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# LOCAL SHARP MAXIMAL FUNCTIONS, GEOMETRICAL MAXIMAL FUNCTIONS AND ROUGH MAXIMAL FUNCTIONS ON LOCAL MORREY SPACES WITH VARIABLE EXPONENTS

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*Abstract.* We study the local Morrey spaces with variable exponents. We show that the local block space with variable exponents are pre-duals of the local Morrey spaces with variable exponents. Using this duality, we establish the extrapolation theory for the local Morrey spaces with variable exponents. The extrapolation theory gives the mapping properties for the local sharp maximal functions, the geometric maximal functions and the rough maximal function on the local Morrey spaces with variable exponents.

## 1. Introduction

The main theme of this paper is the mapping properties local sharp maximal functions, geometrical maximal functions and rough maximal functions on local Morrey spaces with variable exponents. We obtain these results by using the duality and the extrapolation theory for local Morrey spaces with variable exponents.

The local Morrey space is an extension of the classical Morrey space introduced by Morrey in [35]. One of the remarkable features of the local Morrey spaces is that the real interpolation of local Morrey spaces can be explicitly identified [7]. In contrast to the real interpolation of the local Morrey space, the complete description of the real interpolation of the classical Morrey spaces is yet to be found [1].

Other than the interpolation of the local Morrey spaces, a number of important results from harmonic analysis had been extended to the local Morrey spaces. The boundedness of the Hardy-Littlewood maximal operator on the local Morrey spaces was given in [2]. The mapping properties of the singular integral operators were established in [4]. For the mapping properties of the Riesz potential and the fractional maximal operator on the local Morrey spaces, the reader may consult [3, 5, 6, 8]. For the studies of weighted local Morrey spaces, the reader is referred to [39]. In addition, the Stein-Weiss inequalities for the radial functions in the local Morrey spaces are obtained in [28].

We generalized local Morrey spaces to the setting of variable exponents. The local Morrey spaces with variable exponents are extensions of the local Morrey spaces and

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Lebesgue spaces with variable exponents. The reader is referred to [12, 13, 14] for the recent developments and applications of the Lebesgue spaces with variable exponents.

In this paper, we aim to obtain the extrapolation theory for the local Morrey spaces with variable exponents. The extrapolation theory was originated from Rubio de Francia [42, 43, 44]. Recently, it has been generalized to the Lebesgue spaces with variable exponent, Morrey spaces, mixed norm spaces, weighted Hardy spaces with variable exponents, Herz spaces with variable exponents and ball-Banach function spaces [10, 11, 20, 21, 23, 25, 26, 29, 30, 48]. One of the main results in this paper is the extrapolation theory for the local Morrey spaces with variable exponents.

To obtain the extrapolation theory for the local Morrey spaces with variable exponents, we need to identify the pre-dual spaces of the local Morrey spaces with variable exponents. We study the local block spaces with variable exponents. We find that the dual space of the local block space with variable exponent is the local Morrey space with variable exponent. We also obtain the boundedness of the Hardy-Littlewood maximal operators on the local block spaces with variable exponents. With these results, we extend the extrapolation theory to the local Morrey spaces with variable exponents. As applications of this result, we obtain the mapping properties of the rough maximal function, the local sharp maximal function and the geometric maximal operator on local Morrey spaces with variable exponents.

This paper is organized as follows. The definitions of the Lebesgue spaces with variable exponents and the local Morrey spaces with variable exponents are given in Section 2. The local block spaces with variable exponents are introduced in Section 3. The duality and the boundedness of the Hardy-Littlewood maximal operators on local block spaces with variable exponents are also obtained in this section. The main results for local Morrey spaces with variable exponents on the mapping properties of the rough maximal function, the local sharp maximal function, and the geometric maximal operator are presented in Section 4.

## 2. Definitions

Let $\mathscr{M}$ and $L^1_{\mathrm{loc}}$ denote the space of Lebesgue measurable functions and the space of locally integrable functions on $\mathbb{R}^n$, respectively.

For any $x \in \mathbb{R}^n$ and $r > 0$, define $B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$ and $\mathbb{B} = \{B(x,r) : x \in \mathbb{R}^n, r > 0\}$.

We briefly recall the definition of Lebesgue spaces with variable exponents and the class of globally log-Hölder continuous functions in the following.

DEFINITION 1. Let $p(\cdot) : \mathbb{R}^n \to [1,\infty]$ be a Lebesgue measurable function. The Lebesgue space with variable exponent $L^{p(\cdot)}$ consists of all Lebesgue measurable functions $f : \mathbb{R}^n \to \mathbb{C}$ satisfying

$$\|f\|_{L^{p(\cdot)}} = \inf\{\lambda > 0 : \rho_{p(\cdot)}(f/\lambda) \leqslant 1\} < \infty$$

where $\mathbb{R}^n_\infty = \{x \in \mathbb{R}^n : p(x) = \infty\}$ and

$$\rho_{p(\cdot)}(f) = \int_{\mathbb{R}^n \setminus \mathbb{R}^n_\infty} |f(x)|^{p(x)} dx + \operatorname*{ess\,sup}_{\mathbb{R}^n_\infty} |f(x)|.$$

We call $p(x)$ the exponent function of $L^{p(\cdot)}$.

In view of [14, Theorem 3.2.13], $L^{p(\cdot)}$ is a Banach function space. In view of the definition of Banach function spaces, we have

$$\chi_B \in L^{p(\cdot)}, \quad \forall B \in \mathbb{B}. \tag{1}$$

For any Lebesgue measurable function $p(x) : \mathbb{R}^n \to [1,\infty]$, define $p_- = \operatorname{essinf}_{x\in\mathbb{R}^n} p(x)$ and $p_+ = \operatorname{esssup}_{x\in\mathbb{R}^n} p(x)$.

The associate space of $L^{p(\cdot)}$ is given in [14, Theorem 3.2.13].

THEOREM 1. *If $1 < p(x) < \infty$, then the associate space of $L^{p(\cdot)}$ is $L^{p'(\cdot)}(\mathbb{R}^n)$ where $p'$ satisfies $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.*

We call $p'(x)$ the conjugate function of $p(x)$. Whenever $\sup_{x\in\mathbb{R}^n} p(x) < \infty$, the dual space of $L^{p(\cdot)}$ is the associate space of $L^{p(\cdot)}$, see [14, Theorem 3.4.6].

DEFINITION 2. Let $p(\cdot) : \mathbb{R}^n \to [1,\infty]$ be a Lebesgue measurable function. We write $p(\cdot) \in \mathscr{B}$ if the Hardy-Littlewood maximal operator

$$\mathrm{M}f(x) = \sup_{B\ni x} \frac{1}{|B|} \int_B |f(y)|dy$$

where the supremum is taken over all $B \in \mathbb{B}$ containing $x$, is bounded on $L^{p(\cdot)}$. We write $p(\cdot) \in \mathscr{B}'$ if $p'(\cdot) \in \mathscr{B}$.

The following gives the conditions on the exponent functions of $L^{p(\cdot)}$ that guarantee the boundedness of the Hardy-Littlewood maximal operator on $L^{p(\cdot)}$.

DEFINITION 3. A continuous function $g$ on $\mathbb{R}^n$ is locally log-Hölder continuous if there exists $c_{\log} > 0$ such that

$$|g(x) - g(y)| \leqslant \frac{c_{\log}}{\log(e + 1/|x-y|)}, \quad \forall x, y \in \mathbb{R}^n. \tag{2}$$

We denote the class of locally log-Hölder continuous function by $C^{\log}_{\mathrm{loc}}(\mathbb{R}^n)$.

Furthermore, a continuous function is globally log-Hölder continuous if $g \in C^{\log}_{\mathrm{loc}}(\mathbb{R}^n)$ and there exists $g_\infty \in \mathbb{R}$ so that

$$|g(x) - g_\infty| \leqslant \frac{c_{\log}}{\log(e + |x|)}, \quad \forall x \in \mathbb{R}^n. \tag{3}$$

The class of globally log-Hölder continuous function is denoted by $C^{\log}(\mathbb{R}^n)$.

