}}{|x|^{2p_k}} dx \geqslant C \int_{A_k} \left(\frac{|u_k(x)|}{|x|^2 k}\right)^{\frac{N}{2}-\frac{1}{2k}} dx \\
&\geqslant C \int_{A_k} \frac{|u_k(x)|^{\frac{N}{2}}}{|x|^N (f^{-1}(|x|))^{\frac{N}{2}}} \left(\frac{|x|^2 k}{|u_k(x)|}\right)^{\frac{1}{2k}} dx \\
&= C\|\Delta u_k\|_{L^{\frac{N}{2}}(A_k)}^{-\frac{1}{2k}} \int_{A_k} \frac{|u_k(x)|^{\frac{N}{2}}}{|x|^N (f^{-1}(|x|))^{\frac{N}{2}}} \left(f(k+2)^2 \left(\log \frac{f(k)}{f(k+2)}\right)^{-1}\right)^{\frac{1}{2k}} dx.
\end{aligned}$$

If we choose $f(t) = e^{-t}$, then the left-hand side of (18) is not vanishing as $k \to \infty$. Thus we set $f(t) = e^{-t}$ hereinafter. In the similar way to it in §3.1, for $a > 1$, $k \in \mathbb{Z}$, and $\beta > N+2$ we have

$$C \int_{A_k} \frac{|u_k|^{\frac{N}{2}}}{|x|^N \left(\log \frac{a}{|x|}\right)^{\beta}} dx \leqslant b_k \int_{A_k} |\Delta u_k|^{\frac{N}{2}} dx, \tag{19}$$

where b_k is given by

$$b_k = \begin{cases} k^{\frac{N}{2}-\beta} & \text{if } k \geqslant 2, \\ 1 & \text{if } k = 1, 0, -1, \\ 0 & \text{if } k \leqslant -2. \end{cases}$$

Note that we used the second order Poincaré inequality $C\|u\|_q \leqslant \|\Delta u\|_q$ to show (19) in the case where $k \leqslant 1$, see e.g. [19]. Then we have

$$C \sum_{k \in \mathbb{Z}} \int_{B_1} \frac{|u\phi_k|^{\frac{N}{2}}}{|x|^N \left(\log \frac{a}{|x|}\right)^{\beta}} dx \leqslant \sum_{k \in \mathbb{Z}} b_k \int_{A_k} |\Delta(u\phi_k)|^{\frac{N}{2}} dx$$

which yields that

$$\begin{aligned} C \int_{B_1} \frac{|u|^{\frac{N}{2}}}{|x|^N \left(\log \frac{a}{|x|}\right)^{\beta}} dx &\leqslant C \sum_{k=2}^{\infty} b_k \int_{A_k} |\Delta \phi_k|^{\frac{N}{2}} |u|^{\frac{N}{2}} + \phi_k^{\frac{N}{2}} |\Delta u|^{\frac{N}{2}} + |\nabla u|^{\frac{N}{2}} |\nabla \phi_k|^{\frac{N}{2}} dx \\ &=: C \sum_{k=2}^{\infty} (I_1 + I_2 + I_3). \end{aligned} \tag{20}$$

Since $|\Delta \phi_k(x)| \leqslant Ce^{2(k+1)}$ for $x \in A_k$, by (8) in Lemma 3 we have

$$\begin{aligned} I_1 &\leqslant Ck^{\frac{N}{2}-\beta} e^{N(k+1)} \int_{A_k} |u|^{\frac{N}{2}} dx \\ &\leqslant Ck^{\frac{N}{2}-\beta} e^{N(k+1)} \|\Delta u\|_{\frac{N}{2}}^{\frac{N}{2}} \int_{A_k} \left(\log \frac{1}{|x|}\right)^{\frac{N}{2}} dx \\ &\leqslant Ck^{\frac{N}{2}-\beta} e^{N(k+1)} \|\Delta u\|_{\frac{N}{2}}^{\frac{N}{2}} \int_k^{k+2} t^{\frac{N}{2}} e^{-Nt} dt \\ &\leqslant Ck^{N+1-\beta} \|\Delta u\|_{\frac{N}{2}}^{\frac{N}{2}}. \end{aligned}$$

In the similar way, we obtain the following estimates of I_2 and I_3.

