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LINEAR MAPS OF POSITIVE PARTIAL TRANSPOSE MATRICES AND SINGULAR VALUE INEQUALITIES

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Dedicated to Professor Roger Horn in honor of his numerous contributions to Matrix Analysis and the mathematical community.

(*Communicated by M. S. Moslehian*)

Abstract. Linear maps $\Phi : \mathbb{M}_n \to \mathbb{M}_k$ are called m-PPT if $[\Phi(A_{ij})]_{i,j=1}^m$ are positive partial transpose matrices for all positive semi-definite matrices $[A_{ij}]_{i,j=1}^m \in \mathbb{M}_m(\mathbb{M}_n)$. In this paper, connections between m-PPT maps, m-positive maps and m-copositive maps are given. In consequence, characterizations of completely PPT maps are obtained. The results are applied to study two linear maps $X \mapsto X + a(\mathrm{tr}X)I$ and $X \mapsto a(\mathrm{tr}X)I - X$ for $a \geqslant 0$. Moreover, singular values inequalities of 2×2 positive block matrices under these two linear maps are given. In particular, we prove an open singular values inequality formulated by Lin [Linear Algebra Appl, 520 (2017)] for $n \leqslant 3$.

1. Introduction

Let $\mathbb{M}_n$ be the set of all $n\times n$ complex matrices and $\mathbb{M}_m(\mathbb{M}_n)$ be the set of all $m\times m$ block matrices with entries in $\mathbb{M}_n$. For $A,B \in \mathbb{M}_n$, we write by $A \geqslant B$ ($A > B$) if $A-B$ is positive semi-definite (positive definite). In particular, $A \geqslant 0$ ($A > 0$) if A is positive semi-definite (positive definite). Denote by A^t, $\bar{A}$ and A^* the transpose, conjugate and conjugate transpose of the matrix A, respectively. The trace of a matrix A is denoted by $\mathrm{tr}A$. We denote by $\{E_{ij},\ i,j=1,\dots,n\}$ the standard basis of $\mathbb{M}_n$.

The partial transpose of $\mathbf{A} = [A_{ij}]_{i,j=1}^m \in \mathbb{M}_m(\mathbb{M}_n)$ is defined by $\mathbf{A}^\tau = [A_{ji}]_{i,j=1}^m$. For $m=1$, we set $\mathbf{A}^\tau = \mathbf{A}$. A matrix $\mathbf{A} \in \mathbb{M}_m(\mathbb{M}_n)$ is called partial positive transpose (PPT) if $\mathbf{A} \geqslant 0$ and $\mathbf{A}^\tau \geqslant 0$. For example, if $m=2$, $A,B,C \in \mathbb{M}_n$ and

$$\mathbf{M} = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \in \mathbb{M}_2(\mathbb{M}_n),$$

then $\mathbf{M}$ is called PPT if $\mathbf{M} \geqslant 0$ and

$$\mathbf{M}^\tau = \begin{bmatrix} A & B^* \\ B & C \end{bmatrix} \geqslant 0.$$

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It is clear that all PPT matrices are positive semi-definite, however the converse does not hold. For example,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{M} = \begin{bmatrix} A & B^* \\ B & C \end{bmatrix} \in \mathbb{M}_2(\mathbb{M}_2), \tag{1}$$

then $\mathbf{M} \geqslant 0$ but $\mathbf{M}^\tau \ngeqslant 0$. We refer the reader to [2, 17] for some characterizations of positive semi-definite 2×2 block matrices. The study of PPT matrices arises in quantum information theory in distinguishing the separability of quantum states; see [4, 9]. In particular, it connects to the PPT criterion which gives a necessary condition for the joint quantum state to be separable; see [10, 20]. On the other hand, PPT matrices carry many algebraic properties which are interesting in their own right. Recently, singular values and eigenvalues inequalities of PPT matrices have been studied by many researchers; see [1, 12, 15, 16, 25].

A linear map $\Phi : \mathbb{M}_n \to \mathbb{M}_k$ is positive if $\Phi(A) \geqslant 0$ for all $A \geqslant 0$. A positive map $\Phi : \mathbb{M}_n \to \mathbb{M}_k$ is called m-positive if its induced map $I_m \otimes \Phi : \mathbb{M}_m(\mathbb{M}_n) \to \mathbb{M}_m(\mathbb{M}_k)$ is positive, that is, $[\Phi(A_{ij})]_{i,j=1}^m \geqslant 0$ for all positive semi-definite matrices $[A_{ij}]_{i,j=1}^m \in \mathbb{M}_m(\mathbb{M}_n)$. A linear map is called completely positive if it is m-positive for every positive integer m. The concept of completely positive maps is introduced by Steinspring [21] in studying dilation problems of operators; see also [2, Chapter 3]. In application, the class of completely positive maps plays an important role in the development of quantum computing; see [18].

