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SHARP SOBOLEV INEQUALITIES ON THE COMPLEX SPHERE

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Abstract. This paper is devoted to establish a class of sharp Sobolev inequalities on the unit complex sphere as follows:

1) **Case** $0<d<Q=2n+2$: for any $f\in C^\infty$ and $2\leqslant q\leqslant \frac{2Q}{Q-d}$,

$$\|f\|_q^2 \leqslant \frac{8(q-2)}{d(Q-d)}\frac{\Gamma^2((Q-d)/4+1)}{\Gamma^2((Q+d)/4)}\left(\int_{\mathbb{S}^{2n+1}} f\mathscr{A}_d f\,d\xi - \frac{\Gamma^2((Q+d)/4)}{\Gamma^2((Q-d)/4)}\int_{\mathbb{S}^{2n+1}}|f|^2 d\xi\right)+\int_{\mathbb{S}^{2n+1}}|f|^2 d\xi;$$

2) **Case** $d=Q$: for any $f\in C^\infty\cap\mathbb{R}\mathscr{P}$ and $2\leqslant q<+\infty$,

$$\|f\|_q^2\leqslant \frac{q-2}{(n+1)!}\int_{\mathbb{S}^{2n+1}} f\mathscr{A}'_Q f\,d\xi+\int_{\mathbb{S}^{2n+1}}|f|^2 d\xi,$$

where $\mathscr{A}_d(0<d<Q)$ are the intertwining operator, $\mathscr{A}'_Q$ is the conditional intertwinor introduced in [2], and $d\xi$ is the normalized surface measure of $\mathbb{S}^{2n+1}$.

1. Introduction

It is well known that the classical Sobolev inequalities and Hardy-Littlewood-Sobolev(HLS) inequalities are basic tools in analysis and geometry and their sharp constants play an essential role because they contain geometric and probabilistic information (see e.g., [1, 3, 14, 15]). Recently, many interesting and challenging results on Riemannian geometry and sub-Riemannian manifolds (such as Heisenberg Group, CR sphere) were also obtained to understand different geometry framework. In particular, many interesting geometric inequalities, Sobolev-type inequalities and HLS inequality on the sub-Riemannian manifolds attracted the attention of analysts (see e.g., [2, 5, 6, 8, 9]). Based on the work of Frank and Lieb [6] this paper establishes the CR-sphere counterpart of the Sobolev inequalities discussed in [1] in the Euclidean-sphere setting.

For convenience, we firstly introduce some notations and known facts about the complex sphere $\mathbb{S}^{2n+1}$. More details can be found in [2] and references therein.

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Denote by $\mathbb{S}^{2n+1}$ the complex sphere

$$\mathbb{S}^{2n+1}=\Big\{\xi=(\xi_1,\xi_2,\cdots,\xi_{n+1})\in\mathbb{C}^{n+1}:\ \sum_{j=1}^{n+1}|\xi_j|^2=1\Big\}.$$

Then $\mathbb{C}T\mathbb{S}^{2n+1}$ is generated by the vectors $T_j,\overline{T}_j,\ j=1,2,\cdots,n+1$ and $\mathscr{T}$, where

$$T_j=\frac{\partial}{\partial\xi_j}-\overline{\xi_j}\sum_{j=1}^{n+1}\xi_k\frac{\partial}{\partial\xi_k},\ j=1,2,\cdots,n+1,\text{ and }\mathscr{T}=\frac{i}{2}\sum_{k=1}^{n+1}\Big(\xi_k\frac{\partial}{\partial\xi_k}-\overline{\xi_k}\frac{\partial}{\partial\overline{\xi_k}}\Big).$$

Let $Q=2n+2$ be the homogeneous dimension induced from Heisenberg group by Cayley transformation and denote by $d\xi$ the normalized surface measure on $\mathbb{S}^{2n+1}$.

It is known that $L^2(\mathbb{S}^{2n+1})$ can be decomposed into its $U(n+1)$-irreducible components

$$L^2(\mathbb{S}^{2n+1})=\bigoplus_{j,k\geqslant 0}\mathscr{H}_{jk},\tag{1.1}$$

where $\mathscr{H}_{jk}$ is the space of restrictions to $\mathbb{S}^{2n+1}$ of harmonic polynomials $p(z,\bar{z})$ on $\mathbb{C}^{n+1}$ which are homogeneous of degree j in z and degree k in $\bar{z}$. Take $\{Y_{jk}\}$ as an orthonormal basis of $\mathscr{H}_{jk}$. Moreover, denote the *Hardy spaces* as follows:

$$\begin{aligned}
\mathscr{H}&=\bigoplus_{j\geqslant 0}\mathscr{H}_{j0}\\
&=\{L^2\text{ boundary values of holomorphic functions on the unit ball}\},\\
\overline{\mathscr{H}}&=\bigoplus_{j\geqslant 0}\mathscr{H}_{0j}\\
&=\{L^2\text{ boundary values of antiholomorphic functions on the unit ball}\},\\
\mathscr{P}&=\bigoplus_{j>0}(\mathscr{H}_{j0}\oplus\mathscr{H}_{0j})\bigoplus\mathscr{H}_{00}=\{L^2\text{ CR-pluriharmonic functions}\},\\
\mathbb{R}\mathscr{P}&=\{L^2\text{ real-valued CR pluriharmonic functions}\}.
\end{aligned}$$

