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SOME PROPERTIES OF THE GENERALIZED GAUSSIAN RATIO AND THEIR APPLICATIONS

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Abstract. We are devoted to an integral, asymptotic expansion and Maclaurin series representation for the generalized Gaussian ratio, and find their various related properties such as complete monotonicity and some useful inequalities. As applications, several simple approximations for its inverse function are presented, which may be essential to the estimations for the shape parameter of the generalized Gaussian distribution.

1. Introduction

Recall that a function f is called completely monotonic (for short, CM) on an interval I if f has derivatives of all orders on I and satisfies

$$(-1)^k (f(x))^{(k)} \geqslant 0$$

for all $k \geqslant 0$ on I (see [4], [29]). A positive function f is called logarithmically completely monotonic (for short, LCM) on an interval I if f has derivatives of all orders on I and its logarithm $\ln f$ satisfies

$$(-1)^k (\ln f(x))^{(k)} \geqslant 0$$

for all $k \in \mathbb{N}$ on I (see [3], [23]). For convenience, we denote the sets of the completely and logarithmically completely monotonic functions on I by $\mathscr{C}[I]$ and $\mathscr{L}[I]$, respectively.

The celebrated Bernstein theorem [29, p. 161, Theorem 12b] showed that $f(x)$ is completely monotonic for $0 < x < \infty$, if and only if

$$f(x) = \int_0^\infty e^{-xt} d\mu(t),$$

where $\mu(t)$ is nondecreasing and the integral converges for $0 < x < \infty$. We would like to remark that a continuous function $f : (0,\infty) \to [0,\infty)$ is said to be a Bernstein function if f is of class $\mathbf{C}^\infty$ and

$$(-1)^{k-1} (f(x))^{(k)} \geqslant 0$$

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for all $k \in \mathbb{N}$ and $x > 0$ (see [25, Chapter 3]). Clearly, f is a Bernstein function if and only if it is nonnegative, and f' is a completely monotone function.

On the other hand, the Euler's gamma function Γ is defined by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \tag{1}$$

for $x > 0$, and its logarithmic derivative $\psi(x) = \Gamma'(x)/\Gamma(x)$ is known as the psi or digamma function, while ψ', ψ'', ... are called polygamma functions.

Since the 1980s, the complete monotonicity of certain ratios of gamma functions has been investigated widely and deeply, see for example, [12], [7], [23], [2], [17],[6] [19], [20], [24], [22], [34], [35], [8], [32], [33]. In particular, the following ratio of gamma functions

$$T(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma((x+y)/2)^2}, \qquad \forall x,y > 0$$

is called the Gurland ratio [11], whose property can be found in [17]. An interesting relation between $T(u,v)$ and the modified Bessel functions $I_v(x)$ and $K_v(x)$ was established in recent papers [36], [37]. Additionally, in the field of probability theory and its applications, the ratio

$$T(x,x+2u) = \frac{\Gamma(x)\Gamma(x+2u)}{\Gamma(x+u)^2}, \qquad \forall x, x+2u > 0$$

is related to the variance of an estimator involving gamma distribution, which satisfies the inequality

$$T(x,x+2u) = \frac{\Gamma(x)\Gamma(x+2u)}{\Gamma(x+u)^2} > 1 + \frac{u^2}{x} \tag{2}$$

for any $x, x+2u > 0$ (see [11]).

While the ratio

$$\mathscr{M}(p) = \frac{\Gamma(1/p)\Gamma(3/p)}{\Gamma(2/p)^2} = T\left(\frac{1}{p}, \frac{3}{p}\right), \quad p > 0$$

appears in the form of ratio of the variance and the squared absolute expectation for the generalized Gaussian random variable with a shape parameter p, and also known as Mallat ratio [16].

It is also well known that the kurtosis ratio is defined by the ratio of the fourth moment and variance squared. Therefore, the kurtosis ratio of the generalized Gaussian random variable with the shape parameter p can be expressed by

$$\mathscr{K}(p) = \frac{\Gamma(1/p)\Gamma(5/p)}{\Gamma(3/p)^2} = T\left(\frac{1}{p}, \frac{5}{p}\right), \quad p > 0.$$

The above two ratios $\mathscr{M}(p)$ and $\mathscr{K}(p)$ have interesting applications in the domain of image recognition and signal processing, etc. for more details see [27], [18], [26], [9], [14], [10], [15]. Some related properties involved $T(1/p, 3/p)$ can be found in [17], [8].