In [11], Lin introduced the concept of completely PPT maps which are linear maps $\Phi : \mathbb{M}_n \to \mathbb{M}_k$ such that $[\Phi(A_{ij})]_{i,j=1}^m$ are PPT matrices for all positive semi-definite matrices $[A_{ij}]_{i,j=1}^m \in \mathbb{M}_m(\mathbb{M}_n)$ and all positive integers m. In particular, Lin showed that $\Phi(X) = X + (\operatorname{tr} X)I_n$ is a completely PPT map. Note that the same result was also observed in [5, Lemma 6]. In [25], Zhang considered the linear map $\Phi(X) = \min\{m,n\}(\operatorname{tr} X)I_n - X$ and showed that $[\Phi(A_{ij})]_{i,j=1}^m$ are PPT matrices for all positive semi-definite matrices $[A_{ij}]_{i,j=1}^m \in \mathbb{M}_m(\mathbb{M}_n)$. Motivated by the result of Zhang, it is natural to extend the concept of completely PPT maps and define m-PPT maps. This is a goal in this paper. Precisely, we will introduce the concept of m-PPT maps in Section 2. One will see that completely PPT maps are m-PPT maps for every positive integer m. Moreover, a characterization of such maps will be given by connecting to m-positive maps and m-copositive map (see Section 2 for definition). In addition, characterizations of completely PPT maps are obtained. As an application, generalizations of Lin's and Zhang's results are given. The proofs given by Lin and Zhang are specific to the prescribed linear maps. They both apply matrix decompositions on positive semi-definite matrices and then simplify the problem to 2×2 block matrices. The characterizations obtained in Section 2 provide a general way to determine m-PPT maps and completely PPT maps. We would like to point out that Lin's and Zhang's results become immediate consequences as well as [5, Lemma 6]. In Section 3, we study singular values inequalities connecting to m-PPT maps and completely PPT maps. In particular, we provide a partial answer on an open singular values inequality of 2×2 positive block matrices arised by Lin [14]. In addition, more singular values inequalities are proposed.

2. m-PPT maps and Completely PPT maps

We begin with the definitions of m-PPT maps and completely PPT maps.

DEFINITION 1. A linear map $\Phi : \mathbb{M}_n \to \mathbb{M}_k$ is said to be m-PPT if $[\Phi(A_{ij})]_{i,j=1}^m$ are PPT matrices for all positive semi-definite matrices $[A_{ij}]_{i,j=1}^m \in \mathbb{M}_m(\mathbb{M}_n)$. It is said to be completely PPT if it is m-PPT for all positive integers m.

In other words, a linear map $\Phi : \mathbb{M}_n \to \mathbb{M}_k$ is m-PPT if it maps the set of all positive semi-definite matrices in $\mathbb{M}_m(\mathbb{M}_n)$ to the set of all PPT matrices in $\mathbb{M}_m(\mathbb{M}_k)$. It is clear to see that the class of 1-PPT maps coincides with the class of positive maps.

Let $T : \mathbb{M}_n \to \mathbb{M}_n$ be the transposition map, that is, $T(X) = X^t$. A linear map $\Phi : \mathbb{M}_n \to \mathbb{M}_k$ is called m-copositive if $\Phi \circ T$, the composition of Φ and T, is m-positive. Similarly, it is completely copositive if $\Phi \circ T$ is completely positive. The study of m-copositive maps arises in the study on decomposibility of positive maps; see [19, 22]. The following result provides a connection between m-PPT maps, m-copositive maps and m-positive maps.

LEMMA 1. *Let $\Phi : \mathbb{M}_n \to \mathbb{M}_k$ be linear. Then Φ is an m-PPT map if and only if Φ is m-positive and m-copositive. In particular, Φ is a completely PPT map if and only if Φ is completely positive and completely copositive.*

Proof. ($\Rightarrow$) It is clear that every m-PPT map is m-positive. Let $[A_{ij}]_{i,j=1}^m \in \mathbb{M}_m(\mathbb{M}_n)$ be positive semi-definite and $T : \mathbb{M}_n \to \mathbb{M}_n$ be the transposition map. Then

$$[\Phi \circ T(A_{ij})]_{i,j=1}^m = \left[\Phi(A_{ij}^t)\right]_{i,j=1}^m = \left[\Phi\left(\overline{A_{ji}}\right)\right]_{i,j=1}^m = \left(\left[\Phi\left(\overline{A_{ij}}\right)\right]_{i,j=1}^m\right)^\tau .$$