For $0<d<Q$, the general *intertwining operator* $\mathscr{A}_d$ of order d is defined with respect to the spherical harmonics as

$$\mathscr{A}_dY_{j,k}=\lambda_j(d)\lambda_k(d)Y_{j,k},\quad j,k=0,1,2,\cdots,\tag{1.2}$$

where

$$\lambda_j(d)=\frac{\Gamma((Q+d)/4+j)}{\Gamma((Q-d)/4+j)},\quad j=0,1,2,\cdots.$$

In particular, $\mathscr{A}_2$ is the *conformal sublaplacian* $\mathscr{D}=\mathscr{L}+\frac{n^2}{4}=\mathscr{L}+(\lambda_0(2))^2$ with

$$\mathscr{L}=-\frac{1}{2}\sum_{j=1}^{n+1}(T_j\overline{T}_j+\overline{T}_jT_j).$$

Recently, Branson et al [2] introduced a class of intertwinors $\mathscr{A}'_Q$ of order Q, named *conditional intertwinors* and defined on $\mathscr{P}$ as

$$\mathscr{A}'_Q Y_{j0} = \lambda_j(Q) Y_{j0} = j(j+1)\cdots(j+n) Y_{j0}, \quad \mathscr{A}'_Q Y_{0k} = \lambda_k(Q) Y_{0k}. \tag{1.3}$$

In [10] and [6], two classes of Sobolev inequalities (see Theorem 3.1 and Corollary 2.3 of [6]) were established as follows:

$$\mathscr{E}[u] \geqslant \frac{n^2}{4}\left(\int_{\mathbb{S}^{2n+1}} |u|^{2Q/(Q-2)} d\xi\right)^{(Q-2)/Q} \tag{1.4}$$

and

$$\frac{4(q-2)}{Q-2}\mathscr{E}_0[u] + \int_{\mathbb{S}^{2n+1}} |u|^2 d\xi \geqslant \left(\int_{\mathbb{S}^{2n+1}} |u|^q d\xi\right)^{2/q}, \quad 2 < q < \frac{2Q}{Q-2} \tag{1.5}$$

where $\mathscr{E}[u] = \mathscr{E}_0[u] + \frac{n^2}{4}u$ and

$$\mathscr{E}_0[u] = \frac{1}{2}\sum_{j=1}^{n+1}(|T_j u|^2 + |\overline{T}_j u|^2).$$

If we adopt the notations of intertwining operator, inequalities (1.4) and (1.5) can be rewrote as:

$$\int_{\mathbb{S}^{2n+1}} u\mathscr{D}u d\xi \geqslant \frac{n^2}{4}\left(\int_{\mathbb{S}^{2n+1}} |u|^{2Q/(Q-2)} d\xi\right)^{(Q-2)/Q} \tag{1.6}$$

and, for $2 < q < \frac{2Q}{Q-2}$,

$$\frac{4(q-2)}{Q-2}\int_{\mathbb{S}^{2n+1}} u\mathscr{L}u d\xi + \int_{\mathbb{S}^{2n+1}} |u|^2 d\xi \geqslant \left(\int_{\mathbb{S}^{2n+1}} |u|^q d\xi\right)^{2/q}, \tag{1.7}$$

respectively.

What is the Sobolev inequality corresponding to the general intertwining operator $\mathscr{A}_d$?

To answer this question and motivated by the idea "fractional integration controls Sobolev inequality", we establish firstly the following HLS inequalities.

THEOREM 1.1. (Subcritical HLS inequalities) *Let $0 < \lambda < Q = 2n+2$ and $\frac{2Q}{2Q-\lambda} < p \leqslant 2$. Then for any $f, g \in L^p(\mathbb{S}^{2n+1})$, it holds*

$$\left|\int_{\mathbb{S}^{2n+1}}\int_{\mathbb{S}^{2n+1}} \frac{\overline{f(\xi)}g(\eta)}{|1-\xi\cdot\bar{\eta}|^{\lambda/2}} d\xi d\eta\right| \leqslant C_{\lambda,n}\|h\|_p\|g\|_p, \tag{1.8}$$

where

$$C_{\lambda,n} = \int_{\mathbb{S}^{2n+1}} |1-\xi\cdot\bar{\eta}|^{-\lambda/2} d\eta = \frac{\Gamma(Q/2)\Gamma((Q-\lambda)/2)}{\Gamma^2((2Q-\lambda)/4)}.$$

Moreover, Equality in (1.8) holds if and only if f and g are all constants.

