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SHARP OFF–DIAGONAL WEIGHTED WEAK TYPE ESTIMATES FOR SPARSE OPERATORS

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Abstract. We prove sharp weak type weighted estimates for a class of sparse operators that includes majorants of standard singular integrals, fractional integral operators, and square functions. These bounds are known to be sharp in many cases, and our main new result is the optimal bound

$$[w,\sigma]_{A_{p,q}^{\alpha}}^{\frac{1}{q}}[w^q]_{A_\infty}^{\frac{1}{\nu}-\frac{1}{p}} \lesssim [w]_{A_{p,q}}^{\frac{1}{q}}[w]_{A_{p,q}}^{\frac{1}{\nu}-\frac{1}{p}} = [w]_{A_{p,q}}^{\frac{1}{\nu}-\frac{\alpha}{d}}$$

for $p>\nu$ and Sobolev type condition $\frac{1}{q}+\frac{\alpha}{d}=\frac{1}{p}$. For $\nu \leqslant q \leqslant \frac{\nu}{1-\frac{\nu\alpha}{d}}$, we also obtain the bounds $[w]_{A_{p,q}}^{\frac{1}{q}}$ and it has an additional logarithmic factor, taking the form $(1+\log[w^q]_{A_\infty})^{\frac{1}{\nu}}$. Moreover, we study a class of sparse maximal operators and give the weak type off-diagonal two-weight sharp bound.

1. Introduction

In this paper, we study weighted inequalities for sparse operators, which can be defined by

$$\mathscr{A}_{\alpha,\nu}^{\mathscr{S}}(f) := \Big(\sum_{Q\in\mathscr{S}} \langle f\rangle_{\alpha,Q}^{\nu}\mathbf{1}_Q\Big)^{\frac{1}{\nu}}, \quad \langle f\rangle_{\alpha,Q} = \frac{1}{|Q|^{1-\frac{\alpha}{d}}}\int_Q f, \tag{1}$$

where $\nu>0$, $0\leqslant\alpha<d$ and $\mathscr{S}$ is a η-sparse collection of dyadic cubes, i.e. for all cubes $Q\in\mathscr{S}$, there exist $E_Q\subset Q$ which are pairwise disjoint and $|E_Q|\geqslant\gamma|Q|$ with $0<\gamma<1$. Note that $\langle f\rangle_Q$ denotes $\langle f\rangle_{\alpha,Q}$ with $\alpha=0$. By now it is known that such the operator $\mathscr{A}_{\alpha,\nu}^{\mathscr{S}}$ dominates large classes of classical operators T, relying upon the sparse domination formula

$$|Tf(x)| \lesssim \sum_{i=1}^{N}\mathscr{A}_{\alpha,\nu}^{\mathscr{S}_i}(|f|)(x), \tag{2}$$

where the collections $\mathscr{S}_i$ depend on the function f. For $\nu=1$ and $\nu=2$ with $\alpha=0$, T becomes the Calderón-Zygmund singular integrals [3, 22] and Littlewood-Paley square functions [19, 20], respectively. Thus, the various norm inequalities that we

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study for $\mathscr{A}_{\alpha,\nu}^{\mathscr{S}}$ immediately translate to corresponding estimates for these classses of classical operators.

A weight w on $\mathbb{R}^d$ is a locally integrable function $w\colon \mathbb{R}^d \to (0,+\infty)$. The class of all A_∞ weights consists of all weights w for which their A_∞ characteristic

$$[w]_{A_\infty} := \sup_Q \frac{1}{w(Q)} \int_Q M(\mathbf{1}_Q w) < \infty,$$

where M is the Hardy-Littlewood maximal function and the supremum take over cubes of sides parallel to the coordinate axes.

More precisely, we are concerned with quantifying the dependence of various weighted operator norms on a mixture of the two weight $A_{p,q}^\alpha$ characteristic

$$[w,\sigma]_{A_{p,q}^\alpha} := \sup_{Q\in\mathscr{S}} |Q|^{q(\frac{\alpha}{d}-1)} w(Q)\sigma(Q)^{\frac{q}{p'}} < \infty.$$

The study of such mixed bounds was initiated in [13]. All our estimates will be stated in a dual-weight formulation, in which the classical off-diagonal one-weight case corresponds to the choice $\sigma = w^{-1/(p-1)}$. Note that becomes the usual one-weight $A_{p,q}$ characteristic $[w]_{A_{p,q}} := [w^q, w^{-p'}]_{A_{p,q}^\alpha}$ with this choice. The properties of one-weight $A_{p,q}$ are introduced in Section 5.

Since we are dealing with dyadic operators, we also consider the dyadic versions of the weight characteristics, where the supremums above are over dyadic cubes only and M_α denotes the dyadic fractional maximal operator. This is a standing convention throughout this paper without further notice.

Throughout this paper, $1 < p, p', q < \infty$, p and p' are conjugate indices, i.e. $1/p + 1/p' = 1$. Formally, we will also define $p = 1$ as conjugate to $p' = \infty$ and vice versa.

Now, we formulate our main results as follows.

THEOREM 1. *Let $0 < \nu < \infty$, $0 \leqslant \alpha < d$ and $1 < p \leqslant q < \infty$. Let w, σ be a pair of weights. Then*

$$\|\mathscr{A}_{\alpha,\nu}^{\mathscr{S}}(\cdot\sigma)\|_{L^p(\sigma)\to L^{q,\infty}(w)} \lesssim [w,\sigma]_{A_{p,q}^\alpha}^{\frac{1}{q}} \begin{cases} [w]_{A_\infty}^{\frac{1}{\nu}(1-(\frac{\nu}{p})^2)} [\sigma]_{A_\infty}^{\frac{1}{\nu}(\frac{\nu}{p})^2}, & p = q > \nu \text{ and } \alpha > 0, \\ [\sigma]_{A_\infty}^{\frac{1}{q}}, & p \leqslant \nu \leqslant q, \\ [w]_{A_\infty}^{(\frac{1}{\nu}-\frac{1}{p})_+}, & \text{other case}. \end{cases} \tag{3}$$

where $x_+ := \max(x,0)$ in the exponent. Here and below, we simplify case analysis by interpreting $[w]_{A_\infty}^0 = 1$, whether or not $[w]_{A_\infty}$ is finite.

Letting $\nu \to \infty$, we write $\mathscr{A}_{\alpha,\nu}^{\mathscr{S}}(f)$ by the following form

$$\mathscr{A}_\alpha(f) = \sup_{\mathscr{S}\ni Q\ni x} \frac{1}{|Q|^{1-\frac{\alpha}{d}}} \int_Q f, \quad 0 \leqslant \alpha < d,$$

and called *sparse fractional maximal function*. For above sparse fractional maximal function $\mathscr{A}_\alpha$, we have the following sharp estimate.

THEOREM 2. *Let $0 \leqslant \alpha < d$ and $1 < p \leqslant q < \infty$. Let w, σ be a pair of weights. Then*

$$\|\mathscr{A}_\alpha(f\sigma)\|_{L^{q,\infty}(w)} \lesssim [w,\sigma]_{A^\alpha_{p,q}}^{\frac{1}{q}} \|f\|_{L^p(\sigma)},$$

where the exponent $\frac{1}{q}$ is sharp.

REMARK 1. We also note that there is a weak-type estimate for M_α. For $1 < p \leqslant q < \infty$ and $0 \leqslant \alpha < d$, standard covering methods give

$$\|M_\alpha(f\sigma)\|_{L^{q,\infty}(w)} \lesssim [w,\sigma]_{A^\alpha_{p,q}}^{\frac{1}{q}} \|f\|_{L^p(\sigma)},$$

where the bound is also sharp. This result recovers the one-weight results due to Muckenhoupt [24], Lacey et al [15] and Pereyra [25].

Lacey and Scurry [17] provided an idea to prove Theorem 1 with $q < \nu$, and we merely repeat their method to two-weight off-diagonal case. For $p > \nu$, the bound

$$[w,\sigma]_{A^\alpha_{p,q}}^{\frac{1}{q}} [w^q]_{A_\infty}^{\frac{1}{\nu}-\frac{1}{p}} \lesssim [w]_{A_{p,q}}^{\frac{1}{q}} [w]_{A_{p,q}}^{\frac{1}{\nu}-\frac{1}{p}} = [w]_{A_{p,q}}^{\frac{1}{\nu}-\frac{\alpha}{d}} \tag{4}$$

is new even in the one weight case for $\frac{1}{q} + \frac{\alpha}{d} = \frac{1}{p}$. For $\nu \leqslant q \leqslant \frac{\nu}{1-\frac{\nu\alpha}{d}}$, we also obtain the bounds $[w]_{A_{p,q}}^{\frac{1}{q}}$ and it has an additional logarithmic factor, taking the form $(1 + \log[w^q]_{A_\infty})^{\frac{1}{\nu}}$. This form bounds which will be proved in Section 5.