$$I_2 + I_3 \leqslant Ck^{\frac{N}{2}-\beta} \|\Delta u\|_{\frac{N}{2}}^{\frac{N}{2}}.$$

Here we used (9) in Lemma 3 to show the estimate of I_3. From (20) and the estimates of I_i for $i = 1,2,3$ we have

$$C\int_{B_1} \frac{|u|^{\frac{N}{2}}}{|x|^N \left(\log \frac{a}{|x|}\right)^{\beta}} dx \leqslant C\left(\sum_{k=2}^{\infty} k^{N+1-\beta}\right)\int_{B_1} |\Delta u|^{\frac{N}{2}} dx \leqslant C\int_{B_1} |\Delta u|^{\frac{N}{2}} dx. \quad \square$$

Let $1 < p < \frac{N}{m}$ and $m \geqslant 2$. The higher order Rellich inequality

$$C_{m,p}^p \int_{B_R} \frac{|u|^p}{|x|^{mp}} dx \leqslant |u|_{m,p}^p$$

holds for all $u \in W_0^{m,p}(B_R)$, see [28], [17], [27]. Here we set

$$|u|_{m,p}^p = \begin{cases} \int_{B_R} |\Delta^{\ell} u|^p \, dx & \text{if } m = 2\ell, \\ \int_{B_R} |\nabla(\Delta^{\ell} u)|^p \, dx & \text{if } m = 2\ell + 1, \end{cases}$$

$$C_{m,p} = \begin{cases} p^{-2\ell} \prod_{j=1}^{\ell} \{N - 2pj\}\{N(p-1) + 2p(j-1)\} & \text{if} m = 2\ell, \\ \frac{(N-p)}{p^{2(\ell+1)}} \prod_{j=1}^{\ell} (N - (2j+1)p)\{N(p-1) + (2j-1)p\} & \text{if} m = 2\ell + 1, \end{cases}$$

for $m, \ell \in \mathbb{N}, \ell \geqslant 1$.

In the higer order case where $m \geqslant 3$, it is difficult to show the pointwise estimate corresponding to Lemma 3 by the same method in Lemma 3. Due to the lack of good pointwise estimate for radial functions, our limiting procedure as $p \nearrow \frac{N}{m}$ does not work well in the higher order case. However, we can show at least the following pointwise estimates for radial functions in $W_0^{m,p}(B_R)$ for $m \geqslant 2$ via iteration method. The following pointwise estimates might be not optimal. We expect that the pointwise estimates in Proposition 2 will be applicable somewhere.

PROPOSITION 2. *Let $N, m \geqslant 3, p \in [1, \frac{N}{2})$ if m is even, and $p \in [1, N)$ if m is odd. Then the following pointwise estimates hold for any radial functions $u \in C_{c,rad}^m(B_R)$ and any $r \in (0, R)$.*

$$|u(r)| \leqslant C|u|_{m,p} r^{2-N} \tag{21}$$

Here, $C > 0$ is a constant which is independent of u

Proof. We shall show (21) for $p \in [1, N)$ and odd number m inductively. First, we show the case where $m = 3$. By the transformation (10) for radial functions u and the pointwise estimate for radial function $v := \Delta u \in W_0^{1,p}$, we obtain

$$|v(r)| \leqslant C\|\nabla v\|_p r^{-\frac{N-p}{p}} = C\|\nabla \Delta u\|_p (t+1)^{\frac{N-p}{(N-2)p}}.$$

By (11) we have

$$|w''(t)| \leqslant C\|\nabla\Delta u\|_p (t+1)^a,$$

where $a = \frac{N-(2N-1)p}{(N-2)p} < -1$. Therefore we have

$$\begin{aligned}|w(t)| &\leqslant \int_0^t \int_0^s |w''(u)|\,du\,ds\\ &\leqslant C\|\nabla\Delta u\|_p \int_0^t \int_0^s (u+1)^a\,du\,ds\\ &\leqslant C\|\nabla\Delta u\|_p \max\{(t+1)^{a+2}, t+1\} \leqslant C\|\nabla\Delta u\|_p (t+1).\end{aligned}$$