REMARK 1.2. When $p = \frac{2Q}{2Q-\lambda}$ for $0 < \lambda < Q$, then (1.8) is the classical HLS inequalities

$$\left| \int_{\mathbb{S}^{2n+1}} \int_{\mathbb{S}^{2n+1}} \frac{\overline{f(\xi)} g(\eta)}{|1-\xi\cdot\overline{\eta}|^{\lambda/2}} d\xi d\eta \right| \leqslant C_{\lambda,n} \|h\|_p \|g\|_p. \tag{1.9}$$

Moreover, by Theorem 2.2 of [6], we know that equality in (1.9) holds if and only if

$$f(\xi) = \frac{c}{|1-\overline{\zeta}\cdot\xi|^{(2Q-\lambda)/2}}, \quad g(\eta) = \frac{c'}{|1-\overline{\zeta}\cdot\xi|^{(2Q-\lambda)/2}} \tag{1.10}$$

for some $c, c' \in \mathbb{C}$ and some $\zeta \in \mathbb{C}^{n+1}$ with $|\zeta| < 1$ (unless $f \equiv 0$ or $g \equiv 0$).

Take $f = g = \sum_{j,k\geqslant 0} Y_{j,k}$ in (1.8) and (1.9). Then, we have by (A.5) that

$$\sum_{j,k\geqslant 0} \gamma_{j,k}^{\lambda} \int_{\mathbb{S}^{2n+1}} |Y_{j,k}|^2 d\xi \leqslant \|f\|_p^2, \quad \frac{2Q}{2Q-\lambda} \leqslant p \leqslant 2. \tag{1.11}$$

By a duality argument and letting $\lambda = Q-d$, we get the following Sobolev inequalities on the $\mathbb{S}^{2n+1}$:

$$\begin{aligned} \|f\|_q^2 &\leqslant \sum_{j,k\geqslant 0} \frac{1}{\gamma_{j,k}^{\lambda}} \int_{\mathbb{S}^{2n+1}} |Y_{j,k}|^2 d\xi \\ &= \sum_{j,k\geqslant 0} \frac{\Gamma(j+(2Q-\lambda)/4)\Gamma(k+(2Q-\lambda)/4)\Gamma^2(\lambda/4)}{\Gamma^2((2Q-\lambda)/4)\Gamma(j+\lambda/4)\Gamma(k+\lambda/4)} \int_{\mathbb{S}^{2n+1}} |Y_{j,k}|^2 d\xi \\ &= \sum_{j,k\geqslant 0} \frac{\Gamma(j+(Q+d)/4)\Gamma(k+(Q+d)/4)\Gamma^2((Q-d)/4)}{\Gamma(j+(Q-d)/4)\Gamma(k+(Q-d)/4)\Gamma^2((Q+d)/4)} \int_{\mathbb{S}^{2n+1}} |Y_{j,k}|^2 d\xi \\ &= \frac{1}{(\lambda_0(d))^2} \int_{\mathbb{S}^{2n+1}} f \mathscr{A}_d f d\xi, \quad 2 \leqslant q \leqslant \frac{2Q}{Q-d}. \end{aligned} \tag{1.12}$$

Particularly, if $d = 2$ and $q = \frac{2Q}{Q-2}$, then (1.12) is Sobolev inequality (1.6). While for $d = 2$ and $2 < q < \frac{2Q}{Q-2}$, we find that the constant $\frac{1}{(\lambda_0(2))^2}$ is strictly bigger than the constant $\frac{4(q-2)}{Q-2}$ of (1.7) and therefore not sharp. Next theorem gives the sharp form of the Sobolev inequalities on the CR-sphere.

THEOREM 1.3. *For any* $f \in C^\infty(\mathbb{S}^{2n+1})$ *and* $0 < d < Q$, *we have:*
1) **Conformal Sobolev inequalities**: *For* $q = \frac{2Q}{Q-d}$,

$$\|f\|_q^2 \leqslant \frac{\Gamma^2((Q-d)/4)}{\Gamma^2((Q+d)/4)} \int_{\mathbb{S}^{2n+1}} f \mathscr{A}_d f d\xi. \tag{1.13}$$

Moreover, equality holds if and only if

$$f(\xi) = c|1-\overline{\zeta}\cdot\xi|^{(d-Q)/2} \tag{1.14}$$

for some $c \in \mathbb{C}$ *and some* $\zeta \in \mathbb{C}^{n+1}$ *with* $|\zeta| < 1$.

2) **Subcritical Sobolev inequalities***: For* $2 \leqslant q < \frac{2Q}{Q-d}$*,*

$$\|f\|_q^2 \leqslant \frac{8(q-2)}{d(Q-d)} \frac{\Gamma^2((Q-d)/4+1)}{\Gamma^2((Q+d)/4)} \left(\int_{\mathbb{S}^{2n+1}} f \mathscr{A}_d f d\xi \right. \\ \left. -(\lambda_0(d))^2 \int_{\mathbb{S}^{2n+1}} |f|^2 d\xi \right) + \int_{\mathbb{S}^{2n+1}} |f|^2 d\xi. \tag{1.15}$$

Moreover, for $2 < q < \frac{2Q}{Q-d}$*, equality holds if and only if* f *is constant.*

REMARK 1.4. The conformal Sobolev inequalities (1.13) and their derivation from Frank and Lieb HLS inequality on the Heisenberg group ([6]) are well known within the group of researchers interested in conformal geometry (see [2] for further details). We provide concise proof for completeness. On the other hand the subcritical Sobolev inequalities (1.15) are new. Their Euclidean counterpart can be found in [1].