Whenever $p(\cdot) \in C^{\log}(\mathbb{R}^n)$, the Hardy-Littlewood maximal operator is bounded on $L^{p(\cdot)}$.

THEOREM 2. *If $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ and $1 < p_-$, then $p(\cdot) \in \mathscr{B}$.*

The reader is referred to [10, 13, 40] and [14, Theorem 4.3.8] for the proof of the preceding theorem.

We now give the definition of local Morrey spaces with variable exponents.

DEFINITION 4. Let $p(\cdot):\mathbb{R}^n \to (1,\infty)$ and $u:(0,\infty)\to(0,\infty)$ be Lebesgue measurable functions. The local Morrey space with variable exponent $LM_u^{p(\cdot)}$ consists of all $f\in\mathscr{M}$ satisfying

$$\|f\|_{LM_u^{p(\cdot)}} = \sup_{r>0}\frac{1}{u(r)}\|\chi_{B(0,r)}f\|_{L^{p(\cdot)}} < \infty.$$

When $p(\cdot)=p$, $1\leqslant p<\infty$, the local Morrey space with variable exponent becomes the local Morrey space $LM_u^p$. The local Morrey space with variable exponent is also a generalization of the generalized Morrey spaces in [36]. For the studies of local Morrey spaces, the reader is referred to [3, 5, 6, 8, 17].

We have a brief discussion on the conditions satisfied by $u$ so that $LM_u^{p(\cdot)}$ is nontrivial. For any $f\in LM_u^{p(\cdot)}$, we have $\|\chi_{B(0,r)}f\|_{L^{p(\cdot)}}\leqslant \|f\|_{LM_u^{p(\cdot)}}u(r)$, $\forall r>0$.

If $\inf_{r\geqslant a}u(r)=0$ for all $a>0$, then $\lim_{r\to\infty}\|\chi_{B(0,r)}f\|_{L^{p(\cdot)}}=0$, that is, $f=0$ a.e. Thus, we can assume that $\inf_{r\geqslant a}u(r)>0$ for some $a>0$. Let

$$U(r)=\inf_{s\geqslant r}u(s),\quad r>0.$$

For any $f\in LM_U^{p(\cdot)}$,

$$\|\chi_{B(0,r)}f\|_{L^{p(\cdot)}}\leqslant \|f\|_{LM_U^{p(\cdot)}}U(r)\leqslant \|f\|_{LM_U^{p(\cdot)}}u(r),\quad r>0.$$

Thus, $LM_U^{p(\cdot)}\hookrightarrow LM_u^{p(\cdot)}$.

For any $f\in LM_u^{p(\cdot)}$, we have

$$\begin{aligned}\|\chi_{B(0,r)}f\|_{L^{p(\cdot)}} &= \inf_{s\geqslant r}\|\chi_{B(0,s)}f\|_{L^{p(\cdot)}}\\ &\leqslant \|f\|_{LM_u^{p(\cdot)}}\inf_{s\geqslant r}u(s)=\|f\|_{LM_u^{p(\cdot)}}U(r).\end{aligned}$$

That is, $LM_u^{p(\cdot)}\hookrightarrow LM_U^{p(\cdot)}$ and, hence, $LM_U^{p(\cdot)}=LM_u^{p(\cdot)}$. Therefore, we can assume that $u$ is increasing in the rest of this paper. The reader is also referred to [37, p.446], [45, Proposition 3.16] and [46, (1.2)] for more results on the conditions imposed on $u$ so that $LM_u^{p(\cdot)}$ is nontrivial.

We now define the class of weight functions used in this paper.

DEFINITION 5. Let $q_0\in(0,\infty)$, $p(\cdot):\mathbb{R}^n\to[1,\infty]$. We say that a Lebesgue measurable function, $u(r):(0,\infty)\to(0,\infty)$, belongs to $\mathbb{LW}^{q_0}_{p(\cdot)}$ if there exists a constant

$C > 0$ such that for any $r > 0$, $u$ fulfills

$$C \leqslant u(r), \quad \forall r \geqslant 1, \tag{4}$$

$$\|\chi_{B(0,r)}\|_{L^{p(\cdot)}} \leqslant Cu(r), \quad \forall r < 1, \tag{5}$$

$$\sum_{j=0}^{\infty} \frac{\|\chi_{B(0,r)}\|_{L^{p(\cdot)/q_0}}}{\|\chi_{B(0,2^{j+1}r)}\|_{L^{p(\cdot)/q_0}}} (u(2^{j+1}r))^{q_0} < C(u(r))^{q_0} \tag{6}$$

for all $r > 0$.

When $q_0 = 1$, we write $\mathbb{LW}_{p(\cdot)} = \mathbb{LW}^1_{p(\cdot)}$.

Roughly speaking, (4)-(5) are used to guarantee that $LM_u^{p(\cdot)}$ is non-trivial and (6) is related to the boundedness of the Hardy-Littlewood maximal operator on the pre-dual of $LM_u^{p(\cdot)}$. There are some equivalent relations between $\mathbb{LW}^{q_0}_{p(\cdot)}$ with different $q_0$, the reader is referred to [38, Proposition 2.7] for details.

We now use (4)-(5) to show that $\chi_B \in LM_u^{p(\cdot)}$, $B \in \mathbb{B}$.

PROPOSITION 1. *Let $p(\cdot) : \mathbb{R}^n \to [1,\infty]$. If $u$ is increasing and satisfies (4) and (5), then for any $B \in \mathbb{B}$, $\chi_B \in LM_u^{p(\cdot)}$.*

*Proof.* It suffices to show that for any $s > 0$, $\chi_{B(0,s)} \in LM_u^{p(\cdot)}$. Let $B = B(0,r)$, $r > 0$. When $r \geqslant 1$, (4) assures that

$$\frac{1}{u(r)} \|\chi_{B(0,r)}\chi_{B(0,s)}\|_{L^{p(\cdot)}} \leqslant \frac{1}{u(r)} \|\chi_{B(0,s)}\|_{L^{p(\cdot)}} \leqslant C\|\chi_{B(0,s)}\|_{L^{p(\cdot)}} \tag{7}$$

for some $C > 0$. When $r < 1$, (5) guarantees that

$$\frac{1}{u(r)} \|\chi_{B(0,s)}\chi_{B(0,r)}\|_{L^{p(\cdot)}} \leqslant \frac{1}{u(r)} \|\chi_{B(0,r)}\|_{L^{p(\cdot)}} \leqslant C. \tag{8}$$

Consequently, (7) and (8) yield

$$\|\chi_{B(0,s)}\|_{LM_u^{p(\cdot)}} = \sup_{r>0} \frac{1}{u(r)} \|\chi_{B(0,s)}\chi_{B(0,r)}\|_{L^{p(\cdot)}} < C + C\|\chi_{B(0,s)}\|_{L^{p(\cdot)}}.$$

Therefore, (1) guarantees that $\chi_B \in LM_u^{p(\cdot)}$.

The above proposition shows that $LM_u^{p(\cdot)}$ is nontrivial whenever $u$ satisfies (4) and (5). We now go to show that $\mathbb{LW}_{p(\cdot)} \neq \emptyset$. We use the ideas from [22, 24] to obtain the following results. For the corresponding results of Proposition 1 and the non-emptiness of the class of weight functions for Morrey spaces built on Banach function space, the reader is referred to [22, 24].

