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SOME GENERALIZATIONS AND COMPLEMENTS OF DETERMINANTAL INEQUALITIES

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Abstract. K. Audenaert in [1] formulated a determinantal inequality arising from diffusion tensor imaging. Very recently M. Lin proved in [6] a complement and proposed a conjecture. In this short note, we generalize his conjecture and we prove it in a wild case, when the matrix is singular. We also present a refinement of the complement found by Lin and finally we present a series of determinantal inequalities followed by a conjecture.

1. Introduction

Audenaert formulated in his work [1] the following inequality for all $A,B \geqslant 0$ of same size $n \geqslant 1$,

$$\det(A^2 + |BA|) \leqslant \det(A^2 + AB) \tag{1}$$

He proved it in order to get the following determinantal inequality arising from diffusion tensor imaging

$$\det(A + U^*B) \leqslant \det(A + B)$$

where A and B are two n-square positive semi-definite matrices and U is a specified n-square unitary matrix arising from the polar decomposition of the matrix BA. Throughout this paper, let M_n be the space of $n \times n$ complex matrices. I_n denotes the identity matrix in M_n. The modulus of a complex matrix X is the unique positive semi-definite square root of the X^*X denoted by $|X| = (X^*X)^{1/2}$. For $X,Y \in M_n$ Hermitian matrices we say $X \geqslant Y$ if $X - Y$ is positive semi-definite matrix. The spectrum of X is the multiset of the eigenvalues of X denoted by $Sp(X)$, we can simply rearrange the eigenvalues of X in decreasing order if they are all real, that is

$$\lambda_1(X) \geqslant \lambda_2(X) \geqslant \ldots \geqslant \lambda_n(X).$$

For every $X \in M_n$ Hermitian we have $\lambda(X) = (\lambda_1(X), \lambda_2(X), \ldots, \lambda_n(X))^t$ is a real vector of order n. The spectral norm of a $X \in M_n$ is defined by $|||X|||_{op} = \rho^{1/2}(X^*X)$ where

$$\rho(X) = \sup_{\lambda \in Sp(X)} |\lambda(X)|.$$

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Let us recall some definitions of majorizations: for a vector $x \in \mathbb{R}^n$, the vector obtained after rearranging the components of x in decreasing order is denoted by $x = (x_1^{\downarrow}, x_2^{\downarrow}, \dots, x_n^{\downarrow})^t$, we say $x \in \mathbb{R}^n$ is weakly log majorized by $y \in \mathbb{R}^n$ denoted by $x \prec_{w,log} y$ if

$$\prod_{i=1}^{k}(x_i^{\downarrow}) \leqslant \prod_{i=1}^{k}(y_i^{\downarrow}) \qquad k = 1,2,\dots,n \tag{2}$$

and x is log majorized by y ($x \prec_{log} y$) if (2) is true and equality holds for $k = n$.

M. Lin proved in [6] a complement and a generalization for (1),

$$\det(A^2 + |AB|) \geqslant \det(A^2 + AB) \tag{3}$$

and

$$\det(A^2 + |BA|^p) \leqslant \det(A^2 + A^p B^p) \qquad 0 \leqslant p \leqslant 2 \tag{4}$$

And he also introduced the following conjecture.

CONJECTURE 1.1. Let A, B be two positive semi-definite matrices. Then

$$\det(A^2 + |AB|^p) \geqslant \det(A^2 + A^p B^p), \qquad 0 \leqslant p \leqslant 2$$

In this paper, we will show determinantal inequalities that are inspired by (1), (3) and (4).

2. Main Results

The main result in this paper are the following:

1. Let A, B be two positive semi-definite matrices. Then

$$\det(A^{kp} + |BA|^p) \leqslant \det(A^{kp} + A^p B^p) \qquad k \geqslant 1, \quad 0 \leqslant p \leqslant 2 \tag{5}$$

2. Let A, B be two positive semi-definite matrices. Then, for all $0 \leqslant p \leqslant 2$,
   - $\det(A^p + |BA|^p) \leqslant \det(A^p + A^p B^p)$.
   - $\det(I_n + |BA|^p) \geqslant \det(I_n + A^p B^p)$.

