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MAJORIZATION AND REFINEMENTS OF HERMITE–HADAMARD INEQUALITY

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Abstract. In this note we use majorization methods to derive and analyze refinements of Hermite-Hadamard inequality for a convex function.

1. Introduction and preliminaries

In this expository section we recall some important properties of convex functions. We begin with the following *Hermite-Hadamard inequality* (1).

THEOREM A. [2, p. 137] *Let $f : I \to \mathbb{R}$ be a convex function on an interval $I \subset \mathbb{R}$, $a, b \in I$ with $a < b$. Then the following inequality holds*

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)\,dx \leqslant \frac{f(a)+f(b)}{2}. \tag{1}$$

Hammer-Bullen's inequality (2) provides a refinement of the right-hand side of inequality (1).

THEOREM B. [4, 7] *Assume that $f : I \to \mathbb{R}$ is a convex function on an interval $I \subset \mathbb{R}$, $a, b \in I$ with $a < b$. Then the following inequality holds:*

$$\frac{1}{b-a}\int_a^b f(x)\,dx \leqslant \frac{1}{4}f(a) + \frac{1}{2}f\left(\frac{a+b}{2}\right) + \frac{1}{4}f(b) \leqslant \frac{f(a)+f(b)}{2}. \tag{2}$$

A generalization of (2) is due to Farissi [6].

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THEOREM C. [6] *Assume that* $f : I \to \mathbb{R}$ *is a convex function on an interval* $I \subset \mathbb{R}$*,* $a, b \in I$ *with* $a < b$*. Then for all* $\mu \in [0,1]$ *the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leqslant l(\mu) \leqslant \frac{1}{b-a}\int_a^b f(x)\,dx \leqslant L(\mu) \leqslant \frac{f(a)+f(b)}{2}, \tag{3}$$

where

$$l(\mu) := \mu f\left(\frac{\mu b + (2-\mu)a}{2}\right) + (1-\mu) f\left(\frac{(1+\mu)b + (1-\mu)a}{2}\right),$$

$$L(\mu) := \frac{1}{2}[f(\mu b + (1-\mu)a) + \mu f(a) + (1-\mu) f(b)].$$

The above inequalities provide some bounds for the integral mean $\frac{1}{b-a}\int_a^b f(x)\,dx$ by using the sums of 1-3 components. A related results with n-components is due to Dragomir [5, Theorem 4 and Corollary 4].

In the present note, we continue this approach by employing the *majorization theory* [10].

To give a motivation for our studies, remind that the Riemann integral of a function can be viewed as a result of a limiting process for the corresponding *finite* Riemann sums of some values of a function. In consequence, proving integral inequalities can be reduced to the problem of comparing two sequences of such finite sums.

Some class of important inequalities for convex functions is closely related to the notion of *majorization preorder* on $\mathbb{R}^m$.

DEFINITION 1. ([10, p. 8]) Let $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_m)$ be two given sequences in $\mathbb{R}^m$. We say that $\mathbf{x}$ *majorizes* $\mathbf{y}$ (written as $\mathbf{y} \prec \mathbf{x}$), if the sum of j largest entries of $\mathbf{y}$ does not exceed the sum of j largest entries of $\mathbf{x}$ for all $j = 1, 2, \ldots, m$ with equality for $j = m$.

That is, $\mathbf{y} \prec \mathbf{x}$ iff

$$\sum_{i=1}^{j} y_{[i]} \leqslant \sum_{i=1}^{j} x_{[i]} \quad \text{for } j = 1, 2, \ldots, m, \quad \text{and} \quad \sum_{i=1}^{m} y_i = \sum_{i=1}^{m} x_i,$$

where the symbols $x_{[i]}$ and $y_{[i]}$ stand for the ith largest entry of $\mathbf{x}$ and $\mathbf{y}$, respectively.

DEFINITION 2. An $m \times k$ real matrix $\mathbf{S} = (s_{ij})$ is said to be *column stochastic* if $s_{ij} \geqslant 0$ for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, k$, and $\sum_{i=1}^{m} s_{ij} = 1$ for $j = 1, 2, \ldots, k$.