Let $0 \leqslant s < 1$, $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ and $w_s(r) = \|\chi_{B(0,r)}\|^s_{L^{p(\cdot)}}$. According to [19, Proposition 2.5 and Lemma 6.3], for any $p > p_+$, there is a constant $C > 0$ such that for any $r > 0$ and $j \in \mathbb{N}$,

$$\frac{\|\chi_{B(0,r)}\|_{L^{p(\cdot)}}}{\|\chi_{B(0,2^j r)}\|_{L^{p(\cdot)}}} \leqslant C\left(\frac{|B(0,r)|}{|B(0,2^j r)|}\right)^{1/p} = C\frac{\|\chi_{B(0,r)}\|_{L^p}}{\|\chi_{B(0,2^j r)}\|_{L^p}}. \tag{9}$$

For any $p > p_+$, (9) offers a constant $C > 0$ such that

$$\sum_{j=0}^{\infty} \frac{\|\chi_{B(0,r)}\|_{L^{p(\cdot)}}}{\|\chi_{B(0,2^{j+1}r)}\|_{L^{p(\cdot)}}} \frac{w_s(2^{j+1}r)}{w_s(r)} = \sum_{j=0}^{\infty} \left( \frac{\|\chi_{B(0,r)}\|_{L^{p(\cdot)}}}{\|\chi_{B(0,2^{j+1}r)}\|_{L^{p(\cdot)}}} \right)^{1-s}$$
$$\leqslant C \sum_{j=0}^{\infty} 2^{-jn(1-s)/p} \leqslant C. \tag{10}$$

That is, $w_s$ fulfills (6) with $q_0 = 1$.

In view of [14, Corollary 4.5.9], we obtain

$$\|\chi_{B(0,r)}\|_{L^{p(\cdot)}} \approx \begin{cases} |B(0,r)|^{\frac{1}{p(0)}}, & |B(0,r)| \leqslant 2^n \\ |B(0,r)|^{\frac{1}{p_\infty}}, & |B(0,r)| \geqslant 1, \end{cases} \tag{11}$$

where $p_\infty = \lim_{x\to\infty} p(x)$ and the existence of this limit is guaranteed by the definition of log-Hölder continuous functions.

Hence, (11) yields that when $r \geqslant 1$, we have

$$w_s(r) = \|\chi_{B(0,r)}\|^s_{L^{p(\cdot)}} \geqslant C|B(0,r)|^{\frac{s}{p_+}} > C$$

for some $C > 0$. Consequently, (4) is fulfilled.

In addition, (11) gives a constant $K > 0$ such that $\|\chi_{B(0,r)}\|_{L^{p(\cdot)}} \leqslant K$ for all $r < 1$. Hence, there exists a $C > 0$ such that

$$w_s(r) = \|\chi_{B(0,r)}\|^s_{L^{p(\cdot)}} \geqslant C\|\chi_{B(0,r)}\|_{L^{p(\cdot)}}, \quad 0 < r < 1.$$

Therefore, (5) is fulfilled and $w_s \in \mathbb{LW}_{L^{p(\cdot)}}$.

## 3. Pre-dual of local Morrey spaces with variable exponents

In this section, we study the local block space with variable exponents. We find that this is a pre-dual of the local Morrey space with variable exponent. Furthermore, the Hardy-Littlewood maximal operator is also bounded on the local block space with variable exponent. For the studies on the predual of local Morrey spaces, the reader is referred to [34, 49].

DEFINITION 6. Let $p(\cdot) : \mathbb{R}^n \to (0,\infty)$ and $u(r) : (0,\infty) \to (0,\infty)$ be Lebesgue measurable functions. A $b \in \mathscr{M}$ is a local $(u, L^{p(\cdot)})$-block if it is supported in $B(0,r)$, $r > 0$, and

$$\|b\|_{L^{p(\cdot)}} \leqslant \frac{1}{u(r)}. \tag{12}$$

Define $\mathfrak{LB}_{u,p(\cdot)}$ by

$$\mathfrak{LB}_{u,p(\cdot)} = \left\{ \sum_{k=1}^{\infty} \lambda_k b_k : \sum_{k=1}^{\infty} |\lambda_k| < \infty \text{ and } b_k \text{ is a local } (u, L^{p(\cdot)})\text{-block} \right\}. \tag{13}$$

The space $\mathfrak{LB}_{u,p(\cdot)}$ is endowed with the norm

$$\|f\|_{\mathfrak{LB}_{u,p(\cdot)}} = \inf\left\{ \sum_{k=1}^{\infty} |\lambda_k| \text{ such that } f = \sum_{k=1}^{\infty} \lambda_k b_k \text{ a.e.} \right\}. \tag{14}$$

We call $\mathfrak{LB}_{u,p(\cdot)}$ the local block space with variable exponent.

We now have the first main result of this paper. The following result shows that $\mathfrak{LB}_{u,p(\cdot)}$ is a pre-dual of the local Morrey space. It also guarantees that $LM_u^{p(\cdot)}$ is a Banach space.

THEOREM 3. *Let $p(\cdot) : \mathbb{R}^n \to (1,\infty)$ and $u : (0,\infty) \to (0,\infty)$ be Lebesgue measurable functions. We have*

$$\mathfrak{LB}^*_{u,p(\cdot)} = LM_u^{p'(\cdot)}$$

*where $\mathfrak{LB}^*_{u,p(\cdot)}$ denotes the dual space of $\mathfrak{LB}_{u,p(\cdot)}$.*

*Proof.* Let $b$ be a local $(u, L^{p(\cdot)})$-block supported in $B(0,r)$. For any $f \in LM_u^{p'(\cdot)}$, the Hölder inequality for $L^{p(\cdot)}$ yields

$$\begin{aligned}\int_{\mathbb{R}^n} |f(x)b(x)|dx &\leqslant C\|\chi_{B(0,r)}f\|_{L^{p'(\cdot)}}\|\chi_{B(0,r)}b\|_{L^{p(\cdot)}} \\ &\leqslant C\frac{1}{u(r)}\|\chi_{B(0,r)}f\|_{L^{p'(\cdot)}}\end{aligned}$$

for some $C > 0$.

Consequently, for any $g = \sum_{k\in\mathbb{N}} \lambda_k b_k \in \mathfrak{LB}_{u,p(\cdot)}$, we obtain

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \leqslant \sum_{k=1}^{\infty} |\lambda_k| \int_{\mathbb{R}^n} |f(x)b_k(x)|dx \leqslant C\|g\|_{\mathfrak{LB}_{u,p(\cdot)}}\|f\|_{LM_u^{p'(\cdot)}} \tag{15}$$

for some $C > 0$. Thus, $LM_u^{p'(\cdot)} \hookrightarrow \mathfrak{LB}^*_{u,p(\cdot)}$.

It remains prove the reverse embedding. For any $r > 0$ and $L \in \mathfrak{LB}^*_{u,p(\cdot)}$, define $X = \{g\chi_{B(0,r)} : g \in L^{p(\cdot)}\}$. Obviously, $X$ is a subspace of $L^{p(\cdot)}$. Define the linear functional $l : X \to \mathbb{C}$ by

$$l(h) = L(\chi_{B(0,r)}g)$$

where $h = \chi_{B(0,r)}g \in X$ and $g \in L^{p(\cdot)}$.

For any $r > 0$,

$$G = \frac{1}{\|g\chi_{B(0,r)}\|_{L^{p(\cdot)}}u(r)}g\chi_{B(0,r)}$$

is a local $(u, L^{p(\cdot)})$-block. According to (14), for any local $(u, L^{p(\cdot)})$-block $b$, we have

$$\|b\|_{\mathfrak{LB}_{u,p(\cdot)}} \leqslant 1. \tag{16}$$

Therefore, (16) yields $\|G\|_{\mathfrak{LB}_{u,p(\cdot)}} \leqslant 1$. That is,

$$\|g\chi_{B(0,r)}\|_{\mathfrak{LB}_{u,p(\cdot)}} \leqslant \|g\chi_{B(0,r)}\|_{L^{p(\cdot)}} u(r). \tag{17}$$

In view of (17) and $L \in \mathfrak{LB}^*_{u,p(\cdot)}$, we find that

$$\begin{aligned}|l(h)| = |L(g\chi_{B(0,r)})| &\leqslant C\|g\chi_{B(0,r)}\|_{\mathfrak{LB}_{u,p(\cdot)}} \\ &\leqslant K\|g\chi_{B(0,r)}\|_{L^{p(\cdot)}} = K\|h\|_{L^{p(\cdot)}}\end{aligned}$$

for some $K > 0$. Therefore, $l$ is bounded on $X$. The Hahn-Banach theorem assures that $l$ can be extended to be a member of $(L^{p(\cdot)})^*$. The duality $(L^{p(\cdot)})^* = L^{p'(\cdot)}$ yields a $f_r \in L^{p'(\cdot)}$ such that

$$l(g) = \int_{\mathbb{R}^n} f_r(x)g(x)dx, \quad \forall g \in L^{p(\cdot)}$$

and we can assume that $\operatorname{supp} f_r \subseteq B(0,r)$.