Thus, we obtain (21) for $m=3$. Next, we assume that (21) holds for $m = 2\ell+1$. And we shall show that (21) also holds for $m = 2(\ell+1)+1$. For radial functions $u \in C_c^{2\ell+3}$, set $v := \Delta u \in C_c^{2\ell+1}$. Applying (21) for v, we have

$$|v(r)| \leqslant C\|\nabla\Delta^\ell v\|_{L^p(B_R)} r^{2-N}.$$

By (10) and (11), we have

$$|w''(t)| \leqslant C\|\nabla\Delta^{\ell+1} u\|_{L^p(B_R)} (t+1)^b,$$

where $b = -\frac{2N}{N-2} < -1$. Therefore we have

$$\begin{aligned}|w(t)| &\leqslant \int_0^t \int_0^s |w''(u)|\,du\,ds\\ &\leqslant C\|\nabla\Delta^{\ell+1} u\|_p \int_0^t \int_0^s (u+1)^b\,du\,ds\\ &\leqslant C\|\nabla\Delta^{\ell+1} u\|_p \max\{(t+1)^{b+2}, t+1\} \leqslant C\|\nabla\Delta^{\ell+1} u\|_p (t+1).\end{aligned}$$

Therefore, we observe that (21) holds for $m = 2(\ell+1)+1$.

In the even case, the strategy of the proof is the same as the odd case. In order to obtain (21) for $m=4$, we use the pointwise estimate in Lemma 3 for radial functions $v := \Delta u \in C_c^2$. We omit the proof. $\square$

4. A limiting procedure for the Poincaré inequality

In this section, we apply our limiting procedure to the Poincaré inequality

$$\lambda(\Omega) \int_\Omega |u|^p\,dx \leqslant \int_\Omega |\nabla u|^p\,dx \quad (u \in C_c^1(\Omega), 1 \leqslant p < \infty). \tag{22}$$

The Poincaré inequality (22) does not have a critical exponent with respect to p like the Hardy type inequalities. However the optimal constant $\lambda(\Omega)$ goes to infinity and $\int_\Omega |u|^p dx$ goes to zero as $|\Omega| \searrow 0$. This can be regarded as a kind of limiting situation. Recall that

$$\lambda(\Omega) \geqslant \left(\frac{N}{p}|B_1|\right)^p |\Omega|^{-\frac{p}{N}} \tag{23}$$

see e.g. [23]. By using this growth order of $\lambda(\Omega)$ as $|\Omega| \searrow 0$ and our limiting procedure, we shall consider *a limit* for the Poincaré inequality as $|\Omega| \searrow 0$.

THEOREM 3. *Let $1 \leqslant p < \frac{N^2}{N-1}$. The following Hardy type inequality (24) can be derived by a limiting procedure for the Poincaré inequality (22) as $|\Omega| \searrow 0$.*

$$C \int_{B_1} \frac{|u|^p}{|x|^\beta} dx \leqslant \int_{B_1} |\nabla u|^p \, dx \quad (u \in C_c^1(B_1)). \tag{24}$$

Here the constant $C = C(\beta, p, N) > 0$ is independent of u and $\beta > 0$ satisfies

$$\begin{cases} \beta < \frac{p}{N} & \text{if } 1 \leqslant p \leqslant N, \\ \beta < \frac{p}{N} + N - p & \text{if } N < p < \frac{N^2}{N-1}. \end{cases}$$

REMARK 2. If $p \geqslant \frac{N^2}{N-1}$, then it is difficult to obtain any information which is better than the Poincaré inequality (22) by out limiting procedure as $|\Omega| \searrow 0$, since $\beta = 0$ in that case.

Proof. From Lemma 2, it is enough to show the inequality (24) for any radial functions $u \in C_c^1(B_1)$. Let $1 \leqslant p < N$ and $\{\phi_k\}_{k \in \mathbb{Z}} \subset C_c^\infty(\mathbb{R}^N \setminus \{0\})$ be radial functions which satisfy

$$(i) \sum_{k=-\infty}^{+\infty} \phi_k(x)^p = 1,\, 0 \leqslant \phi_k(x) \leqslant 1 \ \left(\forall x \in \mathbb{R}^N \setminus \{0\}\right),$$

$$(ii) \operatorname{supp} \phi_k \subset B_{1/k} \setminus B_{1/k+2}.$$

Set $u_k = u\phi_k$ and $A_k = \operatorname{supp} u_k \subset B_1 \cap \left(B_{1/k} \setminus B_{1/k+2}\right)$. Applying the Poincaré inequality (22) for u_k and (23), we have

$$Ck^p(k+2)^{\frac{p}{N}} \int_{A_k} |u_k|^p \, dx \leqslant \int_{A_k} |\nabla u_k|^p \, dx.$$