Clearly, the square root concerning Mallat ratio and kurtosis ratio has the form of

$$\mathscr{R}(x) \equiv \mathscr{R}_{a,w}^{[n_0]}(x) = \frac{\prod_{j=1}^{n_0} \Gamma(xa_j)^{w_j}}{\Gamma\left(x\sum_{j=1}^{n_0} a_j w_j\right)}, \tag{3}$$

where $x = 1/p > 0$, $a_j, w_j > 0$ for $1 \leqslant j \leqslant n_0$ with $\sum_{j=1}^{n_0} w_j = 1$ and $\max_{1\leqslant j\leqslant n_0}(a_j) \neq \min_{1\leqslant j\leqslant n_0}(a_j)$. In the context, we call it to be a generalized Gaussian ratio. The aim of this present paper is to investigate some important properties of the function $L(x) = \ln \mathscr{R}(x)$, such as complete monotonicity and inequalities.

The rest of this paper is organized as follows. In Section 2, a few essential lemmas are given, in which Lemmas 1–5 will be used to prove Theorem 6. In Section 3, the integral, asymptotic expansion and Maclaurin series representations of $\ln \mathscr{R}(x)$ are established. Some complete monotonicity results and inequalities involving $\ln \mathscr{R}(x)$ are proved in Section 4. As applications, some elementary function approximations for the inverse of the function $x \mapsto \ln \mathscr{R}(1/x)$ are presented in Section 5.

2. Auxiliary Lemmas

To prove our main results, we need some auxiliary lemmas. The following is rewritten from [30, Lemma 1], which will be used to prove Lemma 3.

LEMMA 1. *Let $A(t) = \sum_{k=0}^{\infty} a_k t^k$, $B(t) = \sum_{k=0}^{\infty} b_k t^k$ and $C(t) = \sum_{k=0}^{\infty} c_k t^k$ be real power series with radii of convergence $R_A = R_B = R > 0$ and $R_c = r < R$, respectively, and $B(t) > 0$ for $t \in (0, R)$. Assume that $A(t)/B(t)$ converges to $C(t)$ for $|t| < r$, then for integer $n \geqslant 0$ we have*

(i) if

$$d_{k,2n} = \sum_{i=0}^{2n} b_{k-i} c_i - a_k = \mathscr{E}_1 + \sum_{j=1}^{n} \mathscr{E}_{2,j} > (<)\, 0 \; for \; k \geqslant 2n+1,$$

where $\mathscr{E}_1 = b_k c_0 - a_k$ and $\mathscr{E}_{2,j} = \left(b_{k-2j+1} c_{2j-1} + b_{k-2j} c_{2j}\right)$, then it holds that

$$\frac{A(t)}{B(t)} < (>) \sum_{k=0}^{2n} c_k t^k \tag{4}$$

for $t \in (0, R)$;

(ii) if

$$d_{k,2n+1} = \sum_{i=0}^{2n+1} b_{k-i}c_i - a_k = \mathscr{O}_1 + \sum_{j=1}^{n} \mathscr{O}_{2,j} > (<)\, 0 \text{ for } k \geqslant 2n+2,$$

where $\mathscr{O}_1 = (b_k c_0 + b_{k-1}c_1 - a_k)$ *and* $\mathscr{O}_{2,j} = \left(b_{k-2j}c_{2j} + b_{k-2j-1}c_{2j+1}\right)$*, then it holds that*

$$\frac{A(t)}{B(t)} < (>) \sum_{k=0}^{2n+1} c_k t^k \tag{5}$$

for $t \in (0,R)$.

REMARK 1. From the proof of [30, Lemma 1], we clearly see that

$$\sum_{i=0}^{k} b_{k-i}c_i = a_k.$$

To prove Lemma 3, we also need the following lemma due to Qi [21].

LEMMA 2. ([21, Theorem 1.1]) *For* $k \in \mathbb{N}$*, the Bernoulli numbers* B_{2k} *satisfy*

$$\frac{2^{2k-1}-1}{2^{2k+1}-1}\frac{(2k+1)(2k+2)}{\pi^2} < \frac{|B_{2k+2}|}{|B_{2k}|} < \frac{2^{2k}-1}{2^{2k+2}-1}\frac{(2k+1)(2k+2)}{\pi^2}. \tag{6}$$

In particular, let $k = 2j+1,\ 2j$ *we have*

$$\frac{2^{4j+1}-1}{2^{4j+3}-1}\frac{(4j+3)(4j+4)}{\pi^2} < \frac{\left|B_{4j+4}\right|}{\left|B_{4j+2}\right|} < \frac{2^{4j+2}-1}{2^{4j+4}-1}\frac{(4j+3)(4j+4)}{\pi^2}, \tag{7}$$

$$\frac{2^{4j-1}-1}{2^{4j+1}-1}\frac{(4j+1)(4j+2)}{\pi^2} < \frac{\left|B_{4j+2}\right|}{\left|B_{4j}\right|} < \frac{2^{4j}-1}{2^{4j+2}-1}\frac{(4j+1)(4j+2)}{\pi^2}. \tag{8}$$

Such inequalities above-mentioned are often used to prove the complete monotonicity of certain special functions, see [30]. Indeed, by means of Lemmas 1 and 2 we can prove an interesting inequalities for the hyperbolic functions as follows.