3. Let A, B be two positive semi-definite matrices. Then, for all $p \geqslant 2$,
   - $\det(A^p + |BA|^p) \geqslant \det(A^p + A^p B^p)$.
   - $\det(I_n + |BA|^p) \leqslant \det(I_n + A^p B^p)$.

4. Let A, B be two n-square hermitian matrices. Then

$$\det(A^4 + |AB|^2) \geqslant \det(A^4 + A^2 B^2).$$

5. Let A,B be two positive semi-definite matrices. Then for every $k \geqslant 1$

$$\det(A^k + |AB|) \geqslant \det(A^k + AB).$$

6. Let A,B be two positive semi-definite matrices. Then,

$$\det(A^2 + |BA|^2) = \det(A^2 + A^2B^2) \leqslant \det(A^2 + (AB)^2) \leqslant \det(A^2 + |AB|^2).$$

We remark that for two positive semi-definite matrices A and B with A singular, the following general result holds

$$\det(A^k + |BA|^p) = \det(A^k + A^pB^p) \text{ and } \det(A^k + |AB|^p) \geqslant \det(A^k + A^pB^p)$$

for all $k > 0$, $p \geqslant 0$. Which gives a partial answer of the positivity of Lin's conjecture.

We can find a generalization for (4), to do this we need the following lemmas where the proof of the first one is in [6] and the proof of the second is a corollary of Furuta's inequality and it can be found in [3, p. 128] and the third lemma proved in the interesting reference [5].

LEMMA 1. *If $\lambda(A), \lambda(B) \in \mathbb{R}^n_+$ such that $\lambda(A) \prec_{w,log} \lambda(B)$ then*

$$\det(I_n + A) \leqslant \det(I_n + B).$$

LEMMA 2. *Let A, B be two positive semi-definite matrices such that $A \geqslant B$. Then, for all $p \geqslant 1$, $r \geqslant 0$,*

$$A^{(p+2r)/p} \geqslant (A^rB^pA^r)^{1/p}.$$

LEMMA 3. *Let X and Y be two positive semi-definite matrices. Then for every unitarily invariant norm, we have*

$$|||X^tY^tX^t||| \leqslant |||(XYX)^t||| \qquad 0 \leqslant t \leqslant 1 \tag{6}$$

and

$$|||X^tY^tX^t||| \geqslant |||(XYX)^t||| \qquad t \geqslant 1 \tag{7}$$

The following theorem is one of our main result.

THEOREM 1. *Let A,B be two positive semi-definite matrices. Then for all $0 \leqslant p \leqslant 2$ and for all $k \geqslant 1$,*

$$\det(A^{kp} + |BA|^p) \leqslant \det(A^{kp} + A^pB^p) \tag{8}$$

Proof. It is enough to prove the result for $k > 1$ as the case $k = 1$ is followed by limit argument. Assume that A is invertible, for A singular the inequality is true. First we need to prove that, for all $0 \leqslant t \leqslant 1$ and $k > 1$, we have

$$\lambda(A^{kt/2}(A^{-1/2}BA^{-1/2})^t A^{kt/2}) \prec_{log} \lambda(A^{(k-1)t}B^t). \tag{9}$$

To achieve (9), it is enough to show that

$$A^{\frac{(k-1)t}{2}} B^t A^{\frac{(k-1)t}{2}} \leqslant I_n \Rightarrow A^{kt/2}(A^{-1/2}BA^{-1/2})^t A^{kt/2} \leqslant I_n.$$

Assume that $A^{\frac{(k-1)t}{2}} B^t A^{\frac{(k-1)t}{2}} \leqslant I_n$, so $0 \leqslant B^t \leqslant A^{-(k-1)t}$ and by applying Lemma 2, we get

$$A^{-(k-1)t(p+2r)/p} \geqslant \left(A^{-(k-1)tr} B^{tp} A^{-(k-1)tr}\right)^{1/p}.$$

Now, by replacing p with $\frac{1}{t} \geqslant 1$ and r with $\frac{1}{2(k-1)t} > 0$ we obtain

$$A^{-(k-1)t^2\left(\frac{1}{t}+\frac{1}{(k-1)t}\right)} \geqslant \left(A^{-1/2}BA^{-1/2}\right)^t$$

which implies

$$A^{-kt} \geqslant \left(A^{-1/2}BA^{-1/2}\right)^t.$$

Therefore $A^{kt/2}(A^{-1/2}BA^{-1/2})^t A^{kt/2} \leqslant I_n$.