Theorem 1 includes several known cases, the Sobolev type case $\frac{1}{q} + \frac{\alpha}{d} = \frac{1}{p}$ of these results, together with strong type estimate and multilinear extensions, can also be recovered from Fackler and Hytönen [6], Zorin-Kranich [26] the recent general framework, respectively.

For $\nu = 1$ and $\alpha = 0$, (2) holds for all Calderón-Zygmund operators. Conde-Alonso and Rey [3] first prove the result, and Lerner-Nazarov [22] give the most general version, with a simplified proof in the paper [21]. The bound (3) in this case was obtained in [13] for $p = q = 1$. In [10], Hänninen and Lorist consider the sparse domination for the lattice Hardy-Littlewood maximal operator, and their obtained sharp weighted weak L^p estimates.

For $\nu = 2$ and $\alpha = 0$, (2) holds for several square function operators of Littlewood-Paley type [7, 17, 19]. For $p = q$, the mixed bound (3), even for general $\nu > 0$, is from [12, 14]. This improves the pure A_p bound of [7, 17, 19].

For $\nu = 1$ and $0 < \alpha < d$, (2) holds for the fractional integral operator [15]

$$I_\alpha f(x) := \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{n-\alpha}} dy. \tag{5}$$

In the case for $p < q$, (3) is due to [4]. The Sobolev type case with $\frac{1}{q} + \frac{\alpha}{d} = \frac{1}{p}$ was obtained by the same authors in [5]. Additional complications with $p = q$, which lead to the weaker version of our bound (3), have been observed and addressed in different ways in [4, 5].

For $\nu>0$ and $\alpha=0$, in the case for the bound (3) was obtained by Hytönen and Li [12] for $p=q\in(1,\infty)$.

Theorem 1 with $\nu=2$ completes the picture of sharp weighted inequalities for fractional square functions, aside from the remaining case of $2\leqslant q\leqslant\frac{2}{1-\frac{2\alpha}{d}}$. Namely, $[w]_{A_{p,q}}^{\max(\frac{1}{q},\frac{1}{2}-\frac{\alpha}{d})}$ is the optimal bound among all possible bounds of form $\Phi([w]_{A_{p,q}})$ with an increasing function Φ. This was shown by Hytönen and Li [12], Lacey and Scurry [17] in the category of power type function $\Phi(t)=t^{\beta}$; a variant of their argument proves the general claim, as we show in the last section.

To prove the above results, we need the following characterization, which is essentially due to Lai [18]; we supply the necessary details to cover the cases that were not explicitly treated in [18].

THEOREM 3. *Let $1<p\leqslant q<\infty$, $\nu>0$, $p>\nu$ and $0\leqslant\alpha<d$. Let w, σ be a pair of weights. Then*

$$\|\mathscr{A}_{\alpha,\nu}^{\mathscr{S}}(\cdot\sigma)\|_{L^p(\sigma)\to L^{q,\infty}(w)}^{\nu}\simeq\mathscr{T}^*,$$

where the testing constant is defined by

$$\mathscr{T}^*:=\sup_{R\in\mathscr{S}}w(R)^{-\frac{1}{(\frac{q}{\nu})'}}\Big\|\sum_{\substack{Q\in\mathscr{S}\\ Q\subset R}}\langle\sigma\rangle_{\alpha,Q}^{\nu-1}\langle w\rangle_{\alpha,Q}\mathbf{1}_Q\Big\|_{L^{(\frac{p}{\nu})'}(\sigma)}.$$

For the testing constant $\mathscr{T}^*$, Fackler and Hytönen [6] give the following result.

PROPOSITION 1. *Let $\nu>0$, $0\leqslant\alpha<d$, $p>\nu$ and $1<p\leqslant q<\infty$. For $\mathscr{T}^*$ as in Theorem 3, we have*

$$\mathscr{T}^*\lesssim[w,\sigma]_{A_{p,q}^{\alpha}}^{\frac{\nu}{q}}\begin{cases}[w]_{A_\infty}^{1-(\frac{\nu}{p})^2}[\sigma]_{A_\infty}^{(\frac{\nu}{p})^2}, & p=q \text{ and } \alpha>0,\\ [w]_{A_\infty}^{1-\frac{\nu}{p}}, & \text{other case}.\end{cases}$$

The plan of the paper is as follows: In Section 2, Theorem 1 with $p>\nu$ will be proved. The remaining case of Theorem 1 is handled in Section 3. In Section 4, we give the proof of Theorem 2. In the final section, we discuss the sharpness of our weak type estimates by modifying the example given by Lacey and Scurry [17].

2. Proof of Theorem 3

As mentioned, Theorem 3 is essentially due to Hytönen and Li [12].
To prove our results, we first give the following lemma.

LEMMA 1. *Let w,σ be a pair of weights and $p>\nu>0$, then*

$$\|\mathscr{A}_{\alpha,\nu}^{\mathscr{S}}(\cdot\sigma)\|_{L^p(\sigma)\to L^{q,\infty}(w)}^{\nu}\simeq\sup_{\|f\|_{L^p(\sigma)}=1}\Big\|\sum_{Q\in\mathscr{S}}\langle\sigma\rangle_{\alpha,Q}^{\nu}\langle f^{\nu}\rangle_Q^{\sigma}\mathbf{1}_Q\Big\|_{L^{\frac{q}{\nu},\infty}(w)}.$$

Proof. By the definition of $\mathscr{A}_{\alpha,\nu}^{\mathscr{S}}$, we have

$$\begin{aligned}
&\|\mathscr{A}_{\alpha,\nu}^{\mathscr{S}}(\cdot\sigma)\|_{L^p(\sigma)\to L^{q,\infty}(w)}^{\nu} = \sup_{\|f\|_{L^p(\sigma)}=1} \Big\| \sum_{Q\in\mathscr{S}} \langle f\sigma\rangle_{\alpha,Q}^{\nu} \mathbf{1}_Q \Big\|_{L^{\frac{q}{\nu},\infty}(w)} \\
&= \sup_{\|f\|_{L^p(\sigma)}=1} \Big\| \sum_{Q\in\mathscr{S}} \langle \sigma\rangle_{\alpha,Q}^{\nu} (\langle f\rangle_Q^{\sigma})^{\nu} \mathbf{1}_Q \Big\|_{L^{\frac{q}{\nu},\infty}(w)} \\
&\leqslant \sup_{\|f\|_{L^p(\sigma)}=1} \Big\| \sum_{Q\in\mathscr{S}} \langle \sigma\rangle_{\alpha,Q}^{\nu} \langle (M_\sigma(f))^{\nu}\rangle_Q^{\sigma} \mathbf{1}_Q \Big\|_{L^{\frac{q}{\nu},\infty}(w)} \\
&= \sup_{\|f\|_{L^p(\sigma)}=1} \Big\| \sum_{Q\in\mathscr{S}} \langle \sigma\rangle_{\alpha,Q}^{\nu} \left\langle \left(\frac{M_\sigma(f)}{\|M_\sigma(f)\|_{L^p(\sigma)}} \right)^{\nu} \right\rangle_Q^{\sigma} \mathbf{1}_Q \Big\|_{L^{\frac{q}{\nu},\infty}(w)} \|M_\sigma(f)\|_{L^p(\sigma)}^{\nu} \\
&\lesssim \sup_{\|g\|_{L^p(\sigma)}=1} \Big\| \sum_{Q\in\mathscr{S}} \langle \sigma\rangle_{\alpha,Q}^{\nu} \langle g^{\nu}\rangle_Q^{\sigma} \mathbf{1}_Q \Big\|_{L^{\frac{q}{\nu},\infty}(w)},
\end{aligned}$$

where in the last step, we used the boundedness of M_σ on $L^p(\sigma)$, and the bound is independent of σ.

On the other hand, notice that

$$\langle f^{\nu}\rangle_Q^{\sigma} \leqslant \inf_{x\in Q} M_\sigma(f^{\nu})(x) = (\inf_{x\in Q} M_{\sigma,\nu}(f)(x))^{\nu} \leqslant (\langle M_{\sigma,\nu}(f)\rangle_Q^{\sigma})^{\nu},$$

where $M_{\sigma,\nu}(f) := (M_\sigma(f^{\nu}))^{1/\nu}$, with this observation, we have

$$\begin{aligned}
&\sup_{\|f\|_{L^p(\sigma)}=1} \Big\| \sum_{Q\in\mathscr{S}} \langle \sigma\rangle_{\alpha,Q}^{\nu} \langle f^{\nu}\rangle_Q^{\sigma} \mathbf{1}_Q \Big\|_{L^{\frac{q}{\nu},\infty}(w)} \\
&\leqslant \sup_{\|f\|_{L^p(\sigma)}=1} \Big\| \sum_{Q\in\mathscr{S}} \langle \sigma\rangle_{\alpha,Q}^{\nu} (\langle M_{\sigma,\nu}(f)\rangle_Q^{\sigma})^{\nu} \mathbf{1}_Q \Big\|_{L^{\frac{q}{\nu},\infty}(w)} \\
&\leqslant \sup_{\|f\|_{L^p(\sigma)}=1} \Big\| \mathscr{A}_{\alpha,\mathscr{S}}^{\nu}(\cdot\sigma) \Big\|_{L^p(\sigma)\to L^{q,\infty}(w)}^{\nu} \|M_{\sigma,\nu}(f)\|_{L^p(\sigma)}^{\nu} \\
&\lesssim \Big\| \mathscr{A}_{\alpha,\nu}^{\mathscr{S}}(\cdot\sigma) \Big\|_{L^p(\sigma)\to L^{q,\infty}(w)}^{\nu},
\end{aligned}$$

where in the last step, we used the boundedness of $M_{\sigma,\nu}$ on $L^p(\sigma)$ with $p>\nu$, and the bound is independent of σ. This completes the proof of Lemma 1. □