Let $r,s > 0$. For any $B \in \mathbb{B}$ with $B \subseteq B(0,r) \cap B(0,s)$,

$$\int_B f_r(x)dx = l(\chi_B) = \int_B f_s(x)dx.$$

That is, $f_r = f_s$ almost everywhere on $B(0,r) \cap B(0,s)$. Therefore, there is an unique Lebesgue measurable function $f$ such that $f(x) = f_r(x)$ on $B(0,r)$ for all $r$.

Next, we show that $f \in LM_u^{p'(\cdot)}$. For any $h \in L^{p(\cdot)}$ and $B(0,r)$,

$$H = \frac{\chi_{B(0,r)}h}{\|\chi_{B(0,r)}h\|_{L^{p(\cdot)}} u(r)} \tag{18}$$

is a local $(u, L^{p(\cdot)})$-block. In view of (16),

$$\|H\|_{\mathfrak{LB}_{u,p(\cdot)}} \leqslant 1.$$

That is, $\|\chi_{B(0,r)}h\|_{\mathfrak{LB}_{u,p(\cdot)}} \leqslant \|\chi_{B(0,r)}h\|_{L^{p(\cdot)}} u(r)$.

Since the function given in (18) is a local $(u, L^{p(\cdot)})$-block,

$$\begin{aligned}\frac{1}{u(r)}\|\chi_{B(x_0,r)}f\|_{L^{p'(\cdot)}} &= \frac{1}{u(r)} \sup_{\|h\|_{L^{p(\cdot)}}=1} \left| \int_{B(0,r)} f(x)h(x)dx \right| \\ &\leqslant \sup_{\|h\|_{L^{p(\cdot)}}=1} \left| \int_{B(0,r)} f_r(x) \frac{\chi_{B(0,r)}(x)h(x)}{u(r)} dx \right| \\ &\leqslant \|L\|_{\mathfrak{LB}^*_{u,p(\cdot)}} \sup_{\|h\|_{L^{p(\cdot)}}=1} \left\| \frac{h\chi_{B(0,r)}}{u(r)} \right\|_{\mathfrak{LB}_{u,p(\cdot)}} \leqslant \|L\|_{\mathfrak{LB}^*_{u,p(\cdot)}}.\end{aligned}$$

The functionals $L_f(g) = \int_{\mathbb{R}^n} f(x)g(x)dx$ and $L$ are identical on the set of local $(u, L^{p(\cdot)})$-blocks and according to Definition 6, the set of finite linear combinations of local $(u, L^{p(\cdot)})$-blocks is dense in $\mathfrak{LB}_{u,p(\cdot)}$, therefore $L_f = L$ and $\mathfrak{LB}^*_{u,p(\cdot)} \hookrightarrow LM_u^{p'(\cdot)}$.

The following proposition is used to show the boundedness of the Hardy-Littlewood maximal operator on $\mathfrak{LB}_{u,p(\cdot)}$.

PROPOSITION 2. *Let* $p(\cdot):\mathbb{R}^n\to(1,\infty)$, $u:(0,\infty)\to(0,\infty)$ *be Lebesgue measurable functions and* $f\in\mathfrak{LB}_{u,p(\cdot)}$. *If* $g\in\mathscr{M}$ *satisfying* $|g|\leqslant|f|$, *then* $g\in\mathfrak{LB}_{u,p(\cdot)}$.

*Proof.* As $f\in\mathfrak{LB}_{u,p(\cdot)}$, for any $\varepsilon>0$, we have a family of local $(u,L^{p(\cdot)})$-blocks $\{b_i\}_{i=1}^{\infty}$ and a family of scalars $\{\lambda_i\}_{i=1}^{\infty}$ such that

$$f=\sum_{i=1}^{\infty}\lambda_i b_i$$

and $\sum_{i=1}^{\infty}|\lambda_i|\leqslant(1+\varepsilon)\|f\|_{\mathfrak{LB}_{u,p(\cdot)}}$. We find that $g=\sum_{i=1}^{\infty}\lambda_i c_i$ where

$$c_i(x)=\begin{cases}\dfrac{g(x)}{f(x)}b_i(x), & f(x)\neq 0,\\ 0, & f(x)=0.\end{cases}$$

As $|g|\leqslant|f|$, $\{c_i\}_{i=1}^{\infty}$ are local $(u,L^{p(\cdot)})$-blocks. Thus, $g\in\mathfrak{LB}_{u,p(\cdot)}$. As $\varepsilon$ is arbitrary, we also establish $\|g\|_{\mathfrak{LB}_{u,p(\cdot)}}\leqslant\|f\|_{\mathfrak{LB}_{u,p(\cdot)}}$.

The next theorem gives the conditions for which $\mathfrak{LB}_{u,p(\cdot)}\subset L^1_{\mathrm{loc}}$ and $\mathfrak{LB}_{u,p(\cdot)}$ is a Banach space. We use the ideas from the proof of [27, Theorem 3.1] to establish the following result.

THEOREM 4. *Let* $p(\cdot):\mathbb{R}^n\to(1,\infty)$ *and* $u:(0,\infty)\to(0,\infty)$ *be Lebesgue measurable functions. If there exists a constant* $C>0$ *such that for any* $r>0$, $u$ *fulfills*

$$C\leqslant u(r),\quad \forall r\geqslant 1, \tag{19}$$

$$\|\chi_{B(0,r)}\|_{L^{p'(\cdot)}}\leqslant Cu(r),\quad \forall r<1, \tag{20}$$

*then* $\mathfrak{LB}_{u,p(\cdot)}\subset L^1_{\mathrm{loc}}$ *and* $\mathfrak{LB}_{u,p(\cdot)}$ *is a Banach space.*

*Proof.* According to Proposition 1, $\chi_B\in LM_u^{p'(\cdot)}$, $\forall B\in\mathbb{B}$. Theorem 3 assures that $\chi_B\in\mathfrak{LB}^*_{u,p(\cdot)}$. Therefore, for any $f\in\mathfrak{LB}_{u,p(\cdot)}$, (15) gives

$$\int_B|f(x)|dx\leqslant C\|\chi_B\|_{LM_u^{p'(\cdot)}}\|f\|_{\mathfrak{LB}_{u,p(\cdot)}}. \tag{21}$$

Hence, $\mathfrak{LB}_{u,p(\cdot)}\hookrightarrow L^1_{\mathrm{loc}}$.

Next, we show that $\mathfrak{LB}_{u,p(\cdot)}$ is a Banach space. Let $f_i\in\mathfrak{LB}_{u,p(\cdot)}$, where $i\in\mathbb{N}$, satisfying

$$\sum_{i=1}^{\infty}\|f_i\|_{\mathfrak{LB}_{u,p(\cdot)}}<\infty.$$

In view of (21), for any $B\in\mathbb{B}$,

$$\int_B\sum_{i=1}^{\infty}|f_i(x)|dx\leqslant C\|\chi_B\|_{LM_u^{p'(\cdot)}}\left(\sum_{i=1}^{\infty}\|f_i\|_{\mathfrak{LB}_{u,p(\cdot)}}\right).$$

Therefore, $f=\sum_{i=1}^{\infty} f_i$ is a well defined Lebesgue measurable function and $f\in L^1_{\mathrm{loc}}$.