Let $a = \lambda_1(A^{\frac{(k-1)t}{2}} B^t A^{\frac{(k-1)t}{2}})$. If $a = 0$, then it is obvious that (9) is true. If $a > 0$, we observe that

$$A^{\frac{(k-1)t}{2}} B^t A^{\frac{(k-1)t}{2}} \leqslant a\, I_n \quad \text{and} \quad \left(\frac{1}{a^{1/kt}}A\right)^{\frac{(k-1)t}{2}} \left(\frac{1}{a^{1/kt}}B\right)^t \left(\frac{1}{a^{1/kt}}A\right)^{\frac{(k-1)t}{2}} \leqslant I_n.$$

This yields

$$\left(\frac{1}{a^{1/kt}}A\right)^{kt/2} \left[\left(\frac{1}{a^{1/kt}}A\right)^{-1/2} \left(\frac{1}{a^{1/kt}}B\right) \left(\frac{1}{a^{1/kt}}A\right)^{-1/2}\right]^t \left(\frac{1}{a^{1/kt}}A\right)^{kt/2} \leqslant I_n.$$

Thus

$$A^{kt/2}(A^{-1/2}BA^{-1/2})^t A^{kt/2} \leqslant a\, I_n.$$

And hence

$$\lambda_1\left(A^{kt/2}(A^{-1/2}BA^{-1/2})^t A^{kt/2}\right) \leqslant \lambda_1\left(A^{\frac{(k-1)t}{2}} B^t A^{\frac{(k-1)t}{2}}\right) \tag{10}$$

Now, using the antisymmetric tensor product, we have

$$\wedge^s(A^{kt/2}(A^{-1/2}BA^{-1/2})^t A^{kt/2}) = (\wedge^s A)^{kt/2}\left((\wedge^s A)^{-1/2}(\wedge^s B)(\wedge^s A)^{-1/2}\right)^t (\wedge^s A)^{kt/2}$$

and

$$\wedge^s\left(A^{\frac{(k-1)t}{2}}B^tA^{\frac{(k-1)t}{2}}\right) = (\wedge^s A)^{\frac{(k-1)t}{2}}(\wedge^s B)^t(\wedge^s A)^{\frac{(k-1)t}{2}} \quad \text{for } 1 \leqslant s \leqslant n.$$

Replacing A and B with $\wedge^s A$ and $\wedge^s B$ respectively in (10) yields

$$\lambda_1\left(\wedge^s(A^{kt/2}(A^{-1/2}BA^{-1/2})^tA^{kt/2})\right) \leqslant \lambda_1\left(\wedge^s(A^{\frac{(k-1)t}{2}}B^tA^{\frac{(k-1)t}{2}})\right).$$

And so, for all $1 \leqslant s \leqslant n-1$, we have

$$\prod_{i=1}^{s}\lambda_i\left(A^{kt/2}(A^{-1/2}BA^{-1/2})^tA^{kt/2}\right) \leqslant \prod_{i=1}^{s}\lambda_i\left(A^{\frac{(k-1)t}{2}}B^tA^{\frac{(k-1)t}{2}}\right).$$

And as in general $\det\left(A^{kt/2}(A^{-1/2}BA^{-1/2})^tA^{kt/2}\right) = \det\left(A^{\frac{(k-1)t}{2}}B^tA^{\frac{(k-1)t}{2}}\right)$, we obtain

$$\prod_{i=1}^{n}\lambda_i\left(A^{kt/2}(A^{-1/2}BA^{-1/2})^tA^{kt/2}\right) = \prod_{i=1}^{n}\lambda_i\left(A^{\frac{(k-1)t}{2}}B^tA^{\frac{(k-1)t}{2}}\right).$$

and by consequently,

$$\lambda(A^{kt/2}(A^{-1/2}BA^{-1/2})^tA^{kt/2}) \prec_{log} \lambda(A^{(k-1)t}B^t).$$