REMARK 1.5. When $d = 2$, (1.13) and (1.15) are (1.6) and (1.7), respectively.

Combining the method of Beckner in [1] with the HLS inequality on the Heisenberg group ([6]) and letting $d \to Q^-$, we have the following sharp inequalities.

THEOREM 1.6. *For any* $f \in C^\infty(\mathbb{S}^{2n+1}) \cap \mathbb{R}\mathscr{P}$*, we have:*
1) **Beckner-Onofri's inequality***:*

$$\frac{1}{2(n+1)!} \int_{\mathbb{S}^{2n+1}} f \mathscr{A}_Q' f d\xi + \int_{\mathbb{S}^{2n+1}} f d\xi - \log \int_{\mathbb{S}^{2n+1}} e^f d\xi \geqslant 0; \tag{1.16}$$

2) **Subcritical Sobolev inequalities***: for* $2 \leqslant q < +\infty$*,*

$$\|f\|_q^2 \leqslant \frac{q-2}{(n+1)!} \int_{\mathbb{S}^{2n+1}} f \mathscr{A}_Q' f d\xi + \int_{\mathbb{S}^{2n+1}} |f|^2 d\xi. \tag{1.17}$$

REMARK 1.7. Note that Beckner-Onofri's inequalities (1.16) is the main result of [2]. The authors of [2] were well aware that (1.16) could be derived from (1.13) and they also say how, but [2] was made available as a preprint several years before [6] was published and at the time (1.13) was only a conjecture. So, for conciseness, we omit the proof.

REMARK 1.8. As in [1], by making the substitution $f \to 1 + \frac{1}{q} f$ in (1.17) and taking the limit $q \to +\infty$ for bounded f, we can obtain (1.16) again.

The plan of the paper is as follows. Section 2 is devoted to the proof of Theorem 1.1, Theorem 1.3 and the subcritical case of Theorem 1.6. Our main tools are the Funck-Heck Theorem on the complex sphere and the duality argument. For completeness, in Appendix A, we state the Fuck-Heck theorem established by Frank and Lieb in [6] and give some applications.

2. Proofs of Theorem 1.1, Theorem 1.3 and Theorem 1.6

Proof of Theorem 1.1. 1) **Case** $\frac{2Q}{2Q-\lambda} < p < 2$**.**

Firstly, we claim that, for any λ_1 and λ_2 satisfying $0 < \lambda_1 < \lambda_2 < Q$ and any $f \in L^2(\mathbb{S}^{2n+1})$, it holds

$$\frac{\int_{\mathbb{S}^{2n+1}} \int_{\mathbb{S}^{2n+1}} \frac{\overline{f(\xi)} f(\eta)}{|1-\xi\cdot\overline{\eta}|^{\lambda_1/2}} d\xi d\eta}{\int_{\mathbb{S}^{2n+1}} |1-\xi\cdot\overline{\eta}|^{-\lambda_1/2} d\eta} \leqslant \frac{\int_{\mathbb{S}^{2n+1}} \int_{\mathbb{S}^{2n+1}} \frac{\overline{f(\xi)} f(\eta)}{|1-\xi\cdot\overline{\eta}|^{\lambda_2/2}} d\xi d\eta}{\int_{\mathbb{S}^{2n+1}} |1-\xi\cdot\overline{\eta}|^{-\lambda_2/2} d\eta}. \tag{2.1}$$

Moreover, equality holds if and only if f is constant.

Now, taking $\lambda_1 = \lambda$ and $\lambda_2 = 2Q(1-1/p)$ in (2.1), noting the positivity of the left side of (1.8) and combining with the classical HLS inequalities (1.9), we can complete the proof of Theorem 1.1 for the case $\frac{2Q}{2Q-\lambda} < p < 2$ since $L^2(\mathbb{S}^{2n+1})$ is dense in $L^p(\mathbb{S}^{2n+1})$. Therefore, it is sufficient to prove (2.1).

To prove inequality (2.1), we only need to show $\gamma_{j,k}^{\lambda_1} \leqslant \gamma_{j,k}^{\lambda_2}$, $j,k = 0,1,2,\cdots$ by (A.5).

Obviously, $\gamma_{0,0}^{\lambda_1} = \gamma_{0,0}^{\lambda_2}$. While for $j+k \geqslant 1$, it is easy to see that

$$\gamma_{j,k}^{\lambda} = \frac{\Gamma^2((2Q-\lambda)/4)\Gamma(j+\lambda/4)\Gamma(k+\lambda/4)}{\Gamma(j+(2Q-\lambda)/4)\Gamma(k+(2Q-\lambda)/4)\Gamma^2(\lambda/4)}$$

is strictly increasing with respect to λ. Therefore, (2.1) holds. Moreover, by the decomposition of L^2 function, we know that equality in (2.1) holds if and only if f is a constant.