An $m \times m$ real matrix $\mathbf{S} = (s_{ij})$ is said to be *doubly stochastic* if $s_{ij} \geqslant 0$ for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, m$, and $\sum_{j=1}^{m} s_{ij} = 1$ for $i = 1, 2, \ldots, m$, and $\sum_{i=1}^{m} s_{ij} = 1$ for $j = 1, 2, \ldots, m$.

It is well known that $\mathbf{y} \prec \mathbf{x}$ iff there exists an $m \times m$ doubly stochastic matrix $\mathbf{S}$ such that $\mathbf{y} = \mathbf{x}\mathbf{S}$ [10, p. 10].

The next result, due to Hardy, Littlewood, Pólya and Karamata, provides a useful inequality for two finite sums involving values of a convex function.

THEOREM D. [10, pp. 92] *Let $f : I \to \mathbb{R}$ be a convex function defined on an interval $I \subset \mathbb{R}$. Let $\mathbf{x} = (x_1, x_2, \ldots, x_m) \in I^m$ and $\mathbf{y} = (y_1, y_2, \ldots, y_m) \in I^m$.*

Then

$$\mathbf{y} \prec \mathbf{x} \quad \textit{implies} \quad \sum_{i=1}^{m} f(y_i) \leqslant \sum_{i=1}^{m} f(x_i). \tag{4}$$

It follows from Theorem D for $m = 2$ that if $f : I \to \mathbb{R}$ is convex, $a, b \in I$ with $a < b$ and $\xi \in [a, \frac{a+b}{2}]$ and $\eta = a + b - \xi$, then

$$f(\xi) + f(\eta) \leqslant f(a) + f(b). \tag{5}$$

To see this, note that the vector $(a, b) \in \mathbb{R}^2$ majorizes the vector $(\xi, \eta) \in \mathbb{R}^2$.

Similarly, if $f : I \to \mathbb{R}$ is convex, and $[c, d] \subset [a, b] \subset I$ and $a \leqslant \xi \leqslant c \leqslant \frac{a+b}{2} = \frac{c+d}{2} \leqslant d \leqslant \eta \leqslant b$ with $d = a + b - c$ and $\eta = a + b - \xi$, then

$$(c, d) \prec (\xi, \eta) \prec (a, b),$$

and, in consequence,

$$f(c) + f(d) \leqslant f(\xi) + f(\eta) \leqslant f(a) + f(b). \tag{6}$$

An extension of Theorem **D** is the following result.

THEOREM E. [13, 1, 3] *Let f be a real convex function defined on an interval $J \subset \mathbb{R}$. Let $\mathbf{a} = (a_1, a_2, \ldots, a_m) \in \mathbb{R}_+^m$, $\mathbf{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{R}_+^n$, $\mathbf{x} = (x_1, x_2, \ldots, x_m) \in J^m$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in J^n$.*

If

$$\mathbf{y} = \mathbf{x}\mathbf{S} \quad \textit{and} \quad \mathbf{a} = \mathbf{b}\mathbf{S}^T \tag{7}$$

for some $m \times n$ column stochastic matrix $\mathbf{S} = (s_{ij})$, then

$$\sum_{j=1}^{n} b_j f(y_j) \leqslant \sum_{i=1}^{m} a_i f(x_i). \tag{8}$$

Inequality (8) is referred as *Sherman's inequality*, and statement (7) determines the notion of *weighted majorization* of the pairs $(\mathbf{b}, \mathbf{y})$ and $(\mathbf{a}, \mathbf{x})$.

2. Refinements of Hermite-Hadamard inequalities for convex functions

The forthcomming theorem is in a line of Dragomir's multipoints improvement of Hermite-Hadamard inequalities for convex functions (see [5, Theorem 4 and Corollary 4]). Here we demonstrate an alternative approach by making use of the majorization preorder on $\mathbb{R}^2$ via the property (6).

THEOREM 1. *Let $f: I \to \mathbb{R}$ be a convex function on an interval $I \subset \mathbb{R}$, $a, b \in I$, $a < b$.*

For a given fixed positive integer r, let

$$a = a_0 < a_1 < \ldots < a_{r-1} < a_r = \frac{a+b}{2} = b_r < b_{r-1} < \ldots < b_1 < b_0 = b, \qquad (9)$$

where $b_i = a+b-a_i$, $I_i = [a_{i-1}, a_i]$, $J_i = [b_i, b_{i-1}]$, $|I_i| = a_i - a_{i-1}$, $|J_i| = b_{i-1} - b_i$ for $i = 1, \ldots, r$.