Since $[A_{ij}]_{i,j=1}^m \geqslant 0$ and Φ is an m-PPT map, we have $\left[\overline{A_{ij}}\right]_{i,j=1}^m \geqslant 0$ and hence $\left(\left[\Phi\left(\overline{A_{ij}}\right)\right]_{i,j=1}^m\right)^\tau \geqslant 0$. This implies $[\Phi \circ T(A_{ij})]_{i,j=1}^m \geqslant 0$ for all $[A_{ij}]_{i,j=1}^m \geqslant 0$. Then Φ is m-copositive.

($\Leftarrow$) As Φ is m-positive, it suffices to check that $\left([\Phi(A_{ij})]_{i,j=1}^m\right)^\tau \geqslant 0$ for all positive semi-definite matrices $[A_{ij}]_{i,j=1}^m \in \mathbb{M}_m(\mathbb{M}_n)$. Note that

$$\left([\Phi(A_{ij})]_{i,j=1}^m\right)^\tau = [\Phi(A_{ji})]_{i,j=1}^m = \left[\Phi(A_{ij}^*)\right]_{i,j=1}^m = \left[\Phi \circ T\left(\overline{A_{ij}}\right)\right]_{i,j=1}^m .$$

Similarly, as $[A_{ij}]_{i,j=1}^m \geqslant 0$ and Φ is m-copositive, we have $\left[\overline{A_{ij}}\right]_{i,j=1}^m \geqslant 0$ and $\left[\Phi \circ T\left(\overline{A_{ij}}\right)\right]_{i,j=1}^m \geqslant 0$. The result follows.

The second statement is clear by the definition of completely PPT map.

We remark that in general there is no implication between $\left([\Phi(A_{ij})]_{i,j=1}^m\right)^\tau \geqslant 0$ and $[\Phi \circ T(A_{ij})]_{i,j=1}^m \geqslant 0$. We illustrate it by the following example.

EXAMPLE 1. Let $\Phi : \mathbb{M}_3 \to \mathbb{M}_3$ be defined by

$$\Phi(X) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i \end{bmatrix}^* X \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i \end{bmatrix} + X.$$

Note that $\Phi(\cdot)$ is a completely positive map. Let $\mathbf{M} = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$, where $A = I_3$, $C = B^*B$ and

$$B = \begin{bmatrix} 0 & -2-3i & 4i \\ 3+i & 1 & 1+4i \\ -2-2i & 4+3i & 1 \end{bmatrix}.$$

By direct computation, $\begin{bmatrix} \Phi(A) & \Phi(B) \\ \Phi(B)^* & \Phi(C) \end{bmatrix}^\tau \not\geqslant 0$, but $\begin{bmatrix} \Phi(A^t) & \Phi(B^t) \\ \Phi(B^t)^* & \Phi(C^t) \end{bmatrix} \geqslant 0$.

In [8], Choi gave the following characterization of completely positive map. It gives a direct way to determine the complete positivity of a given linear map.

PROPOSITION 1. *Let $\Phi : \mathbb{M}_n \to \mathbb{M}_k$ be linear. Then Φ is completely positive if and only if $[\Phi(E_{ij})]_{i,j=1}^n \geqslant 0$, where $\{E_{ij},\ i,j = 1,\dots,n\}$ is the standard basis of $\mathbb{M}_n$.*

The matrix $[\Phi(E_{ij})]_{i,j=1}^n$ is called the Choi matrix in literature. From Proposition 1, it is clear that $\Phi : \mathbb{M}_n \to \mathbb{M}_k$ is completely copositive if and only if $[\Phi \circ T(E_{ij})]_{i,j=1}^n = [\Phi(E_{ji})]_{i,j=1}^n \geqslant 0$. Therefore, by Lemma 1, we have the following characterization of completely PPT maps.

THEOREM 1. *Let $\Phi : \mathbb{M}_n \to \mathbb{M}_k$ be linear. Then Φ is a completely PPT map if and only if $[\Phi(E_{ij})]_{i,j=1}^n \geqslant 0$ and $[\Phi(E_{ji})]_{i,j=1}^n \geqslant 0$.*

Theorem 1 asserts that one can check the positivity of two Choi matrices to determine the completely PPT property of a linear map. In the following, we will consider some linear maps and study their m-PPT or completely PPT properties. In particular, we will generalize the results of Lin [11] and Zhang [25].