LEMMA 3. *For any integer* $n \geqslant 0$*, the double inequality*

$$\sum_{k=0}^{2n+1} \frac{2^{2k+1}}{(2k)!} B_{2k+2} t^{2k} < \frac{t\cosh t - \sinh t}{\sinh^3 t} < \sum_{k=0}^{2n} \frac{2^{2k+1}}{(2k)!} B_{2k+2} t^{2k} \tag{9}$$

holds for all $t > 0$*, where* B_{2k} *is the Bernoulli number.*

Proof. Inequalities (9) can be written as

$$\sum_{k=0}^{2n-1} \frac{2^{2k+1}}{(2k)!} B_{2k+2} t^{2k} < \frac{(t\cosh t - \sinh t)/t^3}{\left(\sinh^3 t\right)/t^3} < \sum_{k=0}^{2n} \frac{2^{2k+1}}{(2k)!} B_{2k+2} t^{2k}.$$

Let

$$A(t) = \frac{t\cosh t - \sinh t}{t^3} = \sum_{k=0}^{\infty} \frac{2k+2}{(2k+3)!} t^{2k} := \sum_{k=0}^{\infty} a_k t^{2k}, \quad |t| < \infty,$$
$$B(t) = \frac{1}{4} \frac{\sinh 3t - 3\sinh t}{t^3} = \sum_{k=0}^{\infty} \frac{3^{2k+3}-3}{4(2k+3)!} t^{2k} := \sum_{k=0}^{\infty} b_k t^{2k}, \quad |t| < \infty,$$
$$C(t) = \frac{A(t)}{B(t)} = \sum_{k=0}^{\infty} \frac{2^{2k+1}}{(2k)!} B_{2k+2} t^{2k} := \sum_{k=0}^{\infty} c_k t^{2k}, \quad |t| < \pi.$$

(1) Let us first prove that for $k \geqslant 2n+1$ there holds

$$d_{k,2n} = \sum_{i=1}^{2n} b_{k-i}c_i - a_k = \mathscr{E}_1 + \sum_{j=1}^{n} \mathscr{E}_{2,j} > 0,$$

where $\mathscr{E}_1 = (b_k c_0 - a_k)$ and $\mathscr{E}_{2,j} = \left(b_{k-2j+1}c_{2j-1} + b_{k-2j}c_{2j}\right)$. We now part it by distinguishing three cases.

Case 1.1. If $k = 2n+1$, by considering $\sum_{i=0}^{k} b_{k-i}c_i = a_k$ we see that

$$d_{2n+1,2n} = \sum_{i=0}^{2n} b_{2n+1-i}c_i - a_{2n+1} = -b_0 c_{2n+1} = -\frac{2^{4n+3}}{(4n+2)!} B_{4n+4} > 0.$$

Case 1.2. If $k = 2n+2$, similarly we have

$$\begin{aligned} d_{2n+2,2n} &= \sum_{i=0}^{2n} b_{2n+2-i}c_i - a_{2n+2} = -b_0 c_{2n+2} - b_1 c_{2n+1} \\ &= -\frac{2^{4n+5}}{(4n+4)!} B_{4n+6} - \frac{1}{2} \frac{2^{4n+3}}{(4n+2)!} B_{4n+4} \\ &= \frac{2^{4n+5}|B_{4n+4}|}{(4n+4)!} \left(\frac{1}{2}(n+1)(4n+3) - \frac{|B_{4n+6}|}{|B_{4n+4}|} \right). \end{aligned}$$

Using the right hand side inequality of (8) for $j = n+1$ we obtain

$$\frac{|B_{4n+6}|}{|B_{4n+4}|} < \frac{(4n+5)(4n+6)}{\pi^2} \frac{2^{4n+4}-1}{2^{4n+6}-1},$$

which yields

$$\frac{1}{2}(n+1)(4n+3) - \frac{|B_{4n+6}|}{|B_{4n+4}|} > \frac{1}{2}(n+1)(4n+3) - \frac{(4n+5)(4n+6)}{\pi^2} \frac{2^{4n+4}-1}{2^{4n+6}-1}$$
$$> \frac{1}{2}(n+1)(4n+3) - \frac{(4n+5)(4n+6)}{8} \frac{1}{4} = \frac{3}{2}n^2 + \frac{17}{8}n + \frac{9}{16} > 0,$$

where the first inequality holds due to $\pi^2 > 8$ and $\left(2^{4n+4}-1\right)/\left(2^{4n+6}-1\right) < 1/4$. It then follows that $d_{2n+2,2n} < 0$.