Then the following Hermite-Hadamard type inequalities hold:

$$\begin{aligned} f\left(\frac{a+b}{2}\right) \quad &\leqslant \quad \sum_{i=1}^{r} \tfrac{|I_i|}{b-a}(f(a_i)+f(b_i)) \quad \leqslant \quad \frac{1}{b-a}\int_a^b f(t)\,dt \\ &\leqslant \sum_{i=1}^{r} \tfrac{|J_i|}{b-a}(f(a_{i-1})+f(b_{i-1})) \leqslant \frac{f(a)+f(b)}{2}. \end{aligned} \qquad (10)$$

Proof. Taking into account (9), for $i = 1, \ldots, r$ let I_{ij}, $j = 1, \ldots, k_i$, be consecutive intervals (from left to right) forming a partition of I_i, and J_{ij}, $j = 1, \ldots, k_i$, be consecutive intervals (from right to left) forming a partition of J_i so that J_{ij} is symmetric to I_{ij} about $\frac{a+b}{2}$. Thus $I_i = \bigcup_{j=1}^{k_i} I_{ij}$ and $J_i = \bigcup_{j=1}^{k_i} J_{ij}$. Evidently, the lengths of I_{ij} and J_{ij} are equal, i.e., $|I_{ij}| = |J_{ij}|$.

For $i = 1, \ldots, r$, $j = 1, \ldots, k_i$, take ξ_{ij} to be an immediate point in I_{ij}, and η_{ij} be the point in J_{ij} symmetric to ξ_{ij} about $\frac{a+b}{2}$, that is, $\eta_{ij} = a+b-\xi_{ij}$.

For $i = 1, \ldots, r$, $j = 1, \ldots, k_i$, since

$$\xi_{ij} \in I_{ij} \subset I_i = [a_{i-1}, a_i],$$

it is readily seen that

$$(a_i, b_i) \prec (\xi_{ij}, \eta_{ij}) \prec (a_{i-1}, b_{i-1}),$$

where $\prec$ denotes the majorization preorder on $\mathbb{R}^2$. Therefore by (6),

$$f(a_i) + f(b_i) \leqslant f(\xi_{ij}) + f(\eta_{ij}) \leqslant f(a_{i-1}) + f(b_{i-1}).$$

Because $|I_{ij}| = |J_{ij}|$ for i = 1,…,r , $j = 1, \ldots, k_i$, it follows that

$$|I_{ij}|f(a_i) + |I_{ij}|f(b_i) \leqslant |I_{ij}|f(\xi_{ij}) + |J_{ij}|f(\eta_{ij}) \leqslant |J_{ij}|f(a_{i-1}) + |J_{ij}|f(b_{i-1}).$$

Hence, for $i = 1, \ldots, r$,

$$\sum_{j=1}^{k_i} |I_{ij}|(f(a_i)+f(b_i)) \leqslant \sum_{j=1}^{k_i} (|I_{ij}|f(\xi_{ij}) + |J_{ij}|f(\eta_{ij})) \leqslant \sum_{j=1}^{k_i} |J_{ij}|(f(a_{i-1})+f(b_{i-1})).$$

Equivalently,

$$|I_i|(f(a_i)+|f(b_i)) \leqslant \sum_{j=1}^{k_i} (|I_{ij}|f(\xi_{ij}) + |J_{ij}|f(\eta_{ij})) \leqslant |J_i|(f(a_{i-1})+f(b_{i-1})).$$

Now, for each $i=1,\dots,r$, by letting k_i tend to ∞, and $|I_{ij}|$ and $|J_{ij}|$ tend to 0, we obtain inequalities

$$|I_i|(f(a_i)+f(b_i)) \leqslant \int\limits_{I_i\cup J_i} f(t)\,dt \leqslant |J_i|(f(a_{i-1})+f(b_{i-1})).$$

By summing over i from 1 to r, we obtain

$$\sum_{i=1}^{r}|I_i|(f(a_i)+f(b_i)) \leqslant \sum_{i=1}^{r}\int\limits_{I_i\cup J_i} f(t)\,dt = \int_a^b f(t)\,dt \leqslant \sum_{i=1}^{r}|J_i|(f(a_{i-1})+f(b_{i-1})).$$

This easily leads to the second and third inequalities in (10).