Now suppose that B is the sharp constant such that

$$\Big\| \sum_{Q\in\mathscr{S}} \langle \sigma\rangle_{\alpha,Q}^{\nu} \langle f^{\nu}\rangle_Q^{\sigma} \mathbf{1}_Q \Big\|_{L^{\frac{q}{\nu},\infty}(w)} \leqslant B\|f\|_{L^p(\sigma)}^{\nu},$$

that is,

$$\Big\| \sum_{Q\in\mathscr{S}} \langle \sigma\rangle_{\alpha,Q}^{\nu} \langle f\rangle_Q^{\sigma} \mathbf{1}_Q \Big\|_{X^{\frac{q}{\nu}}(w)} \leqslant B\|f\|_{L^{\frac{p}{\nu}}(\sigma)}. \tag{6}$$

Then

$$\|\mathscr{A}_{\alpha,\mathscr{S}}(\cdot\sigma)\|_{L^p(\sigma)\to X^q(w)} \simeq B^{\frac{1}{\nu}}.$$

Hence, we have reduced the problem to study (6).

The following proposition was given by Lacey, Sawyer and Uriarte-Tuero [16].

PROPOSITION 2. *Let $\tau = \{\tau : Q \in \mathscr{Q}\}$ be nonnegative constants, w, σ be weights and define linear operators by*

$$T_\tau := \sum_{Q \in \mathscr{Q}} \tau_Q \langle f \rangle_Q \mathbf{1}_Q.$$

Then for $1 < p \leqslant q < \infty$, there holds

$$\|T_\tau(\cdot\sigma)\|_{L^p(\sigma)\to L^{q,\infty}(w)} \simeq \sup_{R\in\mathscr{Q}} w(R)^{-\frac{1}{q'}} \Big\| \sum_{\substack{Q\in\mathscr{Q}\\ Q\subset R}} \tau_Q \langle w\rangle_Q \mathbf{1}_Q \Big\|_{L^{p'}(\sigma)}.$$

Observing that for (6), we have

$$\Big\| \sum_{Q\in\mathscr{S}} \langle\sigma\rangle^{\nu}_{\alpha,Q} \langle f\rangle^{\sigma}_Q \mathbf{1}_Q \Big\|_{L^{\frac{q}{\nu},\infty}(w)} = \|T_\tau(f\sigma)\|_{L^{\frac{q}{\nu},\infty}(w)}$$

with $\tau_Q = \langle\sigma\rangle^{\nu-1}_{\alpha,Q} |Q|^{\frac{\alpha}{d}}$. Theorem 3 follows immediately from Proposition 2. Thus, using Theorem 3 and Proposition 1, the case $p > \nu$ of Theorem 1 is proved.

The following proposition is weighted weak estimate for fractional maximal operator, which can be found in the paper[9].

PROPOSITION 3. *Given $1 < p \leqslant q < \infty$, $0 \leqslant \alpha < d$ and a pair of wights (w,σ). Then for all measurable functions f,*

$$\|M_\alpha(f\sigma)\|_{L^{q,\infty}(w)} \lesssim [w,\sigma]^{\frac{1}{q}}_{A^\alpha_{p,q}} \|f\|_{L^p(\sigma)}.$$

3. Proof of the weak type bound for $1 < p \leqslant \nu$

We are left to prove Theorem 1 in the case that $1 < p \leqslant \nu$. Actually, the method stem from Hytönen and Li [12], they have investigated the two-weight case. Following their method, it is easy to give the off-diagonal two-weight estimate as well. For completeness, we give the deails.

3.1. The case for $1 < p \leqslant q < \nu$

We want to bound the following inequality,

$$\sup_{\lambda>0} \lambda w(\{x\in\mathbb{R}^n : \mathscr{A}^{\mathscr{S}}_{\alpha,\nu}(f\sigma) > \lambda\})^{\frac{1}{q}} \lesssim [w,\sigma]^{\frac{1}{q}}_{A^\alpha_{p.q}} \|f\|_{L^p(\sigma)}.$$

By scaling it suffices to give an uniform estimate for

$$\lambda_0 w(\{x\in\mathbb{R}^n : \mathscr{A}^{\mathscr{S}}_{\alpha,\nu}(f\sigma) > \lambda_0\})^{\frac{1}{q}},$$

where λ_0 is some constant to be determined later. It is also free to further sparsify $\mathscr{S}$ such that

$$\sum_{\substack{Q'\subsetneq Q\\ Q',Q\in\mathscr{S}}} |Q'|^{1-\frac{\alpha}{d}} \leqslant \frac{1}{4}|Q|^{1-\frac{\alpha}{d}}.$$

Now set

$$\mathscr{S}_m := \{Q \in \mathscr{S} : 2^{-m-1} < \langle f\sigma\rangle_{\alpha,Q} \leqslant 2^{-m}\} \quad \text{with} \quad m \geqslant 0, \tag{7}$$

and

$$\mathscr{S}' := \{Q \in \mathscr{S} : \langle f\sigma\rangle_{\alpha,Q} > 1\}. \tag{8}$$

Then for $Q \in \mathscr{S}_m$ with $m \geqslant 0$, denote by $\mathrm{ch}_{\mathscr{S}_m}(Q)$ the maximal subcubes of Q in $\mathscr{S}_m$ and define

$$E_Q := Q \backslash \bigcup_{Q' \in \mathrm{ch}_{\mathscr{S}_m}(Q)} Q'. \tag{9}$$

Then

$$\begin{aligned}
\langle f\sigma \mathbf{1}_{E_Q}\rangle_{\alpha,Q} &= \frac{1}{|Q|^{1-\frac{\alpha}{d}}}\int_Q f\sigma - \frac{1}{|Q|^{1-\frac{\alpha}{d}}}\sum_{Q' \in \mathrm{ch}_{\mathscr{S}_m}(Q)}\int_{Q'} f\sigma \\
&= \frac{1}{|Q|^{1-\frac{\alpha}{d}}}\int_Q f\sigma - \sum_{Q' \in \mathrm{ch}_{\mathscr{S}_m}(Q)}\left(\frac{|Q'|}{|Q|}\right)^{1-\frac{\alpha}{d}}\frac{1}{|Q'|^{1-\frac{\alpha}{d}}}\int_{Q'} f\sigma \\
&\geqslant \frac{1}{|Q|^{1-\frac{\alpha}{d}}}\int_Q f\sigma - \frac{1}{4}2^{-m} \geqslant \frac{1}{2}\langle f\sigma\rangle_{\alpha,Q}.
\end{aligned} \tag{10}$$

Also, we set $\mathscr{A}_{\alpha,\nu}^{\mathscr{S}_m}$ and $\mathscr{A}_{\alpha,\nu}^{\mathscr{S}'}$ to be the sparse operators associated with $\mathscr{S}_m$ and $\mathscr{S}'$, respectively

$$(\mathscr{A}_{\alpha,\nu}^{\mathscr{S}_m}(f))^\nu := \sum_{Q \in \mathscr{S}_m}\langle f\rangle_{\alpha,Q}^\nu \mathbf{1}_Q \qquad \text{and} \qquad (\mathscr{A}_{\alpha,\nu}^{\mathscr{S}'}(f))^\nu := \sum_{Q \in \mathscr{S}'}\langle f\rangle_{\alpha,Q}^\nu \mathbf{1}_Q. \tag{11}$$

Thus, it is easy to know that

$$\mathscr{A}_{\alpha,\nu}^{\mathscr{S}} := \sum_{Q \in \mathscr{S}}\langle f\rangle_{\alpha,Q}^\nu \mathbf{1}_Q = \sum_{m \in \mathbb{N}}(\mathscr{A}_{\alpha,\nu}^{\mathscr{S}_m}(f))^\nu + (\mathscr{A}_{\alpha,\nu}^{\mathscr{S}'}(f))^\nu. \tag{12}$$