Case 1.3. If $k \geqslant 2n+3$, it suffices to check that $\mathscr{E}_1 = b_k c_0 - a_k > 0$ and $\mathscr{E}_{2,j} = b_{k-2j+1}c_{2j-1} + b_{k-2j}c_{2j} > 0$ for $1 \leqslant j \leqslant n$ and $k \geqslant 2n+3$. We have

$$\mathscr{E}_1 := b_k c_0 - a_k = \frac{3^{2k+3}-3}{4\,(2k+3)!}\frac{1}{3} - \frac{2k+2}{(2k+3)!} = \frac{3^{2k+2}-8k-9}{4\,(2k+3)!} > 0.$$

Using the left hand side inequality of (8) we have

$$\begin{aligned}\frac{\mathscr{E}_{2,j}}{\left|B_{4j}\right|} &= \frac{b_{k-2j+1}c_{2j-1}+b_{k-2j}c_{2j}}{\left|B_{4j}\right|}\\ &= -\frac{3^{2k-4j+5}-3}{4\,(2k-4j+5)!}\frac{2^{4j-1}}{(4j-2)!} + \frac{3^{2k-4j+3}-3}{4\,(2k-4j+3)!}\frac{2^{4j+1}}{(4j)!}\frac{\left|B_{4j+2}\right|}{\left|B_{4j}\right|}\end{aligned}$$

$$\begin{aligned}&> -\frac{3^{2k-4j+5}-3}{4\,(2k-4j+5)!}\frac{2^{4j-1}}{(4j-2)!} + \frac{3^{2k-4j+3}-3}{4\,(2k-4j+3)!}\frac{2^{4j+1}}{(4j)!}\frac{2^{4j-1}-1}{2^{4j+1}-1}\frac{(4j+1)(4j+2)}{\pi^2}\\ &= \frac{2^{4j-1}\left(3^{2k-4j+5}-3\right)}{4\,(4j-2)!\,(2k-4j+5)!}\left(\frac{4}{\pi^2}f_1(2k-4j+3)\,f_2(4j)-1\right),\end{aligned}$$

where

$$f_1(x) = \frac{(x+2)(x+1)(3^x-3)}{3^{x+2}-3},\ x = 2k-4j+3, \tag{10}$$

$$f_2(y) = \frac{2^{y-1}-1}{2^{y+1}-1}\frac{(y+2)(y+1)}{y(y-1)},\ y = 4j \geqslant 4. \tag{11}$$

Since $x = 2k-4j+3 \geqslant 2(2n+3)-4n+3 = 9$ and

$$f_1(x+1) - f_1(x) = \frac{2}{3}(x+2)\frac{3^{2x+2}+4(2x-1)\times 3^x+3}{(3^{x+2}-1)(3^{x+1}-1)} > 0,$$

it is derived that $f_1(x) \geqslant f_1(9) = 8200/671$. Also, $f_2(y) \geqslant 1/4$ due to

$$f_2(y) - \frac{1}{4} = \frac{4(2y+1)\times 2^y - \left(3y^2+13y+8\right)}{4y\,(2\times 2^y-1)(y-1)} > 0 \ \text{ for } y \geqslant 4.$$

These yield

$$\frac{4}{\pi^2}f_1(2k-4j+3)\,f_2(4j) - 1 > \frac{4}{\pi^2}\frac{8200}{671}\frac{1}{4} - 1 > 0,$$

and so $\mathscr{E}_{2,j} > 0$ for $1 \leqslant j \leqslant n$ and $k \geqslant 2n+3 \geqslant 3$.

From the cases 1.1–1.3, it results in $d_{k,2n} < 0$ for $k \geqslant 2n+1$, and by Lemma 1 the right hand side inequality of (9) holds for all $t > 0$.

(2) We now prove that for $k \geqslant 2n+2$,

$$d_{k,2n+1} = (b_k c_0 + b_{k-1}c_1 - a_k) + \sum_{j=1}^{n}\left(b_{k-2j}c_{2j} + b_{k-2j-1}c_{2j+1}\right) < 0,$$

where $\mathscr{O}_1 = (b_k c_0 + b_{k-1} c_1 - a_k)$ and $\mathscr{O}_{2,j} = \left(b_{k-2j} c_{2j} + b_{k-2j-1} c_{2j+1}\right)$. Similarly, we distinguish three cases to prove it.

Case 2.1. For $k = 2n+2$, since $\sum_{i=0}^{k} b_{k-i} c_i = a_k$, we have

$$d_{2n+2,2n+1} = \sum_{i=0}^{2n+1} b_{2n+2-i} c_i - a_{2n+2} = -b_0 c_{2n+2} = -\frac{2^{4n+5}}{(4n+4)!} B_{4n+6} < 0.$$

Case 2.2. For $k = 2n+3$, likewise, we have

$$\begin{aligned} d_{2n+3,2n+1} &= \sum_{i=0}^{2n+1} b_{2n+3-i} c_i - a_{2n+3} = -b_0 c_{2n+3} - b_1 c_{2n+2} \\ &= -\frac{2^{4n+7}}{(4n+6)!} B_{4n+8} - \frac{1}{2} \frac{2^{4n+5}}{(4n+4)!} B_{4n+6} \\ &= \frac{2^{4n+7} |B_{4n+6}|}{(4n+6)!} \left(\frac{|B_{4n+8}|}{|B_{4n+6}|} - \frac{1}{8} (4n+6)(4n+5) \right). \end{aligned}$$