To show the first and last inequalities in (10), notice that

$$(a_r,b_r)\prec(a_i,b_i)\prec(a_0,b_0) \quad \text{for } i=0,1,\dots,r,$$

where $a_r=b_r=\frac{a+b}{2}$ and $a_0=a$, $b_0=b$. From this via (6) we deduce that

$$f\left(\frac{a+b}{2}\right)+f\left(\frac{a+b}{2}\right)\leqslant f(a_i)+f(b_i)\leqslant f(a)+f(b) \quad \text{for } i=0,1,\dots,r. \tag{11}$$

For this reason,

$$2\sum_{i=1}^{r}|I_i|f\left(\frac{a+b}{2}\right)\leqslant\sum_{i=1}^{r}|I_i|(f(a_i)+f(b_i))$$

However,

$$(b-a)f\left(\frac{a+b}{2}\right)=2\sum_{i=1}^{r}|I_i|f\left(\frac{a+b}{2}\right).$$

So, we get

$$f\left(\frac{a+b}{2}\right)\leqslant\sum_{i=1}^{r}\frac{|I_i|}{b-a}(f(a_i)+f(b_i))$$

completing the proof of the first inequality in (10).

In order to see the last inequality in (10), we use (11) and obtain

$$\sum_{i=1}^{r}\frac{|J_i|}{b-a}(f(a_{i-1})+f(b_{i-1}))\leqslant\sum_{i=1}^{r}\frac{|J_i|}{b-a}(f(a)+f(b))=\frac{1}{2}(f(a)+f(b)).$$

This completes the proof of Theorem 1. □

For equidistant partition (9), Theorem 1 takes the following form.

COROLLARY 1. *Under the hypotheses of Theorem 1, if in addition*

$$|I_1|=|I_2|=\ldots=|I_r|=|J_r|=|J_{r-1}|=\ldots=|J_1|, \tag{12}$$

then

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{2r}\sum_{i=1}^{r}(f(a_i)+f(b_i)) \leqslant \frac{1}{b-a}\int_a^b f(t)\,dt$$
$$\leqslant \frac{1}{2r}\sum_{i=1}^{r}(f(a_{i-1})+f(b_{i-1})) \leqslant \frac{f(a)+f(b)}{2}. \tag{13}$$

Proof. By (12) we have

$$|I_i| = |J_i| = \frac{b-a}{2r} \quad \text{for } i = 1,\dots,r.$$

By putting this to (10), we get inequalities (13). □

REMARK 1. For $r = 1$ we have

$$a = a_0 < a_1 = \frac{a+b}{2} = b_1 < b_0 = b$$

(see (9)), so in this case inequality (13) reduces to the Hermite-Hadamard inequality (1) (see Theorem A).

REMARK 2. Inequality (13) can be restated as

$$\begin{aligned}&\frac{f(a_r)+f(b_r)}{2r}\\ &\leqslant \frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(a_1)+\ldots+f(a_{r-1})}{2r} - \frac{f(b_{r-1})+\ldots+f(b_1)}{2r}\\ &\leqslant \frac{f(a_0)+f(b_0)}{2r}.\end{aligned} \tag{14}$$

We now consider the case of Theorem 1 for $r = 2$.

COROLLARY 2. *Let $f : I \to \mathbb{R}$ be a convex function on an interval $I \subset \mathbb{R}$, $a,b \in I$, $a < b$. Let $a < a_1 < \frac{a+b}{2} < b_1 < b$ with $b_1 = a+b-a_1$.*
Then

$$\frac{a_1-a}{b-a}(f(a_1)+f(b_1)) + 2\frac{\frac{a+b}{2}-a_1}{b-a} f\left(\frac{a+b}{2}\right)$$
$$\leqslant \frac{1}{b-a}\int_a^b f(t)\,dt \leqslant \frac{b-b_1}{b-a}(f(a)+f(b)) + \frac{b_1-\frac{a+b}{2}}{b-a}(f(a_1)+f(b_1)). \tag{15}$$