We now show that $f=\sum_{i=1}^{\infty} f_i$ belongs to $\mathfrak{LB}_{u,p(\cdot)}$. For any $\varepsilon>0$, there exists a $N\in\mathbb{N}$ such that for any $n>N$,

$$\sum_{i=n}^{\infty}\|f_i\|_{\mathfrak{LB}_{u,p(\cdot)}}<\varepsilon. \tag{22}$$

According to the definition of $\mathfrak{LB}_{u,p(\cdot)}$, for any $\varepsilon>0$,

$$f_i=\sum_{k=1}^{\infty}\lambda_{k,i}b_{k,i}$$

where $b_{k,i}$, $i,k\in\mathbb{N}$ are local $(u,L^{p(\cdot)})$-blocks and

$$\sum_{k=1}^{\infty}|\lambda_{k,i}|\leqslant(1+\varepsilon)\|f_i\|_{\mathfrak{LB}_{u,p(\cdot)}}.$$

Moreover, for any $1\leqslant i\leqslant n$, there exists a $N_i\in\mathbb{N}$ such that

$$\left\|f_i-\sum_{k=1}^{N_i}\lambda_{k,i}b_{k,i}\right\|_{\mathfrak{LB}_{u,p(\cdot)}}\leqslant\sum_{k=N_i+1}^{\infty}|\lambda_{k,i}|<2^{-i}\varepsilon. \tag{23}$$

Therefore, for any $B\in\mathbb{B}$,

$$\begin{aligned}
&\int_B\left|f(x)-\sum_{i=1}^{N}\sum_{k=1}^{N_i}\lambda_{k,i}b_{k,i}(x)\right|dx\\
&\leqslant\int_B\left|f(x)-\sum_{i=1}^{N}f_i(x)\right|dx+\int_B\left|\sum_{i=1}^{N}f_i(x)-\sum_{i=1}^{N}\sum_{k=1}^{N_i}\lambda_{k,i}b_{k,i}(x)\right|dx\\
&\leqslant\int_B\sum_{i=N+1}^{\infty}|f_i(x)|dx+\sum_{i=1}^{N}\int_B\left|f_i(x)-\sum_{k=1}^{N_i}\lambda_{k,i}b_{k,i}(x)\right|dx.
\end{aligned}$$

By using (21), (22) and (23),

$$\begin{aligned}
&\int_B\left|f(x)-\sum_{i=1}^{N}\sum_{k=1}^{N_i}\lambda_{k,i}b_{k,i}(x)\right|dx\\
&\leqslant C\|\chi_B\|_{LM_u^{p'(\cdot)}}\left(\sum_{i=N+1}^{\infty}\|f_i\|_{\mathfrak{LB}_{u,p(\cdot)}}+\sum_{i=1}^{N}\left\|f_i-\sum_{k=1}^{N_i}\lambda_{k,i}b_{k,i}\right\|_{\mathfrak{LB}_{u,p(\cdot)}}\right)\\
&\leqslant C\|\chi_B\|_{LM_u^{p'(\cdot)}}\left(\varepsilon+\sum_{i=1}^{N}2^{-i}\varepsilon\right)<2C\|\chi_B\|_{LM_u^{p'(\cdot)}}\varepsilon.
\end{aligned}$$

Consequently,

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\lambda_{k,i}b_{k,i}$$

converges to $f$ in $L^1_{\mathrm{loc}}$. Hence, $\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\lambda_{k,i}b_{k,i}$ converges to $f$ locally in measure. Therefore, a subsequence of $\{\sum_{i=1}^{N}\sum_{k=1}^{M}\lambda_{k,i}b_{k,i}\}_{N,M}$ converges to $f$ a.e. Furthermore, $\lambda_{k,i}$, where $i,k\in\mathbb{N}$, satisfies

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}|\lambda_{k,i}|\leqslant(1+\varepsilon)\sum_{i=1}^{\infty}\|f_i\|_{\mathfrak{LB}_{u,p(\cdot)}}<\infty.$$

That is, $\sum_{i=1}^{\infty}f_i$ converges to $f$ in $\mathfrak{LB}_{u,p(\cdot)}$. Since $\varepsilon>0$ is arbitrary,

$$\Big\|\sum_{i=1}^{\infty}f_i\Big\|_{\mathfrak{LB}_{u,p(\cdot)}}\leqslant\sum_{i=1}^{\infty}\|f_i\|_{\mathfrak{LB}_{u,p(\cdot)}}.$$

Therefore, $\mathfrak{LB}_{u,p(\cdot)}$ is a Banach space.

The following result presents the boundedness of the Hardy-Littlewood maximal operator on $\mathfrak{LB}_{u,p(\cdot)}$. It is used to obtain the extrapolation theory for $LM_u^{p(\cdot)}$. We use the ideas from the proof of [9, Theorem 3.1] to establish the following theorem.

THEOREM 5. *Let $p(\cdot):\mathbb{R}^n\to(1,\infty)$ and $u:(0,\infty)\to(0,\infty)$ be Lebesgue measurable functions. If $p(\cdot)\in\mathscr{B}$ and $u\in\mathbb{LW}_{p'(\cdot)}$, then the Hardy-Littlewood maximal operator* $\mathrm{M}$ *is bounded on $\mathfrak{LB}_{u,p(\cdot)}$.*

*Proof.* In view of Theorem 4, we have $\mathfrak{LB}_{u,p(\cdot)}\subset L^1_{loc}$, therefore the Hardy-Littlewood maximal operator is well defined on $\mathfrak{LB}_{u,p(\cdot)}$.

Let $b$ be a local $(u,L^{p(\cdot)})$-block with support $B(0,r)$, $r>0$. For any $k\in\mathbb{N}$, write $B_k=B(0,2^kr)$. Define $m_k=\chi_{B_{k+1}\backslash B_k}\mathrm{M}b$, $k\in\mathbb{N}\backslash\{0\}$ and $m_0=\chi_{B(0,2r)}\mathrm{M}b$. We have $\mathrm{supp}\, m_k\subseteq B_{k+1}\backslash B_k$ and $\mathrm{M}(b)=\sum_{k=0}^{\infty}m_k$.

As $p(\cdot)\in\mathscr{B}$, we obtain

$$\|m_0\|_{L^{p(\cdot)}}\leqslant C\|\mathrm{M}b\|_{L^{p(\cdot)}}\leqslant\frac{C}{u(r)}\leqslant\frac{C}{u(2r)}$$

for some constant $C>0$ independent $r$. We have the last inequality because (6) assures that $\dfrac{\|\chi_{B(0,r)}\|_{L^{p(\cdot)}}}{\|\chi_{B(0,2r)}\|_{L^{p(\cdot)}}}u(2r)\leqslant Cu(r)$ and [24, (2.2)] guarantees that $\|\chi_{B(0,2r)}\|_{L^{p(\cdot)}}\leqslant C\|\chi_{B(0,r)}\|_{L^{p(\cdot)}}$ for some $C>0$ independent of $r>0$.

Consequently, $m_0$ is a constant-multiple of a local $(u,L^{p(\cdot)})$-block.

The Hölder inequality for $L^{p(\cdot)}$ yields

$$\begin{aligned}m_k=\chi_{B_{k+1}\backslash B_k}\mathrm{M}b&\leqslant\frac{\chi_{B_{k+1}\backslash B_k}}{2^{kn}r^n}\int_{B(0,r)}|b(x)|dx\\&\leqslant C\chi_{B_{k+1}\backslash B_k}\frac{1}{2^{kn}r^n}\|b\|_{L^{p(\cdot)}}\|\chi_{B(0,r)}\|_{L^{p'(\cdot)}}\end{aligned}$$

for some $C>0$ independent of $k$.