THEOREM 2. *Let $\Phi : \mathbb{M}_n \to \mathbb{M}_k$ be the linear map $\Phi(X) = (\mathrm{tr} X) I_k$. Then $\Phi(\cdot)$ is a completely PPT map.*

Proof. It is not hard to see that

$$[\Phi(E_{ij})]_{i,j=1}^n = [\Phi(E_{ji})]_{i,j=1}^n = I_{nk} \geqslant 0.$$

Hence by Theorem 1, Φ is a completely PPT map.

Let $\mathbf{A} = [A_{ij}]_{i,j=1}^m \in \mathbb{M}_m(\mathbb{M}_n)$. A partial trace of $\mathbf{A}$ is defined by $\mathrm{tr}_2 \mathbf{A} = [\mathrm{tr} A_{ij}]_{i,j=1}^m$. Setting $k = 1$ in Theorem 2, we have the following result which was first given by Choi [6, Theorem 2].

COROLLARY 1. *Let* $\mathbf{A} = [A_{ij}]_{i,j=1}^m \in \mathbb{M}_m(\mathbb{M}_n)$ *be positive semi-definite. Then*

$$\mathrm{tr}_2(\mathbf{A}^\tau) = [\mathrm{tr} A_{ji}]_{i,j=1}^m \geqslant 0.$$

For every $a \geqslant 0$, we consider the linear map $\Phi_a : \mathbb{M}_n \to \mathbb{M}_n$ defined by $\Phi_a(X) = X + a(\mathrm{tr} X)I_n$. It is known that $\Phi_a(\cdot)$ is completely positive for all $a \geqslant 0$. As a consequence of Theorem 1, we have the following result.

THEOREM 3. *Let* $\Phi_a : \mathbb{M}_n \to \mathbb{M}_n$ *be defined by* $\Phi_a(X) = X + a(\mathrm{tr} X)I_n$. *If* $n = 1$, *then* $\Phi_a(X)$ *is completely PPT map if and only if* $a \geqslant 0$. *If* $n \geqslant 2$, *then* $\Phi_a(X)$ *is completely PPT map if and only if* $a \geqslant 1$.

Proof. The case $n = 1$ is trivial and we omit it here. Now consider $n \geqslant 2$. Note that $\Phi_a(\cdot)$ is completely positive for $a \geqslant 0$. Therefore, by Theorem 1, it remains to show that $[\Phi_a(E_{ji})]_{i,j=1}^n \geqslant 0$ if and only if $a \geqslant 1$. Observe that $[\Phi_a(E_{ji})]_{i,j=1}^n = [E_{ji}]_{i,j=1}^n + aI_n$ and $[E_{ji}]_{i,j=1}^n$ is a symmetric permutation matrix. It has eigenvalues 1 with multiplicity $n(n+1)/2$ and -1 with multiplicity $n(n-1)/2$. As $n \geqslant 2$, we have $[E_{ji}]_{i,j=1}^n + aI_n \geqslant 0$ if and only if $a \geqslant 1$.

By putting $a = 1$, we have the following result of Lin [11].

COROLLARY 2. *Let* $\Phi : \mathbb{M}_n \to \mathbb{M}_n$ *be defined by* $\Phi(X) = X + (\mathrm{tr} X)I_n$. *Then* $\Phi(\cdot)$ *is a completely PPT map.*

In the following, we consider another linear map $\tilde{\Phi}_a : \mathbb{M}_n \to \mathbb{M}_n$ defined by $\tilde{\Phi}_a(X) = a(\mathrm{tr} X)I_n - X$. We have the following result.

THEOREM 4. *Let* $\tilde{\Phi}_a : \mathbb{M}_n \to \mathbb{M}_n$ *be defined by* $\Phi_a(X) = a(\mathrm{tr} X)I_n - X$. *Then* $\Phi_a(\cdot)$ *is an* m*-PPT map if and only if* $a \geqslant m$. *In particular, it is completely PPT if and only if* $a \geqslant n$.

Proof. Let $T : \mathbb{M}_n \to \mathbb{M}_n$ be the transpose map. Since $\left[\tilde{\Phi}_a(E_{ji})\right]_{i,j=1}^n = aI_n - [E_{ji}]_{i,j=1}^n$ and $[E_{ji}]_{i,j=1}^n$ has eigenvalues ± 1, then $\tilde{\Phi}_a$ is completely copositive if and only if $a \geqslant 1$. Therefore , it remains to show that $\tilde{\Phi}_a(\cdot)$ is m-positive if and only if $a \geqslant m$. This is given by [23, Theorem 2(i)].