By (11) and (12), we conclude that

$$\begin{aligned}
&w(\{x \in \mathbb{R}^n : \mathscr{A}_{\alpha,\nu}^{\mathscr{S}}(f\sigma) > \lambda_0\}) \\
&\leqslant w\Big(\Big\{x \in \mathbb{R}^n : \sum_{m \geqslant 0}(\mathscr{A}_{\alpha,\nu}^{\mathscr{S}_m}(f))^\nu > \frac{\lambda_0^\nu}{2}\Big\}\Big) + w\Big(\Big\{x \in \mathbb{R}^n : (\mathscr{A}_{\alpha,\nu}^{\mathscr{S}'}(f))^\nu > \frac{\lambda_0^\nu}{2}\Big\}\Big) \\
&= w\Big(\Big\{x \in \mathbb{R}^n : \sum_{m \geqslant 0}\sum_{Q \in \mathscr{S}_m}\langle f\sigma\rangle_{\alpha,Q}^\nu \mathbf{1}_Q > \frac{\lambda_0^\nu}{2}\Big\}\Big) + w\Big(\Big\{x \in \mathbb{R}^n : \sum_{Q \in \mathscr{S}'}\langle f\sigma\rangle_{\alpha,Q}^\nu \mathbf{1}_Q > \frac{\lambda_0^\nu}{2}\Big\}\Big) \\
&=: II_1 + II_2.
\end{aligned}$$

The second term is trival. In fact, it follows immediately from Proposition 3,

$$II_2 \leqslant w\Big(\bigcup_{Q \in \mathscr{S}'} Q\Big) \leqslant w(\{x \in \mathbb{R}^n : M_\alpha(f\sigma) > 1\}) \lesssim [w,\sigma]_{A_{p,q}^\alpha}\|f\|_{L^p(\sigma)}^q.$$

Now let $\frac{\lambda_0^\nu}{2} = \sum_{m\geqslant 0} 2^{-\varepsilon m}$, where $\varepsilon := (\nu - q)/2$. By (10), we obtain the following estimate

$$\begin{aligned}
II_1 &\leqslant \sum_{m\geqslant 0} w(\{x\in\mathbb{R}^n : \sum_{Q\in\mathscr{S}_m} \langle f\sigma\rangle_{\alpha,Q}^\nu \mathbf{1}_Q > 2^{-\varepsilon m}\})\\
&\leqslant \sum_{m\geqslant 0} w(\{x\in\mathbb{R}^n : \sum_{Q\in\mathscr{S}_m} \langle f\sigma\mathbf{1}_Q\rangle_{\alpha,Q}^q \mathbf{1}_Q > 2^{(\nu-q)m}2^{-\varepsilon m}\})\\
&\leqslant \sum_{m\geqslant 0} w(\{x\in\mathbb{R}^n : \sum_{Q\in\mathscr{S}_m} \langle f\sigma\mathbf{1}_{E_Q}\rangle_{\alpha,Q}^q \mathbf{1}_Q > 2^{-q}2^{(\nu-q)m}2^{-\varepsilon m}\})\\
&\leqslant \sum_{m\geqslant 0} 2^{(q-\nu+\varepsilon)m+q}\int_{\mathbb{R}^n}\sum_{Q\in\mathscr{S}_m} \langle f\sigma\mathbf{1}_{E_Q}\rangle_{\alpha,Q}^q \mathbf{1}_Q dw \lesssim [w,\sigma]_{A_{p,q}^\alpha}\|f\|_{L^p(\sigma)}^q,
\end{aligned}$$

where in the last inequality, we used the following the fact

$$\begin{aligned}
&\int_{\mathbb{R}^n}\sum_{Q\in\mathscr{S}_m} \langle f\sigma\mathbf{1}_{E_Q}\rangle_{\alpha,Q}^q \mathbf{1}_Q dw\\
&= \sum_{Q\in\mathscr{S}_m} \langle f\sigma\mathbf{1}_{E_Q}\rangle_{\alpha,Q}^q w(Q)\\
&\leqslant \sum_{Q\in\mathscr{S}_m} \left(\frac{1}{\sigma(E_Q)^{1-\frac{1}{p}+\frac{1}{q}}}\int_{E_Q} f\sigma\right)^q |Q|^{q(\frac{\alpha}{d}-1)} w(Q)\sigma(Q)^{\frac{q}{p'}}\sigma(E_Q)\\
&\leqslant [w,\sigma]_{A_{p,q}^\alpha}\|f\|_{L^p(\sigma)}.
\end{aligned}$$

Combining the above II_1 and II_2, we get

$$\|\mathscr{A}_{\alpha,\nu}^{\mathscr{S}}(f\sigma)\|_{L^{q,\infty}(w)} \lesssim [w,\sigma]_{A_{p,q}^\alpha}^{\frac{1}{q}}\|f\|_{L^p(\sigma)}.$$

3.2. The cases for $p\leqslant q=\nu$ or $p\leqslant \nu<q$

In this case, it can be strainghtforward obtained by [6, Theorem 1.1]

$$\|\mathscr{A}_{\alpha,\nu}^{\mathscr{S}}(f\sigma)\|_{L^{q,\infty}(w)} \leqslant \|\mathscr{A}_{\alpha,\nu}^{\mathscr{S}}(f\sigma)\|_{L^{q}(w)} \lesssim [w,\sigma]_{A_{p,q}^\alpha}^{\frac{1}{q}}[\sigma]_{A_\infty}^{\frac{1}{q}}\|f\|_{L^p(\sigma)}.$$

This completes the proof of Theorem 1. □

4. Proof of Theorem 2

In this section, we key to prove the bound $[w,\sigma]_{A_{p,q}^\alpha}^{\frac{1}{q}}$ is sharp. As mentioned, Theorem 2 is essentially due to Muckenhoupt [24]. Here we extend Muckenhoupt one-weight to off-diagonal two-weight setting.

First, we need to prove

$$\|\mathscr{A}_\alpha(f\sigma)\|_{L^{q,\infty}(w)} \lesssim [w,\sigma]_{A_{p,q}}^{\frac{1}{q}}\|f\|_{L^p(\sigma)}. \tag{13}$$

This weak type estimate is trivial. Indeed, using the fact $|\mathscr{A}_\alpha(f)| \leqslant |M_\alpha(f)|$ and Proposition 3, we can straightforward obtain the desired inequality (13).

Next, we give the proof of the exponent $\frac{1}{q}$ in (13) is sharp. Consider $f = |f|\chi_Q$, then we obtain for $Q \in \mathscr{S}$

$$\mathscr{A}_\alpha(|f|\sigma) \geqslant \langle |f|\sigma\rangle_{\alpha,Q}.$$

Taking $N_0 := \|\mathscr{A}_\alpha(|f|\sigma)\|_{L^p(\sigma)\to L^{q,\infty}(w)}$. Using the weak type norm inequality of $\mathscr{A}_\alpha(|f|\sigma)$, we have

$$N_0^q\left(\int_Q |f|^p\sigma\right)^{\frac{q}{p}} = N_0^q\|f\|_{L^p(\sigma)}^q \geqslant \|\langle |f|\sigma\rangle_{\alpha,Q}\|_{L^{q,\infty}(w)}^q = |Q|^{q(\frac{\alpha}{d}-1)}w(Q)\int_Q |f|\sigma. \quad (14)$$

Consider the specific function $f = \mathbf{1}_Q$ supported on Q and chosen so that both integrands coincide, namely $|f|\sigma = |f|^p\sigma$. Substitute this specific function f into (14) to obtain the following inequality only pertaining the weight σ and the cube Q,

$$N_0^q \geqslant |Q|^{q(\frac{\alpha}{d}-1)}w(Q)\sigma(Q)^{\frac{q}{p'}}.$$

Distribute $|Q|$ and take the supremum over all cubes Q to conclude that $N_0 \geqslant [w,\sigma]_{A_{p,q}^\alpha}^{\frac{1}{q}}$. There is one technicality, the chosen function may not be integrable, choose instead $f_K = \frac{K}{K+\sigma}\mathbf{1}_Q$, run the argument for each $K > 0$ then let K go to infinity. Thus, we finish the proof of Theorem 2. □

5. Sharpness of the weak type bounds for fractional square function

In this section, we will show that the case for $\nu = 2$, which called fractional square function, i.e.

$$\mathscr{A}_{\alpha,2}^{\mathscr{S}}(f) = \Big(\sum_{Q\in\mathscr{S}} \langle f\rangle_{\alpha,Q}^2 \mathbf{1}_Q\Big)^{\frac{1}{2}}, \quad (15)$$

and p,q,α satisfy condition $\frac{1}{q}+\frac{\alpha}{d}=\frac{1}{p}$. We only consider one weight theory estimate for $L^p(w^p) \to L^{q,\infty}(w^q)$ in here. The governing weight class is a generalization of Muckenhoupt A_p weights, and was introduced by Muckenhoupt and Wheeden [23].

$$[w]_{A_{p,q}} := \sup_Q \left(\frac{1}{|Q|}\int_Q w^q\right)\left(\frac{1}{|Q|}\int_Q w^{-p'}\right)^{\frac{q}{p'}} < \infty.$$

Its relation to two weight characteristic is $[w^q, w^{-p'}]_{A_{p,q}^\alpha} = [w]_{A_{p,q}}$ with $\frac{1}{q}+\frac{\alpha}{d}=\frac{1}{p}$. Moreover, it is straightforward to show that the following are equivalent:

$$\text{(a)}\quad w \in A_{p,q}; \qquad \text{(b)}\quad w^q \in A_{1+\frac{q}{p'}} \quad \text{and} \quad w^{-p'} \in A_{1+\frac{p'}{q}}. \quad (16)$$

We will show that the norm bound

$$\|\mathscr{A}_{\alpha,2}^{\mathscr{S}}\|_{L^p(w^p)\to L^{q,\infty}(w^q)} \leqslant [w]_{A_{p,q}}^{\max(\frac{1}{q},\frac{1}{2}-\frac{\alpha}{d})}$$

is unimprovable. Actually, a lower bound with the exponent $\frac{1}{q}$ holds uniformly over all weights, which is the content of the next Theorem. The optimality of the exponent $\frac{1}{2}-\frac{\alpha}{d}$ is slightly more tricky, and is based on a example of a specific weight $A_{p,q}$. Hence, Theorem 1 for $\sigma = w^{-1/(p-1)}$ gives the following mixed $A_{p,q}-A_\infty$ estimate.