[9, Proposition 3.1] asserts that

$$\begin{aligned}\|m_k\|_{L^{p(\cdot)}} &\leqslant C\frac{\|\chi_{B_{k+1}\setminus B_k}\|_{L^{p(\cdot)}}}{2^{kn}r^n}\|b\|_{L^{p(\cdot)}}\|\chi_{B(0,r)}\|_{L^{p'(\cdot)}}\\ &\leqslant C\frac{\|\chi_{B(0,r)}\|_{L^{p'(\cdot)}}}{\|\chi_{B_{k+1}}\|_{L^{p'(\cdot)}}}\frac{u(2^{k+1}r)}{u(r)}\frac{1}{u(2^{k+1}r)}.\end{aligned}$$

Define $m_k = \sigma_k b_k$ where

$$\sigma_k = \frac{\|\chi_{B(0,r)}\|_{L^{p'(\cdot)}}}{\|\chi_{B_{k+1}}\|_{L^{p'(\cdot)}}}\frac{u(2^{k+1}r)}{u(r)}.$$

We find that $b_k$ is a constant-multiple of a local $(u, L^{p(\cdot)})$-block and this constant does not depend on $k$. As $u \in \mathbb{LW}_{p'(\cdot)}$, we have

$$\sum_{j=0}^{\infty}\frac{\|\chi_{B(0,r)}\|_{L^{p'(\cdot)}}}{\|\chi_{B(0,2^{j+1}r)}\|_{L^{p'(\cdot)}}}u(2^{j+1}r) \leqslant Cu(r).$$

Consequently, we find that $\sum_{k=0}^{\infty}\sigma_k < C$ for some $C > 0$. Therefore, $\mathrm{M}b \in \mathfrak{LB}_{u,p(\cdot)}$. Furthermore, there exists a constant $C_0 > 0$ so that for any local $(u, L^{p(\cdot)})$-block $b$,

$$\|\mathrm{M}b\|_{\mathfrak{LB}_{u,p(\cdot)}} < C_0.$$

We now consider $f \in \mathfrak{LB}_{u,p(\cdot)}$. The definition of $\mathfrak{LB}_{u,p(\cdot)}$ yields a family of local $(u, L^{p(\cdot)})$-blocks $\{c_k\}_{k=1}^{\infty}$ and a sequence $\Lambda = \{\lambda_k\}_{k=1}^{\infty} \in l^1$ such that $f = \sum_{k=1}^{\infty}\lambda_k c_k$ with $\|\Lambda\|_{l^1} \leqslant 2\|f\|_{\mathfrak{LB}_{u,p(\cdot)}}$. We have

$$\begin{aligned}\left\|\sum_{k=1}^{\infty}\lambda_k \mathrm{M}c_k\right\|_{\mathfrak{LB}_{u,p(\cdot)}} &\leqslant \sum_{k=1}^{\infty}|\lambda_k|\|\mathrm{M}c_k\|_{\mathfrak{LB}_{u,p(\cdot)}}\\ &\leqslant C_0\sum_{k=1}^{\infty}|\lambda_k| \leqslant 2C_0\|f\|_{\mathfrak{LB}_{u,p(\cdot)}}.\end{aligned}$$

Since $\mathrm{M}f \leqslant \sum_{k=1}^{\infty}|\lambda_k|\mathrm{M}c_k$, Proposition 2 guarantees that $\mathrm{M}f \in \mathfrak{LB}_{u,p(\cdot)}$ and $\|\mathrm{M}f\|_{\mathfrak{LB}_{u,p(\cdot)}} \leqslant C\|f\|_{\mathfrak{LB}_{u,p(\cdot)}}$ for some $C > 0$.

## 4. Main results

In this section, we present the mapping properties of the local sharp maximal function, the geometric maximal function and the rough maximal function on $LM_u^{p(\cdot)}$.

In order to establish the main results, we first extend the extrapolation theory to local Morrey spaces with variable exponents, we recall the definition of the Muckenhoupt classes of weight functions.

DEFINITION 7. For $1<p<\infty$, a locally integrable function $\omega:\mathbb{R}^n\to[0,\infty)$ is said to be an $A_p$ weight if

$$[\omega]_{A_p}=\sup_{B\in\mathbb{B}}\left(\frac{1}{|B|}\int_B\omega(x)dx\right)\left(\frac{1}{|B|}\int_B\omega(x)^{-\frac{p'}{p}}dx\right)^{\frac{p}{p'}}<\infty$$

where $p'=\frac{p}{p-1}$. A locally integrable function $\omega:\mathbb{R}^n\to[0,\infty)$ is said to be an $A_1$ weight if

$$\frac{1}{|B|}\int_B\omega(y)dy\leqslant C\omega(x),\quad a.e. x\in B$$

for some constants $C>0$. The infimum of all such $C$ is denoted by $[\omega]_{A_1}$. We define $A_\infty=\cup_{p\geqslant1}A_p$.

Notice that we have $A_p\subseteq A_q$ whenever $1\leqslant p\leqslant q$.

By $\mathscr{F}$ we mean a family of pairs $(f,g)$ of non-negative, Lebesgue measurable functions that are not identically zero. Given such a family $\mathscr{F}$, $p>0$ and a weight $\omega\in A_q$, if we say that

$$\int_{\mathbb{R}^n}f(x)^p\omega(x)dx\leqslant C\int_{\mathbb{R}^n}g(x)^p\omega(x)dx,\quad(f,g)\in\mathscr{F},$$

then we mean that this inequality holds for all pairs $(f,g)\in\mathscr{F}$ such that the left-hand side is finite, and that the constant C depends only on $p$ and $[\omega]_{A_p}$.

The extrapolation theory for local Morrey spaces with variable exponents is given in the following theorem.

THEOREM 6. *Let $0<p_0<\infty$ and $p(\cdot):\mathbb{R}^n\to(0,\infty)$ be a Lebesgue measurable function. Let $f,g\in\mathscr{M}(\mathbb{R}^n)$. Suppose that for every $\omega\in A_1$, we have*

$$\int_{\mathbb{R}^n}f(x)^{p_0}\omega(x)dx\leqslant C\int_{\mathbb{R}^n}g(x)^{p_0}\omega(x)dx,\quad(f,g)\in\mathscr{F}\tag{24}$$

*where $C$ is independent of $f$, $g$ and $p_0$.*

*Suppose that there exists $q_0$ satisfying $q_0\leqslant\min(p_0,p_-)<\infty$ such that $p(\cdot)/q_0\in\mathscr{B}'$ and $u\in\mathbb{LW}^{q_0}_{p(\cdot)}$ is increasing.*

*Then*

$$\|f\|_{LM^{p(\cdot)}_u}\leqslant C\|g\|_{LM^{p(\cdot)}_u},\quad(f,g)\in\mathscr{F}.\tag{25}$$

*Moreover, for every $q$, $1<q<\infty$, and $(f_i,g_i)\in\mathscr{F}$, $i\in\mathbb{N}$, satisfying (24), we have*

$$\left\|\left(\sum_{i\in\mathbb{N}}|f_i|^q\right)^{1/q}\right\|_{LM^{p(\cdot)}_u}\leqslant C\left\|\left(\sum_{i\in\mathbb{N}}|g_i|^q\right)^{1/q}\right\|_{LM^{p(\cdot)}_u}\tag{26}$$

*for some $C>0$.*

*Proof.* Without loss of generality, we assume that $f$ is non-negative. We follow the Rubio de Francia iteration algorithm presented in [11].

We only give the proof of (25) as the proof of (26) follows similarly [11, Corollary 3.12].

As $u \in \mathbb{LW}^{q_0}_{p(\cdot)}$, we have $u^{q_0} \in \mathbb{LW}_{p(\cdot)/q_0}$, Theorem 5 guarantees that M is bounded on $\mathfrak{LB}_{u^{q_0},(p(\cdot)/q_0)'}$. For any nonnegative function $h$, define

$$\mathscr{R}h(x) = \sum_{k=0}^{\infty} \frac{\mathrm{M}^k h(x)}{2^k \|\mathrm{M}^k\|_{\mathfrak{LB}_{u^{q_0},(p(\cdot)/q_0)'}}}.$$

Since $\mathrm{M} : \mathfrak{LB}_{u^{q_0},(p(\cdot)/q_0)'} \to \mathfrak{LB}_{u^{q_0},(p(\cdot)/q_0)'}$ is bounded, $\|\mathrm{M}\|_{\mathfrak{LB}_{u^{q_0},(p(\cdot)/q_0)'}}$ is well defined.