Since $k \leqslant \frac{1}{|x|} \leqslant k+2$ for $x \in A_k$,

$$C \int_{A_k} \frac{|u_k|^p}{|x|^\beta} dx \leqslant b_k \int_{A_k} |\nabla u_k|^p dx \tag{25}$$

for $k \in \mathbb{Z}$, and $\beta > 2N$, where b_k is given by

$$b_k = \begin{cases} k^{-p}(k+2)^{\beta - \frac{p}{N}} & \text{if } k \geqslant 1, \\ 1 & \text{if } k = 0, -1, \\ 0 & \text{if } k \leqslant -2. \end{cases}$$

Summing both sides on (25), we have

$$C \sum_{k \in \mathbb{Z}} \int_{B_1} \frac{|u\phi_k|^p}{|x|^\beta} dx \leqslant \sum_{k \in \mathbb{Z}} b_k \int_{A_k} |\nabla (u\phi_k)|^N dx.$$

By applying Lemma 1 and calculating in the similar way to it in §3.1, we see that for $\beta < \frac{p}{N}$ the desired inequality (24) can be obtained. In the case where $N \leqslant p < \frac{N^2}{N-1}$, the proof is similar. Therefore, we omit the proof in that case. □

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REFERENCES

[1] ADIMURTHI, K. SANDEEP, *Existence and non-existence of the first eigenvalue of the perturbed Hardy-Sobolev operator*, Proc. Roy. Soc. Edinburgh Sect. A **132** (2002), No. 5, 1021–1043.

[2] ADIMURTHI, S. SANTRA, *Generalized Hardy-Rellich inequalities in critical dimension and its applications*, Commun. Contemp. Math., **11** (2009), No. 3, 367–394.

[3] A. ALVINO, *A limit case of the Sobolev inequality in Lorentz spaces*, Rend. Accad. Sci. Fis. Mat. Napoli (4) **44** (1977), 105–112 (1978).

[4] P. BARAS, J. A. GOLDSTEIN, *The heat equation with a singular potential*, Trans. Amer. Math. Soc., **284** (1984), 121–139.

[5] W. BECKNER, AND M. PEARSON, *On sharp Sobolev embedding and the logarithmic Sobolev inequalitiy*, Bull. London Math. Soc., **30** (1998), 80–84.

[6] C. BENNETT, K. RUDNICK, *On Lorentz-Zygmund spaces*, Dissertationes Math. (Rozprawy Mat.) **175** (1980), 67 pp.

[7] C. BENNETT, R. SHARPLEY, *Interpolation of Operators*, Pure and Applied Mathematics, vol. 129, Boston Academic Press, Inc., (1988).

[8] H. BREZIS, J. L. VÁZQUEZ, *Blow-up solutions of some nonlinear elliptic problems*, Rev. Mat. Univ. Complut. Madrid **10** (1997), No. 2, 443–469.

[9] X. CABRÉ, X. ROS-OTON, *Sobolev and isoperimetric inequalities with monomial weights*, J. Differential Equations. **255**, (2013), 4312–4336.

[10] L. CARLESON, S.-Y. A. CHANG, *On the existence of an extremal function for an inequality of J. Moser*, Bull. Sci. Math. (2) **110** (1986), no. 2, 113–127.

[11] D. CASSANI, B. RUF, C. TARSI, *Best constants for Moser type inequalities in Zygmund spaces*, Mat. Contemp. **36** (2009), 79–90.

[12] D. CASSANI, B. RUF, C. TARSI, *Best constants in a borderline case of second-order Moser type inequalities*, Ann. Inst. H. Poincare Anal. Non Lineaire **27** (2010), no. 1, 73–93.

[13] D. CASSANI, B. RUF, C. TARSI, *Equivalent and attained version of Hardy's inequality in* $\mathbb{R}^n$, J. Funct. Anal. **275** (2018), no. 12, 3303–3324.

[14] D. CASSANI, B. RUF, C. TARSI, *Group invariance and Pohozaev identity in Moser-type inequalities*, Commun. Contemp. Math. **15** (2013), No. 2, 1250054, 20 pp.