COROLLARY 1. *Let $0<\alpha<d$ and $1<p\leqslant q<\infty$ with $\frac{1}{q}+\frac{\alpha}{d}=\frac{1}{p}$. Then*

$$\|\mathscr{A}^{\mathscr{S}}_{\alpha,2}\|_{L^p(w^p)\to L^{q,\infty}(w^q)} \lesssim [w,\sigma]^{\frac{1}{q}}_{A^{\alpha}_{p,q}} \begin{cases} [w^{-p'}]^{\frac{1}{q}}_{A_\infty}, & 2\leqslant q\leqslant \dfrac{2}{1-\frac{2\alpha}{d}}, \\ [w^q]^{(\frac{1}{2}-\frac{1}{p})_+}_{A_\infty}, & \text{other case.} \end{cases}$$

Notice that (16), it is easy to know that

$$[w^q]_{A_{1+\frac{q}{p'}}} = [w]_{A_{p,q}} \qquad \text{and} \qquad [w^{-p'}]_{A_{1+\frac{p'}{q}}} = [w]^{\frac{p'}{q}}_{A_{p,q}}. \tag{17}$$

Since Lerner [20] show that $[w]_{A_\infty}\lesssim [w]_{A_p}$. Using this relation to Corollary 1, we obtain the following pure $A_{p,q}$ estimate.

COROLLARY 2. *Let $0<\alpha<d$ and $1<p\leqslant q<\infty$ with $\frac{1}{q}+\frac{\alpha}{d}=\frac{1}{p}$. Then*

$$\|\mathscr{A}^{\mathscr{S}}_{\alpha,2}\|_{L^p(w^p)\to L^{q,\infty}(w^q)} \lesssim \begin{cases} [w]^{\frac{p'}{q}(1-\frac{\alpha}{d})}_{A_{p,q}}, & 2\leqslant q\leqslant \dfrac{2}{1-\frac{2\alpha}{d}}, \\ [w]^{(\frac{1}{2}-\frac{\alpha}{d})}_{A_{p,q}}, & \dfrac{2}{1-\frac{2\alpha}{d}}<q<\infty, \\ [w]^{\frac{1}{q}}_{A_{p,q}}, & 1\leqslant q<2. \end{cases}$$

However, the exponent $\frac{p'}{q}(1-\frac{\alpha}{d})$ of the case $2\leqslant q\leqslant \frac{2}{1-\frac{2\alpha}{d}}$ is not optimal. We will give the best exponent $\max(\frac{1}{q},\frac{1}{\nu}-\frac{\alpha}{d})$ appear in the following estimate. For general case $\nu\geqslant 1$, we are concerned with the weak-type bounds which have an additional logarithmic factor $(\log_1[w^q]_{A_\infty})^{\frac{1}{\nu}}$ appears in the followig sharp estimate.

THEOREM 4. *Let $\nu\geqslant 1$, $0\leqslant\alpha<d$ and $1\leqslant p\leqslant q<\infty$ with $\frac{1}{p}+\frac{\alpha}{d}=\frac{1}{q}$, there holds for any weight $w\in A_{p,q}$*

$$\|\mathscr{A}^{\mathscr{S}}_{\alpha,\nu}(f)\|_{L^{q,\infty}(w^q)} \lesssim [w]^{\max(\frac{1}{q},\frac{1}{\nu}-\frac{\alpha}{d})}_{A_{p,q}}\phi([w^q]_{A_\infty})\|wf\|_{L^p},$$

where

$$\phi([w^q]_{A_\infty}) = \begin{cases} (\log_1[w^q]_{A_\infty})^{\frac{1}{\nu}}, & \nu\leqslant q\leqslant \dfrac{\nu}{1-\frac{\nu\alpha}{d}}; \\ 1, & \text{other case.} \end{cases}$$

and $\log_1(x)=1+\log_+(x)$.

As a corollary of Theorem 4, the weak type bounds of fractional square function is sharp.

COROLLARY 3. *Let $0 \leqslant \alpha < d$ and $1 \leqslant p \leqslant q < \infty$ with $\frac{1}{p} + \frac{\alpha}{d} = \frac{1}{q}$, there holds for any weight $w \in A_{p,q}$*

$$\|\mathscr{A}^{\mathscr{S}}_{\alpha,2}(f)\|_{L^{q,\infty}(w^q)} \lesssim [w]_{A_{p,q}}^{\max(\frac{1}{q},\frac{1}{2}-\frac{\alpha}{d})} \phi_1([w^q]_{A_\infty})\|wf\|_{L^p},$$

where

$$\phi_1([w^q]_{A_\infty}) = \begin{cases} (\log_1 [w^q]_{A_\infty})^{\frac{1}{2}}, & 2 \leqslant q \leqslant \dfrac{2}{1-\frac{2\alpha}{d}}; \\ 1, & \textit{other case}. \end{cases}$$

A basic tool for us is the following classical reverse Hölder's inequality with optimal bound, which can be found in [13].

PROPOSITION 4. *There is a dimensional constant $c > 0$ such that for $w \in A_\infty$, and $r(w) = 1 + c[w]_{A_\infty}$, there holds*

$$\langle w^{r(w)} \rangle_Q^{\frac{1}{r(w)}} \leqslant 2\langle w \rangle_Q, \qquad Q \text{ a cube.} \tag{18}$$

We also need the following off-diagonal extrapolation given by Duoandikoetxra [8].

PROPOSITION 5. *Let $1 \leqslant p_0 < \infty$ and $0 < q_0 < \infty$. Assume that that for some family of nonnegative couples (f,g) and for all $w \in A_{p_0,q_0}$ we have*

$$\|wg\|_{L^{q_0}} \leqslant CN([w]_{A_{p_0,q_0}})\|wf\|_{L^{p_0}},$$

where N is an increasing function and the constant C does not depend on w. Set $\gamma = \frac{1}{q_0} + \frac{1}{p_0'}$. Then for $1 < p < \infty$ and $0 < q < \infty$, such that

$$\frac{1}{q} - \frac{1}{p} = \frac{1}{q_0} - \frac{1}{q_0},$$

and all $w \in A_{p,q}$ we have

$$\|wg\|_{L^p} \leqslant CK(w)\|wf\|_{L^p},$$

where

$$K(w) = \begin{cases} N([w]_{A_{p,q}}(2\|M\|_{L^{\gamma q}(w^q)})^{\gamma(q-q_0)}), & q < q_0; \\ N([w]_{A_{p,q}}^{\frac{\gamma q_0 - 1}{\gamma q - 1}}(2\|M\|_{L^{\gamma p'}(w^{-p'})})^{\frac{\gamma(q-q_0)}{\gamma q - 1}}), & q > q_0. \end{cases}$$

In particular, $K(w) \leqslant C_1 N(C_2[w]_{A_{p,q}}^{\max(1,\frac{q_0 p'}{q p_0'})})$ for $w \in A_{p,q}$.

The following estimate is based on the work of Domingo-Salazar, Lacey, and Rey [7].

THEOREM 5. *Let* $\nu \geqslant 1$, $0 \leqslant \alpha < d$ *and* $1 \leqslant p \leqslant q < \infty$ *with* $\frac{1}{q} + \frac{\alpha}{d} = \frac{1}{p}$, *there holds for any weight* $w \in A_{p,q}$

$$\|\mathscr{A}_{\alpha,\nu}^{\mathscr{S}}(f)\|_{L^{q,\infty}(w^q)} \lesssim [w]_{A_{p,q}}^{\max(\frac{1}{q},\frac{1}{\nu}-\frac{\alpha}{d})} \phi_2([w^q]_{A_\infty})\|wf\|_{L^p}, \tag{19}$$

where

$$\phi_2([w^q]_{A_\infty}) = \begin{cases} 1, & 1 \leqslant q < \nu; \\ (\log_1 [w^q]_{A_\infty})^{\frac{1}{\nu}}, & \nu \leqslant q < \infty. \end{cases}$$

Theorem 4 follows immediately from Theorems 1 and 5.

To prove Theorem 5, we need the following estimate.