The operator $\mathscr{R}$ fulfills

$$h(x) \leqslant \mathscr{R}h(x), \tag{27}$$

$$\|\mathscr{R}h\|_{\mathfrak{LB}_{u^{q_0},(p(\cdot)/q_0)'}} \leqslant 2\|h\|_{\mathfrak{LB}_{u^{q_0},(p(\cdot)/q_0)'}}, \tag{28}$$

$$[\mathscr{R}h]_{A_1} \leqslant 2\|\mathrm{M}\|_{\mathfrak{LB}_{u^{q_0},(p(\cdot)/q_0)'}}. \tag{29}$$

The proof of (27) is straight-forward. Since

$$\begin{aligned}
\mathrm{M}(\mathscr{R}h) &\leqslant \sum_{k=0}^{\infty} \frac{\mathrm{M}^{k+1} h}{2^k \|\mathrm{M}^k\|_{\mathfrak{LB}_{u^{q_0},(p(\cdot)/q_0)'}}} \\
&\leqslant 2\|\mathrm{M}\|_{\mathfrak{LB}_{u^{q_0},(p(\cdot)/q_0)'}} \sum_{k=1}^{\infty} \frac{\mathrm{M}^k h}{2^k \|\mathrm{M}^k\|_{\mathfrak{LB}_{u^{q_0},(p(\cdot)/q_0)'}}} \\
&\leqslant 2\|\mathrm{M}\|_{\mathfrak{LB}_{u^{q_0},(p(\cdot)/q_0)'}} \mathscr{R}h
\end{aligned}$$

the properties (28) and (29) are consequences of Theorem 5 and the definition of $A_1$.

By applying the standard extrapolation for Lebesgue spaces [11, Corollary 3.14], for any $\omega \in A_1$, we obtain

$$\int_{\mathbb{R}^n} f(x)^{q_0} \omega(x)dx \leqslant C \int_{\mathbb{R}^n} g(x)^{q_0} \omega(x)dx. \tag{30}$$

Lemma 3 guarantees

$$\begin{aligned}
\|f\|^{q_0}_{LM^{p(\cdot)}_u} &= \|f^{q_0}\|_{LM^{p(\cdot)/q_0}_{u^{q_0}}} \\
&\leqslant C \sup\left\{ \int_{\mathbb{R}^n} |f(x)^{q_0} h(x)|dx : \|h\|_{\mathfrak{LB}_{u^{q_0},(p(\cdot)/q_0)'}} \leqslant 1, h \geqslant 0 \right\}
\end{aligned} \tag{31}$$

for some $C > 0$.

(15) and (27) yield

$$\begin{aligned}
\int_{\mathbb{R}^n} f(x)^{q_0} h(x)dx &\leqslant C \int_{\mathbb{R}^n} f(x)^{q_0} \mathscr{R}h(x)dx \\
&\leqslant C\|f^{q_0}\|_{LM^{p(\cdot)/q_0}_{u^{q_0}}} \|h\|_{\mathfrak{LB}_{u^{q_0},(p(\cdot)/q_0)'}} < \infty.
\end{aligned}$$

Moreover, (29) asserts that $\mathscr{R}h \in A_1$. By applying $\omega = \mathscr{R}h$ on (30) and using (27), we find that

$$\int_{\mathbb{R}^n} f(x)^{q_0} h(x)dx \leqslant \int_{\mathbb{R}^n} f(x)^{q_0} \mathscr{R}h(x)dx \leqslant C \int_{\mathbb{R}^n} g(x)^{q_0} \mathscr{R}h(x)dx.$$

Consequently, (15) and (28) give

$$\begin{aligned}\int_{\mathbb{R}^n} f(x)^{q_0} h(x)dx &\leqslant C\|g^{q_0}\|_{LM^{p(\cdot)/q_0}_{u^{q_0}}} \|\mathscr{R}h\|_{\mathfrak{LB}_{u^{q_0},(p(\cdot)/q_0)'}} \\ &\leqslant C\|g\|_{LM^{p(\cdot)}_u} \|h\|_{\mathfrak{LB}_{u^{q_0},(p(\cdot)/q_0)'}} \leqslant C\|g\|^{q_0}_{LM^{p(\cdot)}_u}. \end{aligned} \tag{32}$$

By taking supremum over those $h \in \mathfrak{LB}_{u^{q_0},(p(\cdot)/q_0)'}$, Theorem 3, (31) and (32) yield (25).

We now present the applications of Theorem 6 to the Fefferman-Stein inequalities, the John-Nirenberg inequalities and the boundedness of geometric maximal operators on local Morrey spaces with variable exponents.

We first consider the sharp maximal function. The sharp maximal function for $f \in L^1_{loc}$ is defined as

$$\mathrm{M}^\sharp f(x) = \sup_{x \ni B} \frac{1}{|B|} \int_B |f(y) - f_B| dy$$

where the supremum is taken over all $B \in \mathbb{B}$ containing $x$ and $f_B = \frac{1}{|B|}\int_B f(y)dy$.

Let $L^\infty_c$ denote the set of bounded function with compact support. The Fefferman-Stein inequality [15] states that for any $0 < p < \infty$ and $\omega \in A_\infty$, we have

$$\int_{\mathbb{R}^n} (\mathrm{M}f(x))^p \omega(x)dx \leqslant C \int_{\mathbb{R}^n} (\mathrm{M}^\sharp f(x))^p \omega(x)dx, \quad f \in L^\infty_c. \tag{33}$$

The following Fefferman-Stein inequalities for $LM^{p(\cdot)}_u$ follow from Theorem 6 and the fact that the Hardy-Littlewood maximal operator is bounded on $L^{p(\cdot)}$ whenever $p(\cdot) \in \mathscr{B}$.

THEOREM 7. *Let* $1 < r < \infty$, $p(\cdot) \in \mathscr{B}$ *and* $u : (0,\infty) \to (0,\infty)$ *be a Lebesgue measurable function. If* $u \in \mathbb{LW}^{p_-}_{p(\cdot)}$ *is increasing, then there is a constant* $C > 0$ *such that for any* $\{f_i\}_{i=1}^\infty \subset L^\infty_c$, *we have*

$$\Big\|\Big(\sum_{i=1}^\infty (\mathrm{M}f_i)^r\Big)^{1/r}\Big\|_{LM^{p(\cdot)}_u} \leqslant C\Big\|\Big(\sum_{i=1}^\infty (\mathrm{M}^\sharp f_i)^r\Big)^{1/r}\Big\|_{LM^{p(\cdot)}_u}. \tag{34}$$

Notice that $L^\infty_c$ is not dense in $LM^{p(\cdot)}_u$ in general. For instance, $L^\infty_c$ is not dense in the local Morrey space when $p(\cdot) = p$, $1 < p < \infty$, is a constant function, this result follows from the fact that the power function $|x|^{-n/p}$ belongs to the local Morrey space [33, p.1725].

For the above result for Morrey space, it suffices to show that $f_i$ is a Morrey function, see [45].

The studies of the sharp operator have been extended to the local sharp operator by Strömberg in [51]. To define the local sharp maximal operator, we recall the notation of the decreasing rearrangement, the median value and the local mean oscillation of Lebesgue measurable functions. For any Lebesgue measurable function $f$, the decreasing rearrangement of $f$ is defined as

$$f^*(t) = \inf\{\lambda > 0 : |\{x \in \mathbb{R}^n : |f(x)| > \lambda\}| < t\}, \quad t \in (0,\infty).$$

For any cube $Q$, let $m_f(Q)$ be the median value of $f$ over $Q$. That is, it satisfies

$$|\{x \in Q : f(x) > m_f(Q)\}| \leqslant |Q|/2,$$
$$|\{x \in Q : f(x) < m_f(Q)\}| \leqslant |Q|/2.$$

Notice that the median value of $f$ over $Q$ is not necessary unique.