Using the right hand side inequality of (7) for $j = n+1$ we obtain

$$\frac{|B_{4n+8}|}{|B_{4n+6}|} < \frac{(4n+7)(4n+8)}{\pi^2} \frac{2^{4n+6}-1}{2^{4n+8}-1},$$

which yields

$$\frac{|B_{4n+8}|}{|B_{4n+6}|} - \frac{1}{8}(4n+6)(4n+5) < \frac{(4n+7)(4n+8)}{\pi^2} \frac{2^{4n+6}-1}{2^{4n+8}-1} - \frac{1}{8}(4n+6)(4n+5)$$

$$< \frac{(4n+7)(4n+8)}{8} \frac{1}{4} - \frac{1}{8}(4n+6)(4n+5) = -\frac{1}{8}\left(12n^2 + 29n + 16\right) < 0,$$

where the first inequality holds due to $\pi^2 > 8$ and $\left(2^{4n+6}-1\right) / \left(2^{4n+8}-1\right) < 1/4$. It is deduced that $d_{2n+3,2n+1} < 0$.

Case 2.3. For $k \geqslant 2n+4$, it suffices to check that $\mathscr{O}_1 = b_k c_0 + b_{k-1} c_1 - a_k < 0$ and $\mathscr{O}_{2,j} = b_{k-2j} c_{2j} + b_{k-2j-1} c_{2j+1} < 0$ for $1 \leqslant j \leqslant n$ and $k \geqslant 2n+4$. We easily check that

$$\begin{aligned} \mathscr{O}_1 &= b_k c_0 + b_{k-1} c_1 - a_k = \frac{1}{3} \frac{3^{2k+3}-3}{4(2k+3)!} - \frac{2}{15} \frac{3^{2k+1}-3}{4(2k+1)!} - \frac{2k+2}{(2k+3)!} \\ &= -\frac{\left(8k^2+20k-33\right)3^{2k} - \left(8k^2-20k-33\right)}{20(2k+3)!} < 0 \end{aligned}$$

Using the left hand side inequality (7) we have

$$\begin{aligned} \frac{\mathscr{O}_{2,j}}{|B_{4j+2}|} &= \frac{b_{k-2j} c_{2j} + b_{k-2j-1} c_{2j+1}}{|B_{4j}|} \\ &= \frac{1}{4} \frac{2^{4j+1}\left(3^{2k-4j+3}-3\right)}{(2k-4j+3)!(4j)!} - \frac{1}{4} \frac{2^{4j+3}\left(3^{2k-4j+1}-3\right)}{(2k-4j+1)!(4j+2)!} \frac{|B_{4j+4}|}{B_{4j+2}} \end{aligned}$$

$$< \frac{1}{4}\frac{2^{4j+1}\left(3^{2k-4j+3}-3\right)}{(2k-4j+3)!\,(4j)!} - \frac{1}{4}\frac{2^{4j+3}\left(3^{2k-4j+1}-3\right)}{(2k-4j+1)!\,(4j+2)!}\frac{(4j+4)(4j+3)}{\pi^2}\frac{2^{4j+1}-1}{2^{4j+3}-1}$$
$$= \frac{1}{4}\frac{2^{4j+1}\left(3^{2k-4j+3}-3\right)}{(2k-4j+3)!\,(4j)!}\left(1-\frac{4}{\pi^2}f_1(2k-4j+1)\,f_2(4j)\right),$$

where f_1 and f_2 are given by (10) and (11), respectively. As shown in Case 1.3, $f_1(x)$ is increasing. This in combination with $x = 2k-4j+1 \geqslant 2(2n+4)-2n+1 = 9$ yields $f_1(2k-4j+1) \geqslant f_1(9) = 8200/671$. Also, $f_2(4j) > 1/4$. Therefore, we obtain

$$1-\frac{4}{\pi^2}f_1(2k-4j+1)\,f_2(4j) < 1-\frac{4}{\pi^2}\frac{8200}{671}\frac{1}{4} < 0,$$

and so is $\mathcal{O}_{2,j}$ for $1 \leqslant j \leqslant n$ and $k \geqslant 2n+4$.

Taking into account Cases 2.1–2.3, we arrive at $d_{k,2n+1} < 0$ for $k \geqslant 2n+2$. By Lemma 1 the left hand side inequality of (9) holds for all $t > 0$, which completes the proof.

REMARK 2. The double inequality (9) is equivalent to

$$(-1)^m\left(\frac{t\cosh t-\sinh t}{\sinh^3 t} - \sum_{k=0}^{m}\frac{2^{2k+1}}{(2k)!}B_{2k+2}t^{2k}\right) < 0$$

for any integer $m \geqslant 0$ and all $t > 0$.