2) **Case** $p = 2$

Take the spherical harmonic expansion $f(\xi) = \sum_{j,k\geqslant 0} Y_{j,k}(\xi)$ with $Y_{j,k} \in \mathscr{H}_{j,k}$. Then inequality (1.8) is equivalent to

$$\sum_{j,k\geqslant 0} \gamma_{j,k}^{\lambda} \int_{\mathbb{S}^{2n+1}} |Y_{j,k}(\xi)|^2 d\xi \leqslant \sum_{j,k\geqslant 0} \int_{\mathbb{S}^{2n+1}} |Y_{j,k}(\xi)|^2 d\xi.$$

On the other hand, it is easy to obtain that $\gamma_{0,0}^{\lambda} = 1$ and $\gamma_{j,k}^{\lambda} < 1$ for $j+k \geqslant 1$. So, we complete the proof. □

Proof of Part 1) of Theorem 1.3: Conformal Sobolev inequalities.

By (1.11), we know that, for any $g(\xi) = \sum_{j,k\geqslant 0} Y_{j,k}(\xi) \in C^\infty(\mathbb{S}^{2n+1})$,

$$\sum_{j,k\geqslant 0} \gamma_{j,k}^{\lambda} \int_{\mathbb{S}^{2n+1}} |Y_{j,k}|^2 d\xi \leqslant \|g\|_p^2 \quad \text{with} \quad p = \frac{2Q}{2Q-\lambda}. \tag{2.2}$$

So, for any $f(\xi)=\sum_{j,k\geqslant 0}Z_{j,k}(\xi)\in C^\infty(\mathbb{S}^{2n+1})$,

$$\begin{aligned}
&\left|\int_{\mathbb{S}^{2n+1}}\overline{f(\xi)}g(\xi)d\xi\right| = \left|\sum_{j,k\geqslant 0}\int_{\mathbb{S}^{2n+1}}\overline{Z_{j,k}(\xi)}Y_{j,k}(\xi)d\xi\right|\\
\leqslant&\sqrt{\sum_{j,k\geqslant 0}\frac{1}{\gamma_{j,k}^{\lambda}}\int_{\mathbb{S}^{2n+1}}|Z_{j,k}|^2d\xi}\cdot\sqrt{\sum_{j,k\geqslant 0}\gamma_{j,k}^{\lambda}\int_{\mathbb{S}^{2n+1}}|Y_{j,k}|^2d\xi}\\
\leqslant&\|g\|_p\left(\sum_{j,k\geqslant 0}\frac{1}{\gamma_{j,k}^{\lambda}}\int_{\mathbb{S}^{2n+1}}|Z_{j,k}|^2d\xi\right)^{\frac12} = \|g\|_p\left(\frac{1}{(\lambda_0(d))^2}\int_{\mathbb{S}^{2n+1}}f\mathscr{A}_d f d\xi\right)^{\frac12}, \qquad (2.3)
\end{aligned}$$

where $d=Q-\lambda\in(0,Q)$. Because of the arbitrariness of g and the density, we get

$$\|f\|_q^2\leqslant\frac{1}{(\lambda_0(d))^2}\int_{\mathbb{S}^{2n+1}}f\mathscr{A}_d f d\xi \qquad (2.4)$$

for any $f\in L^q(\mathbb{S}^{2n+1})$ and $q=\frac{2Q}{Q-d}$.

A direct computation shows that, if f is defined as in (1.14), then equality in (2.4) holds. So, the constant $\frac{1}{(\lambda_0(d))^2}$ of (2.4) is sharp. In the following we discuss the extremal functions.

Assume nonnegative function $f_0\in L^q(\mathbb{S}^{2n+1})$ be an extremal function of (2.4), i.e.,

$$\|f_0\|_q^2=\frac{1}{(\lambda_0(d))^2}\int_{\mathbb{S}^{2n+1}}f_0\mathscr{A}_d f_0 d\xi. \qquad (2.5)$$

By (2.3), we have

$$|<f_0,g>|\leqslant\|f_0\|_q\|g\|_{q'}\quad\text{with}\quad q'=\frac{2Q}{Q+d}. \qquad (2.6)$$

It is known that there exists some function $g_0\in L^{q'}(\mathbb{S}^{2n+1})$ such that equality in (2.6) holds. Using the property of Hölder inequality, we know that $f_0=cg_0^{\frac{Q-d}{Q+d}}$, where c is some constant. Substituting f_0 and g_0 into (2.3), we find that g_0 is an extremal function of (1.9). So, the extremal function f_0 must have the form (1.14). □

Proof of Part 2) of Theorem 1.3: Subcritical Sobolev inequalities.