By applying Lemma 1, we get

$$\det(I_n + A^{kt/2}(A^{-1/2}BA^{-1/2})^tA^{kt/2}) \leqslant \det(I_n + A^{(k-1)t}B^t).$$

Now taking $A = A^{-2}$, $B = B^2$ and $t = p/2$ yields

$$\det(I_n + A^{-kp/2}(AB^2A)^{p/2}A^{-kp/2}) \leqslant \det(I_n + A^{-kp+p}B^p) \tag{11}$$

Pre-post multiplying both sides of (11) by $\det(A^{kp/2})$ leads to the result for $k > 1$ and $0 < p \leqslant 2$. Finally, it is easy to see that (8) is true for $p = 0$. □

The next theorem shows some reverse inequalities.

THEOREM 2. *Let A and B be two positive semi-definite matrices. Then for $0 \leqslant p \leqslant 2$,*

$$\det(I_n + |BA|^p) \geqslant \det(I_n + A^pB^p) \tag{12}$$

Proof. Take $Y = B^2$, $X = A$ and the spectral norm in (6) gives

$$\lambda_1(B^tA^{2t}B^t) \leqslant \lambda_1((BA^2B)^t) \qquad 0 \leqslant t \leqslant 1 \tag{13}$$

Note that $\wedge^s(A^tB^{2t}A^t) = (\wedge^s A)^t(\wedge^s B)^{2t}(\wedge^s A)^t$ and $\wedge^s((AB^2A)^t) = \left(\wedge^s A(\wedge^s B)^2 \wedge^s A\right)^t$.

Replacing A and B with $\wedge^s A$ and $\wedge^s B$ respectively in (13) yields

$$\lambda_1(\wedge^s(A^tB^{2t}A^t)) \leqslant \lambda_1(\wedge^s((AB^2A)^t)).$$

And as $\det((AB^2A)^t) = \det(A^tB^{2t}A^t)$, then for $0 \leqslant t \leqslant 1$

$$\lambda(A^tB^{2t}A^t) \prec_{log} \lambda((AB^2A)^t).$$

Assume that A is positive definite matrix. For $t = p/2$ and by Lemma 1 we get

$$\det(I_n + A^{p/2}B^pA^{p/2}) \leqslant \det(I_n + |BA|^p)$$

Therefore we get the desired for A positive semi-definite matrix by continuity argument. $\square$

With a similar proof using (7), we can get the following.

THEOREM 3. *Let A,B be two positive semi-definite matrices. Then for all $p \geqslant 2$,*

- $\det(A^p + |BA|^p) \geqslant \det(A^p + A^pB^p)$.
- $\det(I_n + |BA|^p) \leqslant \det(I_n + A^pB^p)$.

We can find a more general complement for (8) when $k = 2$ and $p = 2$ as the following theorem shows.

THEOREM 4. *Let A,B be two n-square hermitian matrices. Then*

$$\det(A^4 + |AB|^2) \geqslant \det(A^4 + A^2B^2).$$

Proof. Again, assume that A is an invertible matrix, the case of A singular is true by continuity argument. It is well known, in [7, p. 352], that if a matrix $M = \begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix} \in \mathbb{M}_{n+n}(\mathbb{C})$ is positive semi-definite, then

$$|\lambda(Y)| \prec_{w,log} \lambda^{\frac{1}{2}}(X) \circ \lambda^{\frac{1}{2}}(Z).$$

where $X \circ Y$ represents the Hadamard product of the two matrices X and Y.