LEMMA 2. *Let* $\nu \geqslant 1$, $q \geqslant \nu$, $0 \leqslant \alpha < d$ *and* $1 \leqslant p \leqslant q < \infty$ *with* $\frac{1}{q} + \frac{\alpha}{d} = \frac{1}{p}$, *then*

$$\|\mathscr{A}_{\alpha,\nu}^{\mathscr{S}_m}\|_{L^{q,\infty}(w^q)} \lesssim [w]_{A_{p,q}}^{\frac{1}{\nu}-\frac{\alpha}{d}} \|wf\|_{L^p},$$

where $0 < m < \log_1 [w^q]_{A_\infty}$.

Proof. Using the off-diagonal extrapolation in Proposition 5, we only need to prove the case $q = \frac{\nu}{1-\frac{\nu\alpha}{d}}$. By Minkowski's inequality and (10), we obtain the following inequalities

$$\begin{aligned}
\Big(\int_{\mathbb{R}^n} \Big(\sum_{Q\in\mathscr{S}_m} \langle f\rangle_{\alpha,Q}^{\nu} \mathbf{1}_Q\Big)^{\frac{q}{\nu}} \upsilon\Big)^{\frac{1}{q}} &\leqslant \Big(\sum_{Q\in\mathscr{S}_m} \Big(\int_{\mathbb{R}^n} \langle f\rangle_{\alpha,Q}^{q} \mathbf{1}_Q \upsilon\Big)^{\frac{\nu}{q}}\Big)^{\frac{1}{\nu}} \\
&= \Big(\sum_{Q\in\mathscr{S}_m} \langle f\rangle_{\alpha,Q}^{\nu} \upsilon^{\frac{\nu}{q}}(Q)\Big)^{\frac{1}{\nu}} \\
&\lesssim \Big(\sum_{Q\in\mathscr{S}_m} \langle f\mathbf{1}_{E_m(Q)}\rangle_{\alpha,Q}^{\nu} \upsilon^{\frac{\nu}{q}}(Q)\Big)^{\frac{1}{\nu}} \\
&\leqslant \Big(\sum_{Q\in\mathscr{S}_m} \langle f^p \mathbf{1}_{E_m(Q)} w^p\rangle_{\alpha,Q} \langle \sigma^{p'}\rangle_{\alpha,Q}^{\frac{p}{p'}} \upsilon^{\frac{p}{q}}(Q)\Big)^{\frac{1}{p}} \\
&\leqslant [w]_{A_{q,\nu}}^{\frac{1}{q}} \Big(\sum_{Q\in\mathscr{S}_m} \int_{E_m(Q)} f^p \mathbf{1}_{E_m(Q)} w^p\Big)^{\frac{1}{p}} \\
&\leqslant [w]_{A_{q,\nu}}^{\frac{1}{\nu}-\frac{\alpha}{d}} \|wf\|_{L^p},
\end{aligned}$$

where we used the equation $p = \nu$ in the above inequalities. □

The good property of Lebesgue measure appear in the paper [7].

PROPOSITION 6. *Let any $\lambda > 0$, $\mathscr{S}_m$ defined as (7) and $b = \sum_{Q' \in \mathscr{S}_m} \mathbf{1}_{Q'}$, then we have that for any dyadic cube $Q \in \mathscr{S}_m$*

$$|\{x \in Q : b(x) > \lambda\}| \lesssim \exp(-c\lambda)|Q|.$$

For $\log_1 [w^q]_{A_\infty} \leqslant m < \infty$, we also have following estimate.

LEMMA 3. *Let υ denote the weight w^q, for all integers $m_0 > 0$, then*

$$\upsilon\Big(\sum_{m=m_0}^{\infty} (\mathscr{A}_{\alpha,\nu}^{\mathscr{S}_m}(f))^\nu > 1\Big) \lesssim [w]_{A_{p,q}} \left(\frac{[\upsilon]_{A_\infty}}{2^{m_0}} \right)^q \|wf\|_{L^p}^q. \tag{20}$$

Proof. Define

$$\mathscr{S}_m^* := \{Q \text{ maximal s.t. } Q \in \mathscr{S}_m\} \quad \text{and} \quad B_m := \bigcup\{Q : Q \in \mathscr{S}_m^*\}.$$

By the definitions of $\mathscr{S}_m$ and $(\mathscr{A}_{\alpha,\nu}^{\mathscr{S}_m}(f))^\nu$, we can write $(\mathscr{A}_{\alpha,\nu}^{\mathscr{S}_m}(f))^\nu$ as $2^{-\nu m} b_m$, where

$$b_m \leqslant \sum_{Q \in \mathscr{S}_m} \mathbf{1}_Q \quad \text{and} \quad \operatorname{supp}(b_m) \subset B_m.$$

For any dyadic cube $Q \in \mathscr{S}_m$, by Proposition 6, we know that the function b_m is locally exponentially integrable. By the sharp weak-type estimate for the fractional maximal function [15], we know that

$$\upsilon(B_m) \lesssim 2^{qm} [w]_{A_{p,q}} \|wf\|_{L^p}^q.$$

The left hand side of (20) can be estimated as

$$\begin{aligned} \upsilon\Big(\sum_{m=m_0}^{\infty} (\mathscr{A}_{\alpha,\nu}^{\mathscr{S}_m}(f))^\nu > 1\Big) &= \upsilon\Big(\sum_{m=m_0}^{\infty} 2^{-\nu m} b_m > \sum_{m=m_0}^{\infty} 2^{m_0 - m - 1}\Big) \\ &\leqslant \sum_{m=m_0}^{\infty} \upsilon(b_m > 2^{m_0 + (\nu-1)m - 1}). \end{aligned}$$

Taking

$$\beta(Q) := \{x \in Q : b_m(x) > 2^{m_0 + (\nu-1)m - 1}\}$$

for any dyadic cube $Q \in \mathscr{S}_m^*$, by the definition of $\mathscr{S}_m^*$ and Proposition 6, we show that

$$|\beta(Q)| \lesssim \exp(-c 2^{m_0 + (\nu-1)m})|Q|.$$

Using the A_∞ property for $A_{1+\frac{q}{p'}}$ weights with υ-measure and Proposition 18, there holds

$$\begin{aligned} \upsilon(\beta(Q)) = \langle \upsilon \mathbf{1}_{\beta(Q)} \rangle_Q |Q| &\leqslant \langle \mathbf{1}_{\beta(Q)} \rangle_Q^{(\frac{1}{r(\upsilon)})'} \langle \upsilon^{r(\upsilon)} \rangle_Q^{\frac{1}{r(\upsilon)}} |Q| \\ &\lesssim \left[\frac{|\beta(Q)|}{|Q|} \right]^{(c[\upsilon]_{A_\infty})^{-1}} \upsilon(Q) \lesssim \upsilon(Q) \exp\left(-c \frac{2^{m_0 + (\nu-1)m}}{[\upsilon]_{A_\infty}} \right), \end{aligned}$$

where $r(\upsilon)$ as in (18).

Summing over the disjoint cubes in $\mathscr{S}_m^*$, we obtain

$$\upsilon\Big(\sum_{m=m_0}^{\infty}(\mathscr{A}_{\alpha,\nu}^{\mathscr{S}_m}(f))^{\nu}>1\Big)\lesssim[w]_{A_{p,q}}\|wf\|_{L^p}^q\sum_{m=m_0}^{\infty}2^{mq}\exp\left(-c\frac{2^{m_0+(\nu-1)m}}{[\upsilon]_{A_\infty}}\right). \tag{21}$$

The sum in the right hand side of (21), we can be controlled by

$$\begin{aligned}\sum_{m=m_0}^{\infty}2^{mq}\exp\left(-c\frac{2^{m_0+(\nu-1)m}}{[\upsilon]_{A_\infty}}\right)&\leqslant\int_{m_0}^{\infty}2^{qx}\exp\left(-c\frac{2^{m_0+(\nu-1)x}}{[\upsilon]_{A_\infty}}\right)dx\\&\approx\int_{2^{(\nu-1)m_0}}^{\infty}y^q\exp\left(-c\frac{2^{m_0}}{[\upsilon]_\infty}y\right)\frac{dy}{y}\\&=\left(\frac{[\upsilon]_\infty}{2^{m_0}}\right)^q\int_{\frac{2^{\nu m_0}}{[\upsilon]_\infty}}^{\infty}y^qe^{-cy}\frac{dy}{y}\lesssim\left(\frac{[\upsilon]_\infty}{2^{m_0}}\right)^q.\end{aligned}\tag{22}$$

Combining (21) and (22), we obtain the desired result. This completes the proof Lemma 3. □

Proof. of Theorem 5. Since the case $1\leqslant q<\nu$ is contained in Theorem 1, we key to study the case $q\geqslant\nu$. By scaling, the left hand side of (19) suffices to estimate

$$\lambda^q\upsilon(\{x\in\mathbb{R}^n:\mathscr{A}_{\alpha,\nu}^{\mathscr{S}}>\lambda\}). \tag{23}$$