It is known that a linear map $\Phi : \mathbb{M}_n \to \mathbb{M}_n$ is completely positive if and only if it is n-positive; see [2, Theorem 3.1.6]. Therefore, the second statement follows.

The linear map $\tilde{\Phi}_{n-1}(X) = (n-1)(\mathrm{tr} X)I_n - X$ from $\mathbb{M}_n$ to $\mathbb{M}_n$ is the first example of an $(n-1)$-positive map which fails to be n-positive. It was given by Choi in [7]. An interesting consequence from Theorem 4 is that $\tilde{\Phi}_{n-1}(\cdot)$ is an $(n-1)$-PPT map but fails to be an n-PPT map. The following result is given by Zhang [25] which follows immediately from Theorem 4.

COROLLARY 3. *Let* $\Phi : \mathbb{M}_n \to \mathbb{M}_n$ *be defined by* $\Phi(X) = \min\{m,n\}(\mathrm{tr} X)I_n - X$. *Then* $\Phi(\cdot)$ *is an* m*-PPT map.*

3. Singular values inequalities

For any matrix $A \in \mathbb{M}_n$, we denote by $s_1(A) \geqslant s_2(A) \geqslant \cdots \geqslant s_n(A) \geqslant 0$ the singular values of A. In this section, we focus on 2×2 block positive semi-definite matrices. We study singular value inequalities of the block matrices under the maps $\Phi_a(\cdot)$ and $\tilde{\Phi}_a(\cdot)$ defined in Section 2. The following is our main result in this section.

THEOREM 5. Let $\Phi_a : \mathbb{M}_n \to \mathbb{M}_n$ be defined by $\Phi_a(X) = X + a(\mathrm{tr}X)I_n$ and $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$ with $A,B,C \in \mathbb{M}_n$ be positive semi-definite. If $a \geqslant 1/2$, then

$$2s_j(\Phi_a(B)) \leqslant 2(s_j(B) + a|\mathrm{tr}B|) \leqslant s_j\left(\Phi_a(A) + \Phi_a(C)\right), \quad j = 1,\ldots,n.$$

We need the following lemma to prove the result.

LEMMA 2. [3, p. 262] Let X,Y be any $n\times m$ matrices. Then

$$2s_j(XY^*) \leqslant s_j(X^*X + Y^*Y), \quad j = 1,\ldots,n. \tag{2}$$

Now we are ready to prove Theorem 5.

Proof of Theorem 5. The first inequality follows by the classical result of singular value, that is,

$$s_j(X+Y) \leqslant s_j(X) + s_1(Y), \qquad j = 1,\ldots,n,$$

for any $X,Y \in \mathbb{M}_n$; see [24, Theorem 8.13].

Now, we consider the second inequality. Applying the positivity of $\begin{bmatrix} \mathrm{tr}A & \mathrm{tr}B \\ \mathrm{tr}B^* & \mathrm{tr}C \end{bmatrix}$ and using the AM-GM inequality we get

$$|\mathrm{tr}B| \leqslant (\mathrm{tr}A\mathrm{tr}C)^{1/2} \leqslant \frac{\mathrm{tr}A + \mathrm{tr}C}{2}.$$

If the second inequality holds for the case $a = 1/2$, then the general case will follow as

$$\begin{aligned} 2(s_j(B) + a|\mathrm{tr}B|) &= 2\left(s_j(B) + \frac{1}{2}|\mathrm{tr}B|\right) + (2a-1)|\mathrm{tr}B| \\ &\leqslant s_j\left(\Phi_{1/2}(A) + \Phi_{1/2}(C)\right) + (2a-1)\frac{\mathrm{tr}A + \mathrm{tr}C}{2} \\ &= s_j(\Phi_a(A) + \Phi_a(C)). \end{aligned}$$

Therefore, we may set $a = 1/2$ in the following. Moreover, one may replace B by $e^{i\theta}B$ for some real numbers θ and assume without loss of generality that $\mathrm{tr}B \geqslant 0$. We write

$$\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}^* = \begin{bmatrix} XX^* & XY^* \\ YX^* & YY^* \end{bmatrix},$$