Replacing $X = BA^{-2}B$, $Y = B^2A^{-2}$ and $Z = A^{-2}BA^2BA^{-2}$, and by using Schur's complement we get $M = \begin{pmatrix} BA^{-2}B & B^2A^{-2} \\ A^{-2}B^2 & A^{-2}BA^2BA^{-2} \end{pmatrix} \geqslant 0$. Thus

$$|||B^2A^{-2}|||_{op}^2 \leqslant |||BA^{-2}B|||_{op} \cdot |||A^{-2}BA^2BA^{-2}|||_{op}.$$

If $|||B^2A^{-2}|||_{op} = 0$, then the desired determinantal inequality is true.

Noticing that $|||B^2A^{-2}|||_{op} = |||BA^{-2}B|||_{op}$, and dividing both sides by $|||B^2A^{-2}|||_{op} > 0$ gives

$$|||A^{-1}B^2A^{-1}|||_{op} = |||B^2A^{-2}|||_{op} \leqslant |||A^{-2}BA^2BA^{-2}|||_{op}$$

which is

$$\lambda_1(A^{-1}B^2A^{-1}) \leqslant \lambda_1(A^{-2}BA^2BA^{-2}). \tag{14}$$

Observe that $\wedge^s(A^{-1}B^2A^{-1}) = (\wedge^s A)^{-1}(\wedge^s B)^2(\wedge^s A)^{-1}$,

$$\wedge^s(A^{-2}BA^2BA^{-2}) = (\wedge^s A)^{-2}(\wedge^s B)(\wedge^s A)^2(\wedge^s B)(\wedge^s A)^{-2}$$

and replacing A with $\wedge^s A$ and B with $\wedge^s B$ in (14) yields

$$\lambda_1(\wedge^s(A^{-1}B^2A^{-1})) \leqslant \lambda_1(\wedge^s(A^{-2}BA^2BA^{-2})), \qquad 1 \leqslant s \leqslant n-1.$$

Also as $\det(A^{-1}B^2A^{-1}) = \det(A^{-2}BA^2BA^{-2})$ we obtain

$$\lambda(A^{-1}B^2A^{-1}) \prec_{log} \lambda(A^{-2}BA^2BA^{-2}).$$

Using Lemma 1 gives

$$\det(I_n + A^{-1}B^2A^{-1}) \leqslant \det(I_n + A^{-2}BA^2BA^{-2}).$$

Pre-post multiplying by $\det(A^2)$ both sides yields

$$\det(A^4 + A^2B^2) \leqslant \det(A^4 + BA^2B) = \det(A^4 + |AB|^2). \qquad \square$$

We may ask whether the following conjecture is true

CONJECTURE 2.1. Let A and B be two positive semi-definite matrices. Then

$$\det(A^k + A^2B^2) \leqslant \det(A^k + (AB)^2) \text{ for all } k \geqslant 1 \tag{15}$$

If (15) is true we get

$$\det(A^{k'} + A^2B^2) \leqslant \det(A^{k'} + |AB|^2) \text{ for all } k' \geqslant 1 \tag{16}$$

Also, if (16) is true then (15) is true.

The inequality (15) is true for $k = 1, 3$

- When $k = 1$ we have

$$\begin{aligned}\det(A + (AB)^2) &= \det(A) \cdot \det(I_n + BAB) \\ &= \det(A) \cdot \det(I_n + AB^2) \\ &= \det(A + A^2B^2)\end{aligned}$$

- When $k=3$ we have

$$\begin{aligned}\det(A^3+(AB)^2) &= \det(A)\cdot\det((A^{1/2})^4+|A^{1/2}B|^2)\\ &\geqslant \det(A)\cdot\det((A^{1/2})^4+(A^{1/2})^2B^2) \quad \text{(using Theorem 4)}\\ &= \det(A^3+A^2B^2)\end{aligned}$$

The inequality (15) is not valid for $k<1$ as the following example shows.

EXAMPLE 1. For $A=\begin{pmatrix}4&0\\0&1\end{pmatrix}$, $B=\begin{pmatrix}1&1\\1&2\end{pmatrix}$ and $k=1/2$, we have

$$\det(A^{1/2}+A^2B^2)=60>\det(A^{1/2}+(AB)^2)=54.$$

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