For any $\gamma \in (0,1)$ and cube $Q$, the local mean oscillation of $f$ is defined by

$$\omega_\gamma(f,Q) = ((f - m_f(Q))\chi_Q)^*(\gamma|Q|).$$

For any Lebesgue measurable function $f$, the local sharp maximal operator is defined as

$$\mathrm{M}^\sharp_\gamma f(x) = \sup_{B \ni x} \omega_\gamma(f,Q)$$

where the supremum is taken over all $B \in \mathbb{B}$ containing $x$.

The main result in [51] for the local sharp maximal operator is its boundedness on Lebesgue space.

THEOREM 8. *Let $p > 0$ and $\gamma \in (0,\frac{1}{2})$. We have a constant $C > 0$ such that*

$$\|f\|_{L^p} \leqslant C\|\mathrm{M}^\sharp_\gamma f\|_{L^p}.$$

The weighted version of Theorem 8 is given in [31, Theorem 4.6].

THEOREM 9. *Let $\omega \in A_\infty$ and $p > 0$. There exist $\gamma_n, C > 0$ such that for any $\gamma \in (0,\gamma_n)$ and Lebesgue measurable function $f$ satisfying $f^*(+\infty) = 0$, we have*

$$\int_{\mathbb{R}^n} |f(x)|^p \omega(x)dx \leqslant C \int_{\mathbb{R}^n} (\mathrm{M}^\sharp_\gamma f(x))^p \omega(x)dx.$$

Since $A_1 \subset A_\infty$, Theorems 6 and 9 give the following result.

THEOREM 10. *Let $1 < r < \infty$, $p(\cdot) \in \mathscr{B}$ and $u : (0,\infty) \to (0,\infty)$ be a Lebesgue measurable function. If $u \in \mathbb{LW}^{p_-}_{p(\cdot)}$ is increasing, then there is a constant $C > 0$ such that for any $\gamma \in (0,\gamma_n)$ and Lebesgue measurable functions $\{f_i\}_{i=1}^\infty \subset L^\infty_c$ satisfying $f_i^*(+\infty) = 0$, we have*

$$\Big\|\Big(\sum_{i=1}^\infty |f_i|^r\Big)^{1/r}\Big\|_{LM_u^{p(\cdot)}} \leqslant C\Big\|\Big(\sum_{i=1}^\infty (\mathrm{M}^\sharp_\gamma f_i)^r\Big)^{1/r}\Big\|_{LM_u^{p(\cdot)}}. \tag{35}$$

For any $f \in \mathscr{M}$, the geometrical maximal operator is defined as

$$\mathrm{M}_0 f(x) = \sup_{B \ni x} \exp\left(\frac{1}{|B|}\int_B \log|f(y)|dy\right)$$

where the supremum is taken over all balls $B \in \mathbb{B}$ containing $x$. Notice that we have $\mathrm{M}_0|f| = \mathrm{M}_0 f$.

We present the weighted norm inequality for the geometrical maximal operator. According to [50], we have the following weighted norm inequality for $\mathrm{M}_0$.

THEOREM 11. *Let $0 < p < \infty$. We have*

$$\int_{\mathbb{R}^n} (\mathrm{M}_0 f(x))^p \omega(x) \leqslant C \int_{\mathbb{R}^n} |f(x)|^p \omega(x)dx, \quad \forall f \in L^p(\mathbb{R}^n) \tag{36}$$

*for some $C > 0$ if and only if $\omega \in A_\infty$.*

We are now ready to establish the boundedness of the geometric maximal operator on local Morrey spaces with variable exponents.

THEOREM 12. *Let $p(\cdot) \in \mathscr{B}$ and $u : (0,\infty) \to (0,\infty)$ be a Lebesgue measurable function. If $u \in \mathbb{LW}^{p_-}_{p(\cdot)}$ is increasing, then there is a constant $C > 0$ such that for any $f \in LM^{p(\cdot)}_u$, we have*

$$\|\mathrm{M}_0 f\|_{LM^{p(\cdot)}_u} \leqslant C\|f\|_{LM^{p(\cdot)}_u}. \tag{37}$$

*Proof.* For any $f \in LM^{p(\cdot)}_u$ and $N \in \mathbb{N}$, write $f_N = f\chi_{\{x \in B(0,N):|f| \leqslant N\}}$. For any $\omega \in A_1$, Theorem 11 yields

$$\int_{\mathbb{R}^n} (\mathrm{M}_0 f_N(x))^p \omega(x) \leqslant C \int_{\mathbb{R}^n} |f_N(x)|^p \omega(x)dx < \infty.$$

Theorem 6 gives

$$\|\mathrm{M}_0 f_N\|_{LM^{p(\cdot)}_u} \leqslant C\|f_N\|_{LM^{p(\cdot)}_u}. \tag{38}$$

It is easy to see that $f_N \uparrow f$ and $\mathrm{M}_0 f_N \uparrow \mathrm{M}_0 f$ as $N \to \infty$. Consequently, $\|f_N\|_{LM^{p(\cdot)}_u} \uparrow \|f\|_{LM^{p(\cdot)}_u}$ and $\|\mathrm{M}_0 f_N\|_{LM^{p(\cdot)}_u} \uparrow \|\mathrm{M}_0 f\|_{LM^{p(\cdot)}_u}$. Thus, by letting $N \to \infty$ in (38), we obtain (37).

Finally, we apply Theorem 6 to the rough maximal function. Let $\Omega$ be a Lebesgue measurable function satisfying

$$\Omega(\lambda x) = \Omega(x), \quad \lambda > 0, x \in \mathbb{R}^n. \tag{39}$$

Any function satisfying (39) will be identified with a function defined over $\mathbb{S}^{n-1}$ where $\mathbb{S}^{n-1}$ is the unit sphere on $\mathbb{R}^n$.

For any $f \in L^1_{loc}$, the rough maximal function is defined as

$$\mathrm{M}_\Omega f(x) = \sup_{r>0} \frac{1}{r^n} \int_{|y| \leqslant r} |\Omega(y)||f(x-y)|dy, \quad x \in \mathbb{R}^n,$$

see [32, (2.3.1)].

In view of [32, Theorem 2.3.8 (i)], we have the following weighted norm inequalities for the rough maximal function.

THEOREM 13. *Let* $1 < p,q < \infty$ *and* $\Omega \in L^q(\mathbb{S}^{n-1})$ *satisfy* (39)*. If* $q' \leqslant p < \infty$ *and* $\omega \in A_{p/q'}$*, then there is a constant* $C > 0$ *such that*

$$\int_{\mathbb{R}^n} (\mathrm{M}_\Omega f(x))^p \omega(x) \leqslant C \int_{\mathbb{R}^n} |f(x)|^p \omega(x)dx, \quad \forall f \in L^p(\mathbb{R}^n).$$

Theorems 6 and 13 yield the boundedness of the rough maximal function on the local Morrey spaces with variable exponents.

THEOREM 14. *Let* $1 < q < \infty$, $p(\cdot) \in \mathscr{B}$ *and* $u : (0,\infty) \to (0,\infty)$ *be a Lebesgue measurable function. If* $\Omega \in L^q(\mathbb{S}^{n-1})$ *satisfies* (39) *and* $u \in \mathbb{LW}^{q_0}_{p(\cdot)}$ *is increasing where* $q_0 = \min(q', p_-)$*, then there is a constant* $C > 0$ *such that for any* $f \in LM^{p(\cdot)}_u$*, we have*

$$\|\mathrm{M}_\Omega f\|_{LM^{p(\cdot)}_u} \leqslant C\|f\|_{LM^{p(\cdot)}_u}.$$

Since the proof of the preceding theorem is similar to the proof of Theorem 12, for brevity, we omit the proof and leave it to the reader.

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