Note that case $q=2$ is trivial. Therefore, we assume $2<q<\frac{2Q}{Q-d}$ in the sequel. If

$$\begin{aligned}
&\frac{\Gamma^2((Q-d_1)/4)}{\Gamma^2((Q+d_1)/4)}\int_{\mathbb{S}^{2n+1}}f\mathscr{A}_{d_1}f d\xi\\
\leqslant&\frac{8(q-2)}{d(Q-d)}\frac{\Gamma^2((Q-d)/4+1)}{\Gamma^2((Q+d)/4)}\left(\int_{\mathbb{S}^{2n+1}}f\mathscr{A}_d f d\xi\right.\\
&\left.-(\lambda_0(d))^2\int_{\mathbb{S}^{2n+1}}|f|^2d\xi\right)+\int_{\mathbb{S}^{2n+1}}|f|^2d\xi \qquad (2.7)
\end{aligned}$$

holds for $d_1 = Q(1-2/q)$ and $\frac{2Q}{Q-d} > q > 2$, then we can get (1.15) by combining (1.13). For showing inequality (2.7), by the definition of operator $\mathscr{A}_d$, we need to prove

$$\frac{\lambda_j(d_1)\lambda_k(d_1)}{(\lambda_0(d_1))^2} \leqslant 1 + \frac{8(q-2)}{d(Q-d)} \frac{\Gamma^2((Q-d)/4+1)}{\Gamma^2((Q+d)/4)}(\lambda_j(d)\lambda_k(d) - (\lambda_0(d))^2),$$

for $j,k \geqslant 0$, So, we will prove that, for $j,k \geqslant 0$,

$$\begin{aligned}&\frac{\Gamma(j+\frac{Q}{2q'})\Gamma(k+\frac{Q}{2q'})\Gamma^2(\frac{Q}{2q})}{\Gamma(j+\frac{Q}{2q})\Gamma(k+\frac{Q}{2q})\Gamma^2(\frac{Q}{2q'})}\\ &\leqslant 1+\frac{8(q-2)}{d(Q-d)}\frac{\Gamma^2(\frac{Q-d}{4}+1)}{\Gamma^2(\frac{Q+d}{4})}\left(\frac{\Gamma(j+\frac{Q+d}{4})\Gamma(k+\frac{Q+d}{4})}{\Gamma(j+\frac{Q-d}{4})\Gamma(k+\frac{Q-d}{4})} - \frac{\Gamma^2(\frac{Q+d}{4})}{\Gamma^2(\frac{Q-d}{4})}\right),\end{aligned} \tag{2.8}$$

where q' is the conjugate number of q, i.e., $\frac{1}{q}+\frac{1}{q'}=1$. A direct calculation shows that equality in (2.8) occurs at $(j,k)=(0,0),(1,0)$ or $(0,1)$.

To prove (2.8), we differentiate with respect to j and k. If the left derivation is less than the right for $j+k \geqslant 1$, then we can deduce (2.8) for all $j,k \geqslant 0$ from the monotonicity. In fact,

$$\begin{aligned}&\frac{\partial}{\partial k}\left(\frac{\Gamma(j+\frac{Q}{2q'})\Gamma(k+\frac{Q}{2q'})\Gamma^2(\frac{Q}{2q})}{\Gamma(j+\frac{Q}{2q})\Gamma(k+\frac{Q}{2q})\Gamma^2(\frac{Q}{2q'})}\right)\\ &=\frac{\Gamma(j+\frac{Q}{2q'})\Gamma(k+\frac{Q}{2q'})\Gamma^2(\frac{Q}{2q})}{\Gamma(j+\frac{Q}{2q})\Gamma(k+\frac{Q}{2q})\Gamma^2(\frac{Q}{2q'})}\left(\frac{\Gamma'(k+\frac{Q}{2q'})}{\Gamma(k+\frac{Q}{2q'})}-\frac{\Gamma'(k+\frac{Q}{2q})}{\Gamma(k+\frac{Q}{2q})}\right)\\ &=\frac{\Gamma(j+\frac{Q}{2q'})\Gamma(k+\frac{Q}{2q'})\Gamma^2(\frac{Q}{2q})}{\Gamma(j+\frac{Q}{2q})\Gamma(k+\frac{Q}{2q})\Gamma^2(\frac{Q}{2q'})}\sum_{l=0}^{+\infty}\left(\frac{1}{k+\frac{Q}{2q}+l}-\frac{1}{k+\frac{Q}{2q'}+l}\right)\\ &=\frac{\Gamma(j+\frac{Q}{2q'})\Gamma(k+\frac{Q}{2q'})\Gamma^2(\frac{Q}{2q})\frac{Q}{2q}}{\Gamma(j+\frac{Q}{2q})\Gamma(k+\frac{Q}{2q})\Gamma^2(\frac{Q}{2q'})}\sum_{l=0}^{+\infty}\frac{q-2}{(l+k)^2+\frac{Q}{2}(l+k)+(\frac{Q}{2})^2\frac{1}{q}\frac{1}{q'}},\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}&\frac{\partial}{\partial k}\left[1+\frac{8(q-2)}{d(Q-d)}\frac{\Gamma^2(\frac{Q-d}{4}+1)}{\Gamma^2(\frac{Q+d}{4})}\left(\frac{\Gamma(j+\frac{Q+d}{4})\Gamma(k+\frac{Q+d}{4})}{\Gamma(j+\frac{Q-d}{4})\Gamma(k+\frac{Q-d}{4})}-\frac{\Gamma^2(\frac{Q+d}{4})}{\Gamma^2(\frac{Q-d}{4})}\right)\right]\\ &=\frac{8(q-2)}{d(Q-d)}\frac{\Gamma^2(\frac{Q-d}{4}+1)}{\Gamma^2(\frac{Q+d}{4})}\frac{\Gamma(j+\frac{Q+d}{4})\Gamma(k+\frac{Q+d}{4})}{\Gamma(j+\frac{Q-d}{4})\Gamma(k+\frac{Q-d}{4})}\left(\frac{\Gamma'(k+\frac{Q+d}{4})}{\Gamma(k+\frac{Q+d}{4})}-\frac{\Gamma'(k+\frac{Q-d}{4})}{\Gamma(k+\frac{Q-d}{4})}\right)\\ &=\frac{\frac{Q-d}{4}\Gamma^2(\frac{Q-d}{4})}{\Gamma^2(\frac{Q+d}{4})}\frac{\Gamma(j+\frac{Q+d}{4})\Gamma(k+\frac{Q+d}{4})}{\Gamma(j+\frac{Q-d}{4})\Gamma(k+\frac{Q-d}{4})}\sum_{l=0}^{+\infty}\frac{q-2}{(l+k)^2+\frac{Q}{2}(l+k)+\frac{Q-d}{4}\frac{Q+d}{4}}.\end{aligned} \tag{2.10}$$