The following Lemmas 4 and 5 are also useful to prove Theorem 6.

LEMMA 4. *Let $m \geqslant 0$ be an integer. Then the function*

$$\phi_m(x) = \coth\frac{1}{x} - \sum_{k=0}^{m}\frac{2^{2k}B_{2k}}{(2k)!}\frac{1}{x^{2k-1}} \tag{12}$$

is convex on $(0,\infty)$ if m is even and concave on $(0,\infty)$ if m is odd.

Proof. It suffices to prove $(-1)^m\phi_m''(x) > 0$ for $x \in (0,\infty)$. Differentiation yields that for $m \geqslant 1$,

$$\phi_m''(x) = 2\frac{\cosh(1/x)-x\sinh(1/x)}{x^4\sinh^3(1/x)} - \sum_{k=1}^{m}\frac{2^{2k}B_{2k}}{(2k-2)!}\frac{1}{x^{2k+1}}$$
$$= 2t^3\left(\frac{t\cosh t-\sinh t}{\sinh^3 t} - \sum_{k=0}^{m-1}\frac{2^{2k+1}B_{2k+2}}{(2k)!}t^{2k}\right),\ t = \frac{1}{x},$$

which, by Lemma 3, gives the $(-1)^m\phi_m''(x) > 0$ for $x \in (0,\infty)$. Clearly, it is also true for $m = 0$. This completes the proof.

In what follows, we will encounter some classical mean values. The r-th power mean of $a=(a_1,a_2,...a_n)$ with weight $w=(w_1,w_2,...w_n)$ is defined as in [5]

$$M_n^{[r]}(a;w)=\left(\sum_{k=1}^{n} w_k a_k^r\right)^{1/r} \text{ if } r\neq 0 \text{ and } M_n^{[0]}(a;w)=\prod_{k=1}^{n} a_k^{w_k}, \tag{13}$$

where $M_n^{[1]}(a;w)$ and $M_n^{[0]}(a;w)$ are called the weighted arithmetic and geometric means, and also denoted by $A_n(a;w)$ and $G_n(a;w)$, respectively. A special Gini mean, also called the power-exponential mean in [31], is given by

$$Z_n(a;w)=\exp\left(\frac{\sum_{j=1}^{n_0} w_j a_j \ln a_j}{\sum_{j=1}^{n_0} w_j a_j}\right). \tag{14}$$

LEMMA 5. *Let $a_j, w_j>0$ for $1\leqslant j\leqslant n_0$ with $\sum_{j=1}^{n_0} w_j=1$ and $\max_{1\leqslant j\leqslant n_0}(a_j)\neq \min_{1\leqslant j\leqslant n_0}(a_j)$, let h_0 and h_m ($m\in\mathbb{N}$) be defined on $(0,\infty)$ by*

$$h_0(t)=\sum_{j=1}^{n_0} w_j \coth\left(\frac{t}{2a_j}\right)-\coth\left(\frac{t}{2\sum_{j=1}^{n_0} w_j a_j}\right), \tag{15}$$

$$h_m(t)=h_0(t)-\sum_{k=0}^{m}\frac{2d_{1-2k}B_{2k}}{(2k)!}t^{2k-1}, \tag{16}$$

where

$$d_r=M_{n_0}^{[r]}(a;w)^r-A_{n_0}(a;w)^r,\ \ r\in\mathbb{R}. \tag{17}$$

Then we have $(-1)^m h_m(t)>0$ for any integer $m\geqslant 0$ and $t>0$.

Proof. (i) As shown in Lemma 4, $[\coth(1/x)]''>0$ for $x\in(0,\infty)$, which yields $h_0(t)>0$ for $t\in(0,\infty)$.

(ii) If we prove that $h_m(t)$ can be written by

$$h_m(t)=\sum_{j=1}^{n} w_j\phi_m\left(\frac{2a_j}{t}\right)-\phi_m\left(\frac{2\bar{a}}{t}\right),$$

where ϕ_m is defined by (12) and $\bar{a}=\sum_{j=1}^{n_0} w_j a_j$. Then by Lemma 4 we arrive at $(-1)^m h_m(t)>0$ for $m\geqslant 1$ and $t>0$, and the proof is complete. Now using (15) and (17) leads to

$$h_m(t)=\sum_{j=1}^{n} w_j\coth\left(\frac{t}{2a_j}\right)-\coth\left(\frac{t}{2\bar{a}}\right)-2\sum_{k=0}^{m}\left(\sum_{j=1}^{n_0} w_j a_j^{1-2k}-\bar{a}^{1-2k}\right)\frac{B_{2k}}{(2k)!}t^{2k-1}$$

$$=\sum_{j=1}^{n} w_j\coth\left(\frac{t}{2a_j}\right)-\coth\left(\frac{t}{2\bar{a}}\right)-2\sum_{j=1}^{n}\sum_{k=0}^{m} w_j a_j^{1-2k}\frac{B_{2k}}{(2k)!}t^{2k-1}+2\sum_{k=0}^{m}\bar{a}^{1-2k}\frac{B_{2k}}{(2k)!}t^{2k-1}$$