Assume that $\lambda=3^{\frac{1}{\nu}}$ and $\|f\|_{L^p(w^p)}=1$. Thus, by (12), we obtain

$$\begin{aligned}&\upsilon((\mathscr{A}_{\alpha,\nu}^{\mathscr{S}}(f))^{\nu}>3)\\&\leqslant\upsilon((\mathscr{A}_{\alpha,\nu}^{\mathscr{S}'}(f))^{\nu}>1)+\upsilon\left(\sum_{m=0}^{m_0-1}(\mathscr{A}_{\alpha,\nu}^{\mathscr{S}_m}(f))^{\nu}>1\right)+\upsilon\left(\sum_{m=m_0}^{\infty}(\mathscr{A}_{\alpha,\nu}^{\mathscr{S}_m}(f))^{\nu}>1\right).\end{aligned}$$

By the sharp weak-type estimate for the fractional maximal function [15], the first term is controlled as

$$\upsilon((\mathscr{A}_{\alpha,\nu}^{\mathscr{S}'}(f))^{\nu}>1)\lesssim[w]_{A_{p,q}}^{\frac{1}{q}}. \tag{24}$$

By Chebysheff's inequality and Minkowski's inequality for $q\geqslant\nu$, the second term from Lemma 2

$$\begin{aligned}\upsilon\left(\sum_{m=0}^{m_0-1}(\mathscr{A}_{\alpha,\nu}^{\mathscr{S}_m}(f))^{\nu}>1\right)&\leqslant\left\|\sum_{m=0}^{m_0-1}(\mathscr{A}_{\alpha,\nu}^{\mathscr{S}_m}(f))^{\nu}\right\|_{L^{\frac{q}{\nu}}(\upsilon)}^{\frac{q}{\nu}}\\&\leqslant\left(\sum_{m=0}^{m_0-1}\|(\mathscr{A}_{\alpha,\nu}^{\mathscr{S}_m}(f))^{\nu}\|_{L^{\frac{q}{\nu}}(\upsilon)}\right)^{\frac{q}{\nu}}\\&=\left(\sum_{m=0}^{m_0-1}\|\mathscr{A}_{\alpha,\nu}^{\mathscr{S}_m}(f)\|_{L^q(w^q)}^{\nu}\right)^{\frac{q}{\nu}}\lesssim(m_0[w]_{A_{p,q}}^{\frac{1}{\nu}-\frac{\alpha}{d}})^{\frac{q}{\nu}}.\end{aligned}\tag{25}$$

By Lemma 3, the third term can be estimated as

$$\upsilon\left(\sum_{m=m_0}^{\infty}(\mathscr{A}_{\alpha,\nu}^{\mathscr{S}_m}(f))^{\nu}>1\right)\lesssim [w]_{A_{p,q}}\left(\frac{[w^q]_{\infty}}{2^{m_0}}\right)^q. \tag{26}$$

Combining (24), (25) and (26), we get

$$\begin{aligned}\|\mathscr{A}_{\alpha,\nu}^{\mathscr{S}}\|_{L^{q,\infty}(w^q)} &\lesssim [w]_{A_{p,q}}^{\frac{1}{q}}+m_0^{\frac{1}{\nu}}[w]_{A_{p,q}}^{\frac{1}{\nu}-\frac{\alpha}{d}}+[w]_{A_{p,q}}^{\frac{1}{q}}[w^q]_{A_\infty}2^{-m_0}\\ &\approx [w]_{A_{p,q}}^{\max(\frac{1}{q},\frac{1}{\nu}-\frac{\alpha}{d})}(\log_1[w^q]_{A_\infty})^{\frac{1}{\nu}},\end{aligned}$$

due to $m_0\approx\log_1[w^q]_{A_\infty}$. This finishes the proof Theorem 5. □

However, this is not the end of the story; we can prove even more. Here we present our full statement of the main theorem. This estimate is sharp in the following sense.

THEOREM 6. *For any weight w, we have*

$$\|\mathscr{A}_{\alpha,\nu}^{\mathscr{S}}\|_{L^p(w^p)\to L^{q,\infty}(w^q)}\geqslant [w]_{A_{p,q}}^{\frac{1}{q}}.$$

Proof. Let υ denote the weight w^q and consider $f=|f|\chi_Q$, then we obtain for $Q\in\mathscr{S}$

$$\mathscr{A}_{\alpha,\nu}^{\mathscr{S}}(f)\geqslant\langle|f|\rangle_{\alpha,Q}.$$

Taking $N:=\|\mathscr{A}_{\alpha,\nu}^{\mathscr{S}}(f)\|_{L^p(w^p)\to L^{q,\infty}(\upsilon)}$, then we have

$$\begin{aligned}N\|f\|_{L^p(w^p)}&\geqslant\|\mathscr{A}_{\alpha,\nu}^{\mathscr{S}}(f)\|_{L^{q,\infty}(\upsilon)}\geqslant\|\langle|f|\rangle_{\alpha,Q}\|_{L^{q,\infty}(\upsilon)}\\ &=\frac{\upsilon(Q)^{\frac{1}{q}}}{|Q|^{1-\frac{\alpha}{d}}}\int_Q|f|=\frac{\upsilon(Q)^{\frac{1}{q}}}{|Q|^{1-\frac{\alpha}{d}}}\int_Q|f|w^{-p}w^p\end{aligned}$$

for all positive functions $|f|$ on Q. By the converse to Hölder's inequality, this shows that

$$N\geqslant\frac{\upsilon(Q)^{\frac{1}{q}}}{|Q|^{1-\frac{\alpha}{d}}}\|w^{-p}\|_{L^{p'}(w^p)}=\frac{\upsilon(Q)^{\frac{1}{q}}\sigma(Q)^{\frac{1}{p'}}}{|Q|^{1-\frac{\alpha}{d}}},$$

and taking the supremuum over all Q proves this theorem. □

THEOREM 7. *Let $\nu\geqslant1$, $0\leqslant\alpha<d$ and $1\leqslant p\leqslant q<\infty$ with $\frac{1}{q}+\frac{\alpha}{d}=\frac{1}{p}$. If Φ be an increasing function such that*

$$\|\mathscr{A}_{\alpha,\nu}^{\mathscr{S}}\|_{L^p(w^p)\to L^{q,\infty}(w^q)}\leqslant\Phi([w]_{A_{p,q}})$$

for all $w\in A_{p,q}$, then $\Phi(t)\gtrsim ct^{\frac{1}{\nu}-\frac{\alpha}{d}}$.

Lacey and Scurry [17] show that this in class of power functions, namely, they proved that there cannot be a bound the form $\Phi(t)=t^{\frac{1}{2}-\eta}$ for $\eta>0$. We will extend their method to general setting.

Proof. We will consider two cases to prove this theorem: $\nu>1$ and $\nu=1$.

Case 1: $\nu>1$. Following the same arguments as those in [12, 17], the assumption implies

$$\big\|\big(\sum_{Q}\langle a_Q\cdot w^q\rangle_{\alpha,Q}^{\nu}\mathbf{1}_Q\big)^{\frac{1}{\nu}}\big\|_{L^{p'}(w^{-p'})}\lesssim\Phi([w]_{A_{p,q}})\big\|\big(\sum_{Q}a_Q^{\nu}\big)^{\frac{1}{\nu}}\big\|_{L^{q',1}(w^q)} \tag{27}$$

for all sequences of measurable functions a_Q. For $\vartheta\in(0,1)$, we consider $w(x)=|x|^{\frac{\vartheta-1}{q}}$ and a sequence of functions

$$a_{[0,2^{-k})}(x):=a_k(x):=\vartheta^{\frac{1}{\nu-1}-\frac{1}{\nu}}\sum_{j=k+1}^{\infty}2^{-\vartheta(j-k)}\mathbf{1}_{[2^{-j},2^{-j+1})}(x),\qquad k\in\mathbb{N}.$$

Then it is easy to check that

$$[w]_{A_{p,q}}=[w^q]_{A_{1+\frac{q}{p'}}}\simeq\vartheta^{-1}\qquad\text{and}\qquad\sum_k a_k^{\nu}(x)\lesssim\vartheta^{\frac{\nu}{\nu-1}-2}\mathbf{1}_{[0,1]}.$$

In fact, we choose $I_k=[0,2^{-k}]$ and $x\in(2^{-(l+1)},2^{-l}]$ with $l\in\mathbb{N}_0$ such that

$$a_k(x)\simeq\vartheta^{\frac{1}{\nu-1}-\frac{1}{\nu}}|I_k|^{-\vartheta}|x|^{\vartheta}\mathbf{1}_{I_k}(x).$$