Combining the facts: $\frac{d}{dx}\frac{\Gamma(l+x)}{\Gamma(x)} \geqslant 0$ for $x>0$ and $l \geqslant 0$, $\frac{d}{dx}\frac{\Gamma(l+x)}{\Gamma(1+x)} \geqslant 0$ for $x>0$ and

$l \geqslant 1$, and $\frac{2Q}{Q+d} < q' < 2 < q < \frac{2Q}{Q-d}$, we have, for $j+k \geqslant 1$

$$\begin{cases} \frac{\Gamma(j+\frac{Q}{2q'})\Gamma(k+\frac{Q}{2q'})}{\Gamma^2(\frac{Q}{2q'})} \leqslant \frac{\Gamma(j+\frac{Q+d}{4})\Gamma(k+\frac{Q+d}{4})}{\Gamma^2(\frac{Q+d}{4})}, \\ \frac{\Gamma(j+\frac{Q}{2q})\Gamma(k+\frac{Q}{2q})}{\frac{Q}{2q}\Gamma^2(\frac{Q}{2q})} \geqslant \frac{\Gamma(j+\frac{Q-d}{4})\Gamma(k+\frac{Q-d}{4})}{\frac{Q-d}{4}\Gamma^2(\frac{Q-d}{4})}. \end{cases} \tag{2.11}$$

Moreover, since $f(x) = x(1-x)$ is strictly increasing on $[0, \frac{1}{2}]$, then

$$\frac{Q-d}{2Q} \cdot \frac{Q+d}{2Q} = f(\frac{Q-d}{2Q}) < f(\frac{1}{q}) = \frac{1}{q} \cdot \frac{1}{q'},$$

which implies that

$$\frac{q-2}{(l+k)^2 + \frac{Q}{2}(l+k) + (\frac{Q}{2})^2 \frac{1}{q}\frac{1}{q'}} \leqslant \frac{q-2}{(l+k)^2 + \frac{Q}{2}(l+k) + \frac{Q-d}{4}\frac{Q+d}{4}} \tag{2.12}$$

for $k \geqslant 0$ and $l \geqslant 0$. So the k derivative of the LHS of (2.8) is less of the one of the RHS and, the same is true for the j derivative. Then, we get (2.8).

From the above proof, we know that equality of (2.8) occurs only at $(j,k) = (0,0)$, $(1,0)$ or $(0,1)$. Therefore, equality of (2.7) holds if and only if

$$f \in \mathscr{H}_{0,0} \bigoplus \mathscr{H}_{0,1} \bigoplus \mathscr{H}_{1,0}.$$

Combining the extremal result of (1.14), we know that equality of (1.15) for $2 < q < \frac{2Q}{Q-d}$ holds if and only if f is constant. □

Proof of part 2) of Theorem 1.6: Subcritical Sobolev inequalities.