$$= \sum_{j=1}^{n} w_j \left[\coth\left(\frac{t}{2a_j}\right) - \sum_{k=0}^{m} \frac{2^{2k}B_{2k}}{(2k)!}\left(\frac{t}{2a_j}\right)^{2k-1} \right]$$
$$- \left[\coth\left(\frac{t}{2\bar{a}}\right) - \sum_{k=0}^{m} \frac{2^{2k}B_{2k}}{(2k)!}\left(\frac{t}{2\bar{a}}\right)^{2k-1} \right] = \sum_{j=1}^{n} w_j \phi_m\left(\frac{2a_j}{t}\right) - \phi_m\left(\frac{2\bar{a}}{t}\right),$$

which completes the proof.

LEMMA 6. *Let* $L(x) = \ln \mathscr{R}(x)$, *where* $\mathscr{R}(x)$ *is defined on* $(0,\infty)$ *by (3). Then we have*

$$L\left(0^+\right) = \ln \frac{A_{n_0}(a;w)}{G_{n_0}(a;w)} = c_0, \tag{18}$$

$$L^{(m)}\left(0^+\right) = \left(M_{n_0}^{[m]}(a;w)^m - A_{n_0}(a;w)^m\right)\psi^{(m-1)}(1) = d_m \psi^{(m-1)}(1) \tag{19}$$

for $m \in \mathbb{N}$, *where* d_m *is defined by (17).*

Proof. To obtain the desired limit values, we denote $\bar{a} = \sum_{j=1}^{n_0} w_j a_j$ and write $L(x)$ as

$$L(x) = \ln \mathscr{R}(x) = \sum_{j=1}^{n_0} w_j \ln\Gamma(a_j x) - \ln\Gamma(\bar{a}x)$$
$$= \sum_{j=1}^{n_0} w_j \left[\ln\Gamma(a_j x + 1) - \ln(a_j x)\right] - \ln\Gamma(\bar{a}x+1) + \ln(\bar{a}x)$$
$$= \sum_{j=1}^{n_0} w_j \ln\Gamma(a_j x + 1) - \ln\Gamma(\bar{a}x+1) + \left(\ln\bar{a} - \sum_{j=1}^{n_0} w_j \ln a_j\right),$$

which yields $L(0^+) = d_0$. Differentiation leads to

$$L^{(m)}(x) = \sum_{j=1}^{n_0} w_j a_j^m \psi^{(m-1)}(a_j x + 1) - \bar{a}^m \psi^{(m-1)}(\bar{a}x+1),\ m \in \mathbb{N},$$

which implies

$$L^{(m)}\left(0^+\right) = \left(\sum_{j=1}^{n_0} w_j a_j^m - \bar{a}^m\right)\psi^{(m-1)}(1).$$

This completes the proof.

The following lemma is needed to prove Theorem 7.

LEMMA 7. *Let* $n \geqslant 0$ *be an integer. The double inequality*

$$\sum_{k=0}^{2n+1} \frac{(-1)^k x^k}{k!} < e^{-x} < \sum_{k=0}^{2n} \frac{(-1)^k x^k}{k!} \tag{20}$$

holds for $x > 0$.

Proof. For every integer $m \geqslant 0$, let

$$g_m(x) = e^{-x} - \sum_{k=0}^{m} \frac{(-1)^k x^k}{k!}.$$

It suffices to prove $(-1)^m g_m(x) < 0$ for $x > 0$. Differentiation yields

$$(-1)^m g_m^{(j)}(x) = (-1)^{m+j} \left[e^{-x} - \sum_{k=j}^{m} \frac{(-1)^{k-j} x^{k-j}}{(k-j)!} \right], \; j = 1,2,\ldots,m-1,m.$$

In particular, we have

$$(-1)^m g_m^{(m)}(x) = e^{-x} - 1 < 0 \;\text{ for } x > 0,$$

which together with

$$(-1)^m g_m^{(j)}(0) = 0, \; j = 0,1,2,\ldots,m-1$$

gives $(-1)^m g_m(x) < 0$. This completes the proof.

3. Integral and asymptotic expansion representations

In this section, we are devoted to presenting the integral, asymptotic expansion and Maclaurin series representations for the generalized Gaussian ratio.

3.1. Integral representations

THEOREM 1. *Let $x \mapsto \mathscr{R}(x)$ be defined on $(0,\infty)$ by (3). Then $L(x) = \ln \mathscr{R}(x)$ can be expressed as*

$$L(x) = \ln \mathscr{R}(x) = \frac{1}{2} c_0 + c_1 x + \frac{1}{2} \int_0^\infty \frac{h_0(t)}{t} e^{-xt} dt, \tag{21}$$

where c_0 is given by (18) and

$$c_1 = \sum_{j=1}^{n_0} w_j a_j \ln a_j - \left(\sum_{j=1}^{n_0} w_j a_j \right) \ln \left(\sum_{j=1}^{n_0} w_j a_j \right), \tag{22}$$

while $h_0(t)$ is defined by (15).