A simple calculation shows that

$$\begin{aligned}\sum_{k=0}^{\infty}a_k^{\nu}(x)&=\vartheta^{\frac{\nu}{\nu-1}-1}|x|^{\nu\vartheta}\sum_{k=0}^{\infty}|I_k|^{-\nu\vartheta}\mathbf{1}_{I_k}(x)\\&=\vartheta^{\frac{\nu}{\nu-1}-1}|x|^{\nu\vartheta}\sum_{k=0}^{l}(2^{\nu\vartheta})^k\\&=\vartheta^{\frac{\nu}{\nu-1}-1}|x|^{\nu\vartheta}\frac{2^{\nu(l+1)\vartheta}-1}{2^{\nu\vartheta}-1}\\&\lesssim\vartheta^{\frac{\nu}{\nu-1}-2}|x|^{\nu\vartheta}2^{\nu l\vartheta}\lesssim\vartheta^{\frac{\nu}{\nu-1}-2}\mathbf{1}_{[0,1]}.\end{aligned}\tag{28}$$

Using (28), the right hand side of (27) implies that

$$\begin{aligned}\big\|\big(\sum_{k=1}^{\infty}a_k(x)^{\nu}\big)^{\frac{1}{\nu}}\big\|_{L^{q',1}(w^q)}&\lesssim q'\int_0^{\infty}\Big(\int_{\{x\in[0,1]:\,c\vartheta^{\frac{\nu}{\nu-1}-2}>s\}}|x|^{\vartheta-1}dx\Big)^{\frac{1}{q'}}ds\\&\leqslant\int_0^{c\vartheta^{\frac{\nu}{\nu-1}-2}}\Big(\int_0^1|x|^{\vartheta-1}dx\Big)^{\frac{1}{q'}}ds\simeq\vartheta^{\frac{\nu}{\nu-1}-2}\vartheta^{-\frac{1}{q'}}.\end{aligned}$$

On the other hand, the left hand side of (27) can be estimated as

$$\langle a_k \cdot w^q \rangle_{\alpha,[0,2^{-k})} \simeq \vartheta^{\frac{1}{\nu-1}-\frac{1}{\nu}} 2^{k(1-\frac{\alpha}{d})} \sum_{j=k+1}^{\infty} 2^{-\vartheta(j-k)} 2^{-\vartheta j} \simeq \vartheta^{\frac{1}{\nu-1}-\frac{1}{\nu}-1} 2^{k(1-\frac{\alpha}{d}-\vartheta)}.$$

It follows that

$$\begin{aligned} &\int_{[0,1]} \Big(\sum_{k=1}^{\infty} \langle a_k \cdot w^q \rangle^{\nu}_{\alpha,[0,2^{-k})} \mathbf{1}_{[0,2^{-k})} \Big)^{\frac{p'}{\nu}} w^{-p'} \\ &\simeq \vartheta^{\frac{p'}{\nu-1}-\frac{p'}{\nu}-p'} \int_0^1 |x|^{(\vartheta-(1-\frac{\alpha}{d})p'} |x|^{-\frac{(\vartheta-1)p'}{q}} dx \\ &= \vartheta^{\frac{p'}{\nu-1}-\frac{p'}{\nu}-p'} \int_0^1 |x|^{\frac{\vartheta p'}{q'}-1} dx = \frac{q'}{p'} \vartheta^{\frac{p'}{\nu-1}-\frac{p'}{\nu}-p'-1}. \end{aligned}$$

By assumption, this implies

$$\vartheta^{\frac{1}{\nu-1}-\frac{1}{p'}-\frac{1}{\nu}-1} \lesssim \Phi([w]_{A_{p,q}}) \vartheta^{\frac{\nu}{\nu-1}-2} \vartheta^{-\frac{1}{q'}} \lesssim \Phi(c\vartheta^{-1}) \vartheta^{\frac{\nu}{\nu-1}-2} \vartheta^{-\frac{1}{q'}}.$$

Hence, we show that $\Phi(t) \gtrsim t^{\frac{1}{\nu}-\frac{\alpha}{d}}$, this finishes the proof of *Case* 1.

Case 2: $\nu = 1$. The upper bound of this case follows from [15], and we show that

$$\|\mathscr{A}^{\mathscr{S}}_{\alpha,1}(f)\|_{L^{q,\infty}(w^q)} \lesssim \Phi([w]_{A_{p,q}}) \|wf\|_{L^p} \tag{29}$$

holds for $\Phi(t) \geqslant ct^{1-\frac{\alpha}{d}}$.

By (17), we show that

$$\|\mathscr{A}^{\mathscr{S}}_{\alpha,1}(f)\|_{L^{q,\infty}(w^q)} \lesssim \Phi([w^q]_{A_{1+q/p'}}) \|wf\|_{L^p}, \tag{30}$$

and let $u = w^q$, then

$$\|\mathscr{A}^{\mathscr{S}}_{\alpha,1}(f)\|_{L^{q,\infty}(u)} \lesssim \Phi([u]_{A_{1+q/p'}}) \|f\|_{L^p(u^{p/q})}. \tag{31}$$

Assume now that $u \in A_1$, then (31) it yields that

$$\|\mathscr{A}^{\mathscr{S}}_{\alpha,1}(f)\|_{L^{q,\infty}(u)} \lesssim \Phi([u]_{A_1}) \|f\|_{L^p(u^{p/q})}. \tag{32}$$

Since $\frac{p}{q} = 1 - \frac{p\alpha}{d}$, this is equivalent to

$$\|\mathscr{A}^{\mathscr{S}}_{\alpha,1}(u^{\frac{\alpha}{d}} f)\|_{L^{q,\infty}(u)} \lesssim \Phi([u]_{A_1}) \|f\|_{L^p(u)}. \tag{33}$$

Next, we will go to prove (33) holds for $\Phi(t) \geqslant ct^{1-\frac{\alpha}{d}}$. Let

$$u(x) = |x|^{\vartheta-n}$$

with $0 < \vartheta < 1$. Then standard computations shows that

$$[u]_{A_1} \simeq \vartheta^{-1}. \tag{34}$$

Consider the function $f = \chi_B$ where B is the unit ball, we can compute its norm to be

$$\|f\|_{L^p(u)} = u(B)^{\frac{1}{p}} \simeq \vartheta^{-\frac{1}{p}}. \tag{35}$$

By sparse domination formula, then there exists a η-sparse family $\mathscr{S}$ such that

$$\mathscr{A}^{\mathscr{S}}_{\alpha,1}(|f|)(x) \gtrsim |I_\alpha f(x)|, \tag{36}$$

where I_α is defined by (5). Let $0 < x_\vartheta < 1$ be a parameter whose value will be chosen soon. By (36), we have that

$$\begin{aligned}
\|\mathscr{A}^{\mathscr{S}}_{\alpha,1}(u^{\frac{\alpha}{d}} f)\|_{L^{q,\infty}(u)} &\gtrsim \|I_\alpha u^{\frac{\alpha}{d}} f\|_{L^{q,\infty}(u)} \\
&\geqslant \sup_{\lambda>0} \left(u\{|x| < x_\vartheta : \int_B \frac{|y|^{(\vartheta-1)\alpha/d}}{|x-y|^{1-\alpha/d}} dx > \lambda\} \right)^{\frac{1}{q}} \\
&\geqslant \sup_{\lambda>0} \left(u\{|x| < x_\vartheta : \int_{B\setminus B(0,|x|)} \frac{|y|^{(\vartheta-1)\alpha/d}}{|x-y|^{1-\alpha/d}} dx > \lambda\} \right)^{\frac{1}{q}} \\
&\geqslant \sup_{\lambda>0} \left(u\{|x| < x_\vartheta : \int_{B\setminus B(0,|x|)} \frac{|y|^{(\vartheta-1)\alpha/d}}{(2|y|)^{1-\alpha/d}} dx > \lambda\} \right)^{\frac{1}{q}} \\
&= \sup_{\lambda>0} \left(u\{|x| < x_\vartheta : \frac{c_{\alpha,d}}{\vartheta}(1 - |x|^{\vartheta\alpha/d}) > \lambda\} \right)^{\frac{1}{q}} \\
&\geqslant \frac{c_{\alpha,d}}{2\vartheta} \left(u\{|x| < x_\vartheta : \frac{c_{\alpha,d}}{\vartheta}(1 - |x|^{\vartheta\alpha/d}) > \frac{c_{\alpha,d}}{2\vartheta}\} \right)^{\frac{1}{q}} \\
&= \frac{c_{\alpha,d}}{2\vartheta} u(B(0,x_\vartheta))^{\frac{1}{q}},
\end{aligned}$$

where taking $x_\vartheta = (\frac{1}{2})^{d/\alpha\vartheta}$ in the last step. It now follows that for $0 < \vartheta < 1$,

$$\|\mathscr{A}^{\mathscr{S}}_{\alpha,1}(u^{\frac{\alpha}{d}} f)\|_{L^{q,\infty}(u)} \gtrsim \frac{1}{\vartheta} \left(\frac{x_\vartheta}{\vartheta} \right)^{\frac{1}{q}} \simeq \vartheta^{-1-\frac{1}{q}}. \tag{37}$$

Finally, combining (34), (35), (37), and using that $\frac{1}{q} + \frac{\alpha}{d} = \frac{1}{p}$, we have that (33) holds for $\Phi(t) \geqslant ct^{1-\frac{\alpha}{d}}$, which gives the desired bound by the monotonicity of Φ. $\square$

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