for some $n\times 2n$ matrices X,Y. Then for $j = 1,\ldots,n$, we have

$$\begin{aligned}
s_j(XX^*+YY^*) &\geqslant \frac{1}{2}s_j((X+Y)(X+Y)^*)\\
&= \frac{1}{2}s_j((X+Y)^*(X+Y))\\
&= \frac{1}{2}s_j(2(X^*X+Y^*Y)-(X-Y)^*(X-Y))\\
&\geqslant s_j(X^*X+Y^*Y)-\frac{1}{2}\operatorname{tr}[(X-Y)^*(X-Y)]\\
&\geqslant 2s_j(XY^*)-\frac{1}{2}\operatorname{tr}[(X-Y)(X-Y)^*]\\
&= 2s_j(XY^*)-\frac{1}{2}\operatorname{tr}(XX^*+YY^*)+\frac{1}{2}(\operatorname{tr}(XY^*)+\operatorname{tr}(YX^*)),
\end{aligned}$$

where the first and second inequalities follow by $(X-Y)^*(X-Y)\geqslant 0$, and the third inequality follows by Lemma 2. Note that $\operatorname{tr}(XY^*)=\operatorname{tr}(YX^*)=\operatorname{tr}B\geqslant 0$, hence for $j=1,\ldots,n$,

$$s_j\left(A+\frac{1}{2}\operatorname{tr}A+C+\frac{1}{2}\operatorname{tr}C\right)\geqslant 2s_j(B)+\operatorname{tr}B=2\left(s_j(B)+\frac{1}{2}|\operatorname{tr}B|\right).$$

Then the results follows.

The following example shows that the condition $a\geqslant\frac{1}{2}$ in Theorem 5 is necessary.

EXAMPLE 2. Let $\begin{bmatrix}A & B\\ B^* & C\end{bmatrix}\in\mathbb{M}_2(\mathbb{M}_2)$ with

$$A=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix},\quad B=\begin{bmatrix}1 & -1\\ 1 & -1\end{bmatrix},\quad C=\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}.$$

For any $a\geqslant 0$ and $\Phi(X)=X+a(\operatorname{tr}X)I_n$,

$$\begin{bmatrix}\Phi(A) & \Phi(B)\\ \Phi(B^*) & \Phi(C)\end{bmatrix}=\begin{bmatrix}A+2aI & B\\ B^* & C+2aI\end{bmatrix}.$$

By direct computation, $s_1(\Phi(B))=2$ and $s_1(\Phi(A+C))=2+2a$. If $a<1/2$, then $s_1(\Phi(B))>\dfrac{2+2a}{2}=\dfrac{\Phi(A+C)}{2}$. Therefore, $a\geqslant 1/2$ in Theorem 4 is necessary.

By putting $a=1$ in Theorem 5, we have the following result of Lin [13].

COROLLARY 4. Let $\Phi:\mathbb{M}_n\to\mathbb{M}_n$ be defined by $\Phi_a(X)=X+(\operatorname{tr}X)I_n$ and $\begin{bmatrix}A & B\\ B^* & C\end{bmatrix}$ with $A,B,C\in\mathbb{M}_n$ be positive semi-definite. Then

$$2s_j(\Phi(B))\leqslant s_j(\Phi(A)+\Phi(C)),\quad j=1,\ldots,n.$$

In [14], Lin suspected that analogous result as Theorem 5 holds for the linear map $\tilde{\Phi}(X) = 2(\mathrm{tr}X) - X$. Though we have not been able to prove the general case, we show it is true for $n \leqslant 3$.

PROPOSITION 2. *Let $n \leqslant 3$, $\tilde{\Phi} : \mathbb{M}_n \to \mathbb{M}_n$ be defined by $\tilde{\Phi}(X) = 2(\mathrm{tr}X)I_n - X$ and $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$ with $A, B, C \in \mathbb{M}_n$ be positive semi-definite. Then*

$$2s_j(\tilde{\Phi}(B)) \leqslant s_j\left(\tilde{\Phi}(A) + \tilde{\Phi}(C)\right), \quad j = 1, \dots, n. \tag{3}$$

Proof. It suffices to prove the case $n = 3$. Note that the map $X \mapsto (\mathrm{tr}X)I_3 - X^t$ is completely positive. Therefore $\begin{bmatrix} (\mathrm{tr}A)I_3 - A^t & (\mathrm{tr}B)I_3 - B^t \\ (\mathrm{tr}B^*)I_3 - \bar{B} & (\mathrm{tr}C)I_3 - C^t \end{bmatrix} \geqslant 0$. Observe that $\mathrm{tr}\left((\mathrm{tr}X)I_3 - X\right) = 2\mathrm{tr}X$ for any $X \in \mathbb{M}_3$. Hence by Theorem 5, we have for $j = 1, 2, 3$,

$$\begin{aligned} 2s_j(2(\mathrm{tr}B)I_3 - B) &= 2s_j\left((\mathrm{tr}B)I_3 - B^t + \frac{1}{2}\left(\mathrm{tr}\left((\mathrm{tr}B)I_3 - B^t\right)\right)I_3\right) \\ &\leqslant s_j\left((\mathrm{tr}A)I_3 - A^t + (\mathrm{tr}A)I_3 + (\mathrm{tr}C)I_3 - C^t + (\mathrm{tr}C)I_3\right) \\ &= s_j(2(\mathrm{tr}A)I_3 - A + 2(\mathrm{tr}C)I_3 - C). \end{aligned}$$

Then the result follows.