Proof. By putting $r = 2$ into (10) we obtain

$$\frac{|I_1|}{b-a}(f(a_1)+f(b_1))+\frac{|I_2|}{b-a}(f(a_2)+f(b_2))$$
$$\leqslant \frac{1}{b-a}\int_a^b f(t)\,dt \leqslant \frac{|J_1|}{b-a}(f(a_0)+f(b_0))+\frac{|J_2|}{b-a}(f(a_1)+f(b_1)) \tag{16}$$

with

$$a = a_0 < a_1 < a_2 = \frac{a+b}{2} = b_2 < b_1 < b_0 = b$$

and $b_i = a+b-a_i$, $I_i = [a_{i-1}, a_i]$, $J_i = [b_i, b_{i-1}]$, $|I_i| = a_i - a_{i-1}$, $|J_i| = b_{i-1} - b_i$ for $i = 1,2$.

Therefore (16) implies (15), as claimed. □

REMARK 3. It is not hard to verify that the Hermite-Hadamard inequality (1) is a limiting case of the above inequality (16) when $a_1 \to \left(\frac{a+b}{2}\right)^-$ and $b_1 = a+b-a_1 \to \left(\frac{a+b}{2}\right)^+$. Indeed we have

$$\lim_{a_1 \to \left(\frac{a+b}{2}\right)^-} f(a_1) = f\left(\frac{a+b}{2}\right) \quad \text{and} \quad \lim_{b_1 \to \left(\frac{a+b}{2}\right)^+} f(b_1) = f\left(\frac{a+b}{2}\right),$$

as f is convex on $[a,b]$ and therefore f is continuous on (a,b).

3. Comparison results

We are now interested in comparison of bounds for integral mean of a convex function with two partitions of the interval $[a,b]$.

THEOREM 2. *Assume that the hypotheses of Theorem 1 are satisfied.*
If additionally $a_i \leqslant c_i \leqslant \frac{a+b}{2} \leqslant d_i \leqslant b_i$ with $c_i + d_i = a+b$ for $i = 1,\dots,r$, then

$$f\left(\frac{a+b}{2}\right) \leqslant \sum_{i=1}^{r} \frac{|I_i|}{b-a}(f(c_i)+f(d_i))$$
$$\leqslant \sum_{i=1}^{r} \frac{|I_i|}{b-a}(f(a_i)+f(b_i)) \leqslant \frac{1}{b-a}\int_a^b f(t)\,dt. \tag{17}$$

In particular, if $c_i = \frac{a+b}{2} = d_i$ for $i = 1,\dots,r$, then the first inequality of (17) *becomes an equality.*

If $a \leqslant c_{i-1} \leqslant a_{i-1}$ and $b_{i-1} \leqslant d_{i-1} \leqslant b$ with $c_i + d_i = a+b$ for $i = 1,\dots,r$, then

$$\frac{1}{b-a}\int_a^b f(t)\,dt \leqslant \sum_{i=1}^{r} \frac{|J_i|}{b-a}(f(a_{i-1})+f(b_{i-1}))$$
$$\leqslant \sum_{i=1}^{r} \frac{|J_i|}{b-a}(f(c_{i-1})+f(d_{i-1})) \leqslant \frac{f(a)+f(b)}{2}. \tag{18}$$

In particular, if $c_{i-1} = a$ and $d_{i-1} = b$ for $i = 1,\dots,r$, then the last inequality of (18) *becomes an equality.*

Proof. It follows that

$$\left(\frac{a+b}{2},\frac{a+b}{2}\right) \prec (c_i,d_i) \prec (a_i,b_i) \quad \text{for } i=1,\dots,r.$$

So,

$$2f\left(\frac{a+b}{2}\right) \leqslant f(c_i)+f(d_i) \leqslant f(a_i)+f(b_i) \quad \text{for } i=1,\dots,r,$$

and therefore

$$2\sum_{i=1}^{r}\frac{|I_i|}{b-a}f\left(\frac{a+b}{2}\right) \leqslant \sum_{i=1}^{r}\frac{|I_i|}{b-a}(f(c_i)+f(d_i)) \leqslant \sum_{i=1}^{r}\frac{|I_i|}{b-a}(f(a_i)+f(b_i)).$$

This and (10) yield (17), since $\sum\limits_{i=1}^{r}|I_i| = \frac{b-a}{2}$.