[15] D. CASSANI, B. RUF, C. TARSI, *On the capacity approach to non-attainability of Hardy's inequality in* $\mathbb{R}^N$, Discrete Contin. Dyn. Syst. Ser. S **12** (2019), no. 2, 245–250.

[16] D. CASSANI, F. SANI,, C. TARSI, *Equivalent Moser type inequalities in* $\mathbb{R}^2$ *and the zero mass case*, J. Funct. Anal. **267** (2014), no. 11, 4236–4263.

[17] E. B. DAVIES, A. M. HINZ, *Explicit constants for Rellich inequalities in* $L^p(\Omega)$, Math. Z. **227** (1998), no. 3, 511–523.

[18] A. DETALLA, T. HORIUCHI AND H. ANDO, *Missing terms in Hardy-Sobolev inequalities*, Proc. Japan Acad. Ser. A Math. Sci. **80** (2004), no. 8, 160–165.

[19] F. GAZZOLA, H.-C. GRUNAU, G. SWEERS, *Polyharmonic boundary value problems. Positivity preserving and nonlinear higher order elliptic equations in bounded domains*, Lecture Notes in Mathematics, 1991. Springer-Verlag, Berlin, (2010), xviii+423 pp.

[20] T. HORIUCHI, P. KUMLIN, *On the Caffarelli-Kohn-Nirenberg-type inequalities involving critical and supercritical weights*, Kyoto J. Math. **52** (2012), no. 4, 661–742.

[21] N. IOKU, *Attainability of the best Sobolev constant in a ball*, Math. Ann. **375** (2019), no. 1–2, 1–16.

[22] N. IOKU, M. ISHIWATA, *A Scale Invariant Form of a Critical Hardy Inequality*, Int. Math. Res. Not. IMRN (2015), no. 18, 8830–8846.

[23] B. KAWOHL, V. FRIDMAN, *Isoperimetric estimates for the first eigenvalue of the p-Laplace operator and the Cheeger constant*, Comment. Math. Univ. Carolin. **44** (2003), no. 4, 659–667.

[24] O. A. LADYZHENSKAYA, *The mathematical theory of viscous incompressible flow, Second edition, revised and enlarged*, Mathematics and its Applications, Vol. 2, Gordon and Breach, Science Publishers, New York-London-Paris, (1969).

[25] J. LERAY, *Etude de diverses equations integrales non lineaires et de quelques problemes que pose l'hydrodynamique (French)*, (1933), 82 pp.

[26] N. LAM, G. LU, L. ZHANG, *Equivalence of critical and subcritical sharp Trudinger-Moser-Adams inequalities*, Rev. Mat. Iberoam. **33** (2017), no. 4, 1219–1246.

[27] E. MITIDIERI, *A simple approach to Hardy inequalities*, (Russian) Mat. Zametki **67** (2000), no. 4, 563–572; translation in Math. Notes **67** (2000), no. 3–4, 479–486.

[28] F. RELLICH, *Halbbeschränkte Differentialoperatoren höherer Ordnung*, (German) Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 243–250. Erven P. Noordhoff N. V., Groningen; North-Holland Publishing Co., Amsterdam, 1956.

[29] L. SALOFF-COSTE, *Aspects of Sobolev-type inequalities*, London Mathematical Society Lecture Note Series, **289**, Cambridge University Press, Cambridge, 2002. x+190 pp.

[30] M. SANO, *Explicit optimal constants of two critical Rellich inequalities for radially symmetric functions*, arXiv: 2002.04768v2.

[31] M. SANO, *Extremal functions of generalized critical Hardy inequalities*, J. Differential Equations **267** (2019), no. 4, 2594–2615.

[32] M. SANO, *Minimization problem associated with an improved Hardy-Sobolev type inequality*, Nonlinear Anal. **200** (2020), 111965, 16 pp.

[33] M. SANO, *Two limits on Hardy and Sobolev inequalities*, arXiv:1911.04105.

[34] N. S. TRUDINGER, *On imbeddings into Orlicz spaces and some applications*, J. Math. Mech. **17** (1967), 473–483.

[35] S. YANO, *Notes on Fourier analysis XXIX, An extrapolation theorem*, J. Math. Soc. Japan **3**, (1951), 296–305.

[36] A. ZYGMUND, *Trigonometric Series, 2nd ed. Vols. I, II*, Cambridge University Press, New York 1959 Vol. I. xii+383 pp.; Vol. II. vii+354 pp.

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