REMARK 1. Applying similar argument in the proof of Theorem 5, one can see that (3) holds for the linear map $\tilde{\Phi}_a(X) = a(\mathrm{tr}X) - X$ with $a \geqslant 2$. When $n \geqslant 4$, using the method in the proof of Proposition 2, one can show that (3) holds for $\tilde{\Phi}_a(\cdot)$ with $a \geqslant \frac{n+1}{2}$.

The geometric mean of two positive definite matrices $A, B \in \mathbb{M}_n$ is defined by

$$A \sharp B = A^{\frac{1}{2}}(A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^{\frac{1}{2}} A^{\frac{1}{2}}.$$

It is known that $A \sharp B \leqslant \frac{A+B}{2}$. Therefore, one may ask if one can improve Theorem 5 by using geometric mean. The following result is given along this direction for large a. We denote by $\lambda_1(A) \geqslant \lambda_2(A) \geqslant \cdots \geqslant \lambda_n(A) \geqslant 0$ the eigenvalues of positive semi-definite matrices $A \in \mathbb{M}_n$.

PROPOSITION 3. Let $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \in \mathbb{M}_2(\mathbb{M}_n)$ be positive semi-definite. For the linear map $\Phi_a : \mathbb{M}_n \to \mathbb{M}_n$ defined by $\Phi_a(X) = X + a(\mathrm{tr}X)I_n$, if $a \geqslant \frac{n+3}{2}$, then

$$s_j(\Phi_a(B)) \leqslant \lambda_j(\Phi_a(A) \sharp \Phi_a(C)), \quad \text{for } j = 1, \dots, n.$$

For the linear map $\tilde{\Phi}_a : \mathbb{M}_n \to \mathbb{M}_n$ defined by $\tilde{\Phi}_a(X) = a(\mathrm{tr}X)I_n - X$, if $a \geqslant n + \frac{5}{2}$, then

$$s_j(\tilde{\Phi}_a(B)) \leqslant \lambda_j(\tilde{\Phi}_a(A) \sharp \tilde{\Phi}_a(C)), \quad \text{for } j = 1, \dots, n.$$

Proof. We first show that for every PPT matrix $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \in \mathbb{M}_2(\mathbb{M}_n)$ and $a \geqslant 1/2$, we have

$$s_j(\Phi_a(B)) \leqslant \lambda_j(\Phi_a(A)\sharp\Phi_a(C)), \text{ for } j = 1,\dots,n.$$

As $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$ and $\begin{bmatrix} C & B \\ B^* & A \end{bmatrix}$ are positive semi-definite, then a result of Ando [1, Lemma 3.1] asserts that $\begin{bmatrix} A\sharp C & B \\ B^* & C\sharp A \end{bmatrix}$ is positive semi-definite. Hence for $a \geqslant 1/2$,

$$s_j(\Phi_a(B)) \leqslant \lambda_j(\Phi_a(A\sharp C)) \leqslant \lambda_j(\Phi_a(A)\sharp\Phi_a(C)), \quad \text{for } j = 1,\dots,n,$$

the first inequality follows from Theorem 5 and the second inequality follows from [2, Theorem 4.1.5]. Now assume $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \in \mathbb{M}_2(\mathbb{M}_n)$ is positive semi-definite. Then by Theorem 3 and Theorem 4, $\begin{bmatrix} \Phi_1(A) & \Phi_1(B^*) \\ \Phi_1(B) & \Phi_1(C) \end{bmatrix}$ and $\begin{bmatrix} \tilde{\Phi}_2(A) & \tilde{\Phi}_2(B^*) \\ \tilde{\Phi}_2(B) & \tilde{\Phi}_2(C) \end{bmatrix}$ are PPT matrices. Therefore for $a \geqslant 1/2$ and $j = 1,\dots,n$,

$$s_j(\Phi_a(\Phi_1(B))) \leqslant \lambda_j(\Phi_a(\Phi_1(A))\sharp\Phi_a(\Phi_1(C))), \text{ for } j = 1,\dots,n,$$

and

$$s_j(\Phi_a(\tilde{\Phi}_2(B))) \leqslant \lambda_j(\Phi_a(\tilde{\Phi}_2(A))\sharp\Phi_a(\tilde{\Phi}_2(C))), \text{ for } j = 1,\dots,n.$$

The results follow by $\Phi_a(\Phi_1(\cdot)) = \Phi_{a(n+1)+1}(\cdot)$ and $\Phi_a(\tilde{\Phi}_1(\cdot)) = \tilde{\Phi}_{a(2n+1)+2}(\cdot)$.