Likewise, we have

$$(a_{i-1},b_{i-1}) \prec (c_{i-1},d_{i-1}) \prec (a,b) \quad \text{for } i=1,\dots,r.$$

In consequence,

$$f(a_{i-1})+f(b_{i-1}) \leqslant f(c_{i-1})+f(d_{i-1}) \leqslant f(a)+f(b) \quad \text{for } i=1,\dots,r.$$

Hence

$$\sum_{i=1}^{r}\frac{|J_i|}{b-a}(f(a_{i-1})+f(b_{i-1})) \leqslant \sum_{i=1}^{r}\frac{|J_i|}{b-a}(f(c_{i-1})+f(d_{i-1})) \leqslant \sum_{i=1}^{r}\frac{|J_i|}{b-a}f(a)+f(b).$$

This together with (10) imply (18), because $\sum\limits_{i=1}^{r}|J_i| = \frac{b-a}{2}$. This completes the proof of Theorem 2. □

In the next results we apply the weighted majorization method described in Theorem **E**.

THEOREM 3. *Let $f : I \to \mathbb{R}$ be a convex function on an interval $I \subset \mathbb{R}$, $a,b \in I$, $a < b$.*

For a given fixed positive integer r, let

$$a = a_0 < a_1 < \dots < a_{r-1} < a_r = \frac{a+b}{2} = b_r < b_{r-1} < \dots < b_1 < b_0 = b, \tag{19}$$

where $b_i = a+b-a_i$, $I_i = [a_{i-1},a_i]$, $J_i = [b_i,b_{i-1}]$, $|I_i| = a_i - a_{i-1}$, $|J_i| = b_{i-1} - b_i$ for $i = 1,\dots,r$.

For a given fixed positive integer $\widetilde{r}$, let

$$a=\widetilde{a}_0<\widetilde{a}_1<\ldots<\widetilde{a}_{\widetilde{r}-1}<\widetilde{a}_{\widetilde{r}}=\frac{a+b}{2}=\widetilde{b}_{\widetilde{r}}<\widetilde{b}_{\widetilde{r}-1}<\ldots<\widetilde{b}_1<\widetilde{b}_0=b, \tag{20}$$

where $\widetilde{b}_i=a+b-\widetilde{a}_i$, $\widetilde{I}_i=[\widetilde{a}_{i-1},\widetilde{a}_i]$, $\widetilde{J}_i=[\widetilde{b}_i,\widetilde{b}_{i-1}]$, $|\widetilde{I}_i|=\widetilde{a}_i-\widetilde{a}_{i-1}$, $|\widetilde{J}_i|=\widetilde{b}_{i-1}-\widetilde{b}_i$ *for* $i=1,\ldots,\widetilde{r}$.

If there exists a column stochastic matrix $\mathbf{S}$ *of size* $\widetilde{r}\times r$ *such that*

$$(a_1,\ldots,a_r)=(\widetilde{a}_1,\ldots,\widetilde{a}_{\widetilde{r}})\mathbf{S} \quad \textit{and} \quad (|\widetilde{I}_1|,\ldots,|\widetilde{I}_{\widetilde{r}}|)=(|I_1|,\ldots,|I_r|)\mathbf{S}^T, \tag{21}$$

then

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leqslant \sum_{i=1}^{r}\tfrac{|I_i|}{b-a}(f(a_i)+f(b_i)) \\ &\leqslant \sum_{i=1}^{\widetilde{r}}\tfrac{|\widetilde{I}_i|}{b-a}(f(\widetilde{a}_i)+f(\widetilde{b}_i)) \leqslant \frac{1}{b-a}\int_a^b f(t)\,dt. \end{aligned} \tag{22}$$

Proof. We denote

$$\begin{aligned} \mathbf{a}&=(a_1,\ldots,a_r) \quad \text{and} \quad \mathbf{b}=(b_1,\ldots,b_r),\\ \Delta&=(|I_1|,\ldots,|I_r|) \quad \text{and} \quad \Gamma=(|J_1|,\ldots,|J_r|),\\ \widetilde{\mathbf{a}}&=(\widetilde{a}_1,\ldots,\widetilde{a}_{\widetilde{r}}) \quad \text{and} \quad \widetilde{\mathbf{b}}=(\widetilde{b}_1,\ldots,\widetilde{b}_{\widetilde{r}}),\\ \widetilde{\Delta}&=(|\widetilde{I}_1|,\ldots,|\widetilde{I}_{\widetilde{r}}|) \quad \text{and} \quad \widetilde{\Gamma}=(|\widetilde{J}_1|,\ldots,|\widetilde{J}_{\widetilde{r}}|). \end{aligned}$$

The first and last inequalities of (22) are satisfied by Theorem 1 applied to $\mathbf{a}$, $\mathbf{b}$, Δ and $\widetilde{\mathbf{a}}$, $\widetilde{\mathbf{b}}$, $\widetilde{\Delta}$, respectively. It remains to prove the middle inequality of (22).