For any $Y_{j0} \in \mathscr{H}_{j0}$, $j = 0, 1, 2, \cdots$, we have, as $d \to Q^-$,

$$\begin{aligned} &\frac{8(q-2)}{d(Q-d)} \frac{\Gamma^2((Q-d)/4+1)}{\Gamma^2((Q+d)/4} \mathscr{A}_d Y_{j0} \\ =&\frac{2(q-2)}{d} \frac{\frac{Q+d}{4}(\frac{Q+d}{4}+1)\cdots(\frac{Q+d}{4}+j-1)}{(\frac{Q-d}{4}+1)\cdots(\frac{Q-d}{4}+j-1)} Y_{j0} \\ \to&(q-2)\frac{j(j+1)\cdots(j+n)}{(n+1)!} Y_{j0} = \frac{q-2}{(n+1)!} \mathscr{A}'_Q Y_{j0}. \end{aligned}$$

Similarly, the above result holds for any $Y_{0k} \in \mathscr{H}_{0k}$, $k = 0, 1, 2, \cdots$. On the other hand, we have

$$\lambda_0(d) \to 0, \quad \text{as} \quad d \to Q^-.$$

So, we get (1.17) via letting $d \to Q^-$ in (1.15).

A. Appendix The Funk-Hecke Theorem on the complex sphere

In [6], Frank and Lieb established the following two results. Notice that, in the following formulas, the factor $|\mathbb{S}^{2n+1}|$ appears in the denominators because we use the normalized surface measure.

PROPOSITION A.1. (Proposition 5.2 of [6]) *Let K be an integrable function on the unit ball in $\mathbb{C}$. Then the operator on $\mathbb{S}^{2n+1}$ with kernel $K(\xi\cdot\overline{\eta})$ is diagonal with respect to decomposition (1.1), and on the space $\mathscr{H}_{j,k}$ its eigenvalue is given by*

$$\begin{aligned}\frac{1}{|\mathbb{S}^{2n+1}|}\frac{\pi^n m!}{2^{n+|j-k|/2}(m+n-1)!}\int_{-1}^{1} dt(1-t)^{n-1}(1+t)^{|j-k|/2}P_m^{(n-1,|j-k|)}(t)\\ \times\int_{-\pi}^{\pi} d\varphi K(e^{-i\varphi}\sqrt{(1+t)/2})e^{i(j-k)\varphi},\end{aligned} \tag{A.1}$$

where $m:=\min\{j,k\}$ and $P_m^{(\alpha,\beta)}$ are the Jacobi polynomials.

PROPOSITION A.2. (Corollary 5.3 of [6]) *Let $-1<\alpha<\frac{n+1}{2}$.*

(1) The eigenvalue of the operator with kernel $|1-\xi\cdot\overline{\eta}|^{-2\alpha}$ on the subspace $\mathscr{H}_{j,k}$ is

$$E_{j,k}:=\frac{2\pi^{n+1}\Gamma(n+1-2\alpha)}{|\mathbb{S}^{2n+1}|\Gamma^2(\alpha)}\frac{\Gamma(j+\alpha)}{\Gamma(j+n+1-\alpha)}\frac{\Gamma(k+\alpha)}{\Gamma(k+n+1-\alpha)}. \tag{A.2}$$

(2)The eigenvalue of the operator with kernel $|\xi\cdot\overline{\eta}|^2|1-\xi\cdot\overline{\eta}|^{-2\alpha}$ on the subspace $\mathscr{H}_{j,k}$ is

$$E_{j,k}\left(1-\frac{(\alpha-1)(n+1-2\alpha)(2jk+n(j+k-1+\alpha)}{(j-1+\alpha)(j+n+1-\alpha)(k-1+\alpha)(k+n+1-\alpha)}\right). \tag{A.3}$$

When $\alpha=0$ or 1, formula (A.3) and (A.2) are to be understood by taking limits with fixed j and k.

As application, we have the following result.

PROPOSITION A.3. *For $0<\lambda<Q$, we have*

$$\int_{\mathbb{S}^{2n+1}}|1-\xi\cdot\bar{\eta}|^{-\lambda/2}d\eta=\frac{\Gamma(Q/2)\Gamma((Q-\lambda)/2)}{\Gamma^2((2Q-\lambda)/4)}. \tag{A.4}$$

For $f(\xi)=\sum_{j,k\geqslant 0}Y_{j,k}$ with $Y_{j,k}\in\mathscr{H}_{j,k}$, then

$$\frac{\int_{\mathbb{S}^{2n+1}}\int_{\mathbb{S}^{2n+1}}\frac{f(\xi)f(\eta)}{|1-\xi\cdot\bar{\eta}|^{\lambda/2}}d\xi d\eta}{\int_{\mathbb{S}^{2n+1}}|1-\xi\cdot\bar{\eta}|^{-\lambda/2}d\eta}=\sum_{j,k\geqslant 0}\gamma_{j,k}^{\lambda}\int_{\mathbb{S}^{2n+1}}|Y_{j,k}(\xi)|^2d\xi \tag{A.5}$$

with

$$\gamma_{j,k}^{\lambda}=\frac{\Gamma^2((2Q-\lambda)/4)\Gamma(j+\lambda/4)\Gamma(k+\lambda/4)}{\Gamma(j+(2Q-\lambda)/4)\Gamma(k+(2Q-\lambda)/4)\Gamma^2(\lambda/4)},\quad j,k=0,1,2,\cdots.$$

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