Some numerical experiments suggest that in Proposition 3 the singular values inequalities of Φ_a and $\tilde{\Phi}_a$ hold for all $a \geqslant 1$ and $a \geqslant 2$, respectively. However we have not been able to prove it yet. Therefore we leave it as an open problem for further research.

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REFERENCES

[1] T. ANDO, *Geometric mean and norm Schwarz inequality*, Ann. Funct. Anal. 7 (2016), 1–8.

[2] R. BHATIA, *Positive Definite Matrices*, Princeton University Press, 2007.

[3] R. BHATIA, *Matrix Analysis, Grad. Texts in Math., vol. 169*, Springer-Verlag, 1997.

[4] D. BRUSS AND C. MACCHIAVELLO, *How the first partial transpose was written*, Found. Phys. 35 (2005), 1921–1926.

[5] L. CHEN, W.-S. TANG, AND Y. YANG, *Generalized Choi states and 2-distillability of quantum states*, Quantum Inf Process (2018), 17:110.

[6] D. CHOI, *Inequalities related to partial transpose and partial trace*, Linear Algebra Appl. 516 (2017), 1–7.
[7] M.D. CHOI, *Positive linear maps on C^*-algebras*, Canad. J. Math. 24 (1972), 520–529.
[8] M.D. CHOI, *Completely positive linear maps on complex matrices*, Linear Algebra Appl. 10 (1975), 285–290.
[9] T. HIROSHIMA, *Majorization criterion for distillability of abipartite quantum state*, Phys. Rev. Lett. 91 (2003), p. 057902.
[10] M. HORODECKI, P. HORODECKI, AND R. HORODECKI, *Separability of mixed states: necessary and sufficient conditions*, Phys. Lett. A 223 (1996), 1–8.
[11] M. LIN, *A completely PPT map*, Linear Algebra Appl. 459 (2014), 404–410.
[12] M. LIN, *Inequalities related to 2×2 block PPT matrices*, Oper. Matrices 94 (2015), 917–924.
[13] M. LIN, *A singular value inequality related to a linear map*, Electron J. Linear Algebra 31 (2016), 120–124.
[14] M. LIN, *New properties for certain positive semidefinite matrices*, Linear Algebra Appl. 520 (2017), 32–43.
[15] J.T. LIU, Q.W. WANG, AND F.F. SUN, *On Hayajneh and Kittaneh's conjecture on unitarily invariant norm*, Journal of Math. Inequalities 11 (2017), 1019–1022.
[16] E.Y. LEE, *The off-diagonal block of a PPT matrix*, Linear Algebra Appl. 486 (2015), 449–453.
[17] M.S. MOSLEHIAN, M. KIAN, AND Q. XU, *Positivity of 2×2 block matrices of operators*, Banach J. Math. Anal. 13 (2019), 726–743.
[18] M.A. NIELSEN AND I.L. CHUANG, *Quantum Computation and Quantum Information*, Cambridge University Press, 2000.
[19] H. OSAKA, *A series of absolutely indecomposable positive maps in matrix algebras*, Linear Algebra Appl. 186 (1993), 45–53.
[20] ASHER PERES, *Separability criterion for density matrices*, Phys. Rev. Lett. 77 (1996), 1413–1415.
[21] W.F. STEINSPRING, *Positive functions on C^*-algebras*, Proc. Amer. Math. Soc. 6 (1955), 211–216.
[22] E. STORMER, *Decomposable positive maps on C^*-algebras*, Proc. Amer. Math. Soc. 86 (1982), 402–404.
[23] J. TOMIYAMA, *On the geometry of positive maps in matrix algebra. II*, Linear Algebra Appl. 69 (1985), 169–177.
[24] F. ZHANG, *Matrix Theory: Basic Results and Techniques*, Springer, New York, 2011.
[25] P. ZHANG, *On some inequalities related to positive block matrices*, Linear Algebra Appl. 576 (2019), 258–267.

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