To do so, it is enough to show that

$$\sum_{i=1}^{r}|I_i|f(a_i)\leqslant\sum_{i=1}^{\widetilde{r}}|\widetilde{I}_i|f(\widetilde{a}_i), \tag{23}$$

and

$$\sum_{i=1}^{r}|J_i|f(b_i)\leqslant\sum_{i=1}^{\widetilde{r}}|\widetilde{J}_i|f(\widetilde{b}_i). \tag{24}$$

In fact, in light of (21), inequality (23) is an immediate consequence of Sherman's inequality (8) (see Theorem **E**).

To see (24), we now establish an analog of (21) corresponding to $\mathbf{b}$, $\widetilde{\mathbf{b}}$, Γ and $\widetilde{\Gamma}$. We denote

$$\mathbf{e}=(1,\ldots,1)\in\mathbb{R}^r \quad \text{and} \quad \widetilde{\mathbf{e}}=(1,\ldots,1)\in\mathbb{R}^{\widetilde{r}}.$$

Since $\mathbf{S}$ is column stochastic, we get

$$\mathbf{e}=\widetilde{\mathbf{e}}\mathbf{S}.$$

It follows that $\mathbf{b} = (a+b)\mathbf{e} - \mathbf{a}$ and $\widetilde{\mathbf{b}} = (a+b)\widetilde{\mathbf{e}} - \widetilde{\mathbf{a}}$. Hence via (21) we derive

$$\widetilde{\mathbf{b}}\mathbf{S} = (a+b)\widetilde{\mathbf{e}}\mathbf{S} - \widetilde{\mathbf{a}}\mathbf{S} = (a+b)\mathbf{e} - \mathbf{a} = \mathbf{b}.$$

On the other hand, we infer from $\widetilde{\Delta} = \Delta\mathbf{S}^T$ (see (21)) that

$$\widetilde{\Gamma} = (|\widetilde{J}_1|, \ldots, |\widetilde{J}_{\widetilde{r}}|) = (|J_1|, \ldots, |J_r|)\mathbf{S}^T = \Gamma\mathbf{S}^T,$$

because $|J_i| = |I_i|$ for $i = 1, \ldots, r$ and $|\widetilde{J}_i| = |\widetilde{I}_i|$ for $i = 1, \ldots, \widetilde{r}$.

In consequence, by Sherman's inequality (8) (see Theorem **E**), we obtain (24), as wanted. □

THEOREM 4. *Let $f : I \to \mathbb{R}$ be a convex function on an interval $I \subset \mathbb{R}$, $a, b \in I$, $a < b$.*

For a given fixed positive integer r, let

$$a = a_0 < a_1 < \ldots < a_{r-1} < a_r = \frac{a+b}{2} = b_r < b_{r-1} < \ldots < b_1 < b_0 = b, \tag{25}$$

where $b_i = a + b - a_i$, $I_i = [a_{i-1}, a_i]$, $J_i = [b_i, b_{i-1}]$, $|I_i| = a_i - a_{i-1}$, $|J_i| = b_{i-1} - b_i$ for $i = 1, \ldots, r$.

For a given fixed positive integer $\widetilde{r}$, let

$$a = \widetilde{a}_0 < \widetilde{a}_1 < \ldots < \widetilde{a}_{\widetilde{r}-1} < \widetilde{a}_{\widetilde{r}} = \frac{a+b}{2} = \widetilde{b}_{\widetilde{r}} < \widetilde{b}_{\widetilde{r}-1} < \ldots < \widetilde{b}_1 < \widetilde{b}_0 = b, \tag{26}$$

where $\widetilde{b}_i = a + b - \widetilde{a}_i$, $\widetilde{I}_i = [\widetilde{a}_{i-1}, \widetilde{a}_i]$, $\widetilde{J}_i = [\widetilde{b}_i, \widetilde{b}_{i-1}]$, $|\widetilde{I}_i| = \widetilde{a}_i - \widetilde{a}_{i-1}$, $|\widetilde{J}_i| = \widetilde{b}_{i-1} - \widetilde{b}_i$ for $i = 1, \ldots, \widetilde{r}$.

If there exists a column stochastic matrix $\mathbf{S}$ of size $r \times \widetilde{r}$ such that

$$(\widetilde{a}_0, \ldots, \widetilde{a}_{\widetilde{r}-1}) = (a_0, \ldots, a_{r-1})\mathbf{S} \quad \text{and} \quad (|J_1|, \ldots, |J_r|) = (|\widetilde{J}_1|, \ldots, |\widetilde{J}_{\widetilde{r}}|)\mathbf{S}^T, \tag{27}$$

then

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(t)\,dt &\leqslant \sum_{i=1}^{\widetilde{r}} \tfrac{|\widetilde{J}_i|}{b-a}\left(f(\widetilde{a}_{i-1}) + f(\widetilde{b}_{i-1})\right) \\ &\leqslant \sum_{i=1}^{r} \tfrac{|J_i|}{b-a}\left(f(a_{i-1}) + f(b_{i-1})\right) \leqslant \frac{f(a)+f(b)}{2}. \end{aligned} \tag{28}$$

Proof. By denoting

$$\mathbf{a} = (a_0, \ldots, a_{r-1}) \quad \text{and} \quad \mathbf{b} = (b_0, \ldots, b_{r-1}),$$

$$\Delta = (|I_1|, \ldots, |I_r|) \quad \text{and} \quad \Gamma = (|J_1|, \ldots, |J_r|),$$

$$\widetilde{\mathbf{a}} = (\widetilde{a}_0, \ldots, \widetilde{a}_{\widetilde{r}-1}) \quad \text{and} \quad \widetilde{\mathbf{b}} = (\widetilde{b}_0, \ldots, \widetilde{b}_{\widetilde{r}-1}),$$

$$\widetilde{\Delta} = (|\widetilde{I}_1|, \ldots, |\widetilde{I}_{\widetilde{r}}|) \quad \text{and} \quad \widetilde{\Gamma} = (|\widetilde{J}_1|, \ldots, |\widetilde{J}_{\widetilde{r}}|),$$

we see that the first and third inequalities of (28) are valid by Theorem 1 applied to $\widetilde{\mathbf{a}}$, $\widetilde{\mathbf{b}}$, $\widetilde{\Gamma}$, and $\mathbf{a}$, $\mathbf{b}$, Γ, respectively.

In order to prove the second inequality of (28), we employ Theorem **E** in the context of (27) and obtain

$$\sum_{i=1}^{\widetilde{r}} |\widetilde{J}_i| f(\widetilde{a}_{i-1}) \leqslant \sum_{i=1}^{r} |J_i| f(a_{i-1}). \tag{29}$$

We utilize equalities

$$\mathbf{b} = (a+b)\mathbf{e} - \mathbf{a}, \quad \widetilde{\mathbf{b}} = (a+b)\widetilde{\mathbf{e}} - \widetilde{\mathbf{a}} \quad \text{and} \quad \widetilde{\mathbf{e}} = \mathbf{eS},$$

where $\mathbf{e} = (1,\dots,1) \in \mathbb{R}^r$ and $\widetilde{\mathbf{e}} = (1,\dots,1) \in \mathbb{R}^{\widetilde{r}}$. Therefore, by (27),

$$\mathbf{bS} = (a+b)\mathbf{eS} - \mathbf{aS} = (a+b)\widetilde{\mathbf{e}} - \widetilde{\mathbf{a}} = \widetilde{\mathbf{b}}.$$

On account of (27) we find that

$$(|J_1|,\dots,|J_r|) = (|\widetilde{J}_1|,\dots,|\widetilde{J}_{\widetilde{r}}|)\mathbf{S}^T.$$

In summary, by Sherman's inequality (8) we get

$$\sum_{i=1}^{\widetilde{r}} |\widetilde{J}_i| f(\widetilde{b}_{i-1}) \leqslant \sum_{i=1}^{r} |J_i| f(b_{i-1}). \tag{30}$$

This together with (29) completes the proof of (28). □

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