Proof. Using the following Binet's first expression for $\ln \Gamma(z)$ in terms of an infinite integral (see [28, p. 248–250]),

$$\ln \Gamma(z) = \left(z - \frac{1}{2} \right) \ln z - z + \frac{1}{2} \ln(2\pi) + \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} \right) \frac{e^{-zt}}{t} dt \quad \operatorname{Re}(z) > 0,$$

whose variant form (see [30]) is shown as

$$\ln\Gamma(z)=\left(z-\frac{1}{2}\right)\ln z-z+\frac{1}{2}\ln(2\pi)+\int_0^\infty\left(\coth\frac{t}{2}-\frac{2}{t}\right)\frac{e^{-zt}}{2t}dt\quad \operatorname{Re}(z)>0,$$

we obtain that for $a_j,x>0$ there holds

$$\ln\Gamma(a_jx)=\left(a_jx-\frac{1}{2}\right)\ln(a_jx)-a_jx+\frac{1}{2}\ln(2\pi)+\int_0^\infty\left(\coth\frac{t}{2}-\frac{2}{t}\right)\frac{e^{-a_jxt}}{2t}dt$$

$$=(a_j\ln a_j)x-\frac{1}{2}\ln a_j-a_jx+a_jx\ln x-\frac{1}{2}\ln x+\frac{1}{2}\ln(2\pi)+\int_0^\infty\left(\coth\frac{t}{2a_j}-\frac{2a_j}{t}\right)\frac{e^{-xt}}{2t}dt,$$

where we have made a change of variable $a_jt\to t$ in the above integral. It then follows that

$$\begin{aligned}\ln\mathscr{R}(x)&=\sum_{j=1}^{n_0}w_j\ln\Gamma(a_jx)-\ln\Gamma(\bar{a}x)\\&=\left(\sum_{j=1}^{n_0}w_ja_j\ln a_j-\bar{a}\ln\bar{a}\right)x+\frac{1}{2}\left(\ln\bar{a}-\sum_{j=1}^{n_0}w_j\ln a_j\right)\\&\quad+\frac{1}{2}\int_0^\infty\left(\sum_{j=1}^{n_0}w_j\coth\frac{t}{2a_j}-\coth\frac{t}{2\bar{a}}\right)\frac{e^{-xt}}{t}dt\end{aligned}$$

with $\bar{a}=\sum_{j=1}^{n_0}w_ja_j$. This completes the proof.

From Lemma 6 and Theorem 1 we can obtain the integral value of $\int_0^\infty t^{m-1}h_0(t)\,dt$ with every integer $m\geqslant 0$.

PROPOSITION 1. *Let h_0 be defined on $(0,\infty)$ by (15). Then for every integer $m\geqslant 0$, we have*

$$\int_0^\infty t^{m-1}h_0(t)\,dt=\begin{cases}\ln\dfrac{A_{n_0}(a;w)}{G_{n_0}(a;w)}=c_0 & \text{if } m=0,\\ 2A_{n_0}(a;w)\ln\dfrac{Z_{n_0}(a;w)}{A_{n_0}(a;w)}=2c_1 & \text{if } m=1,\\ 2\left(M_{n_0}^{[m]}(a;w)^m-A_{n_0}(a;w)^m\right)\left|\psi^{(m-1)}(1)\right| & \text{if } m\geqslant 2.\end{cases}\tag{23}$$

Proof. The integral representation (21) together with (18) yields

$$L\left(0^+\right)=c_0=\frac{1}{2}c_0+\frac{1}{2}\int_0^\infty\frac{h_0(t)}{t}dt,$$

which implies (23) for $m=0$. Similarly, we easily get

$$L'(0)=0=c_1-\frac{1}{2}\int_0^\infty h_0(t)\,dt,\tag{24}$$

which implies (23) for $m = 1$.

Taking into account the integral representation (21) and (19) we have

$$L^{(m)}\left(0^{+}\right) = \left[\sum_{j=1}^{n_0} w_j a_j^m - \left(\sum_{j=1}^{n_0} w_j a_j\right)^m\right] \psi^{(m-1)}(1) = \frac{(-1)^m}{2} \int_0^{\infty} t^{m-1} h_0(t)\, dt.$$

for $m \geqslant 2$, which indicates (23) for $m \geqslant 2$, and the proof is completed.

REMARK 3. Proposition 1 exhibits an interesting relation connecting the ratio or difference of classical means and integrals $\int_0^\infty t^{m-1} h_0(t)\, dt$ for any integer $m \geqslant 0$. This allows us to establish a few new inequalities of some means values, for example, the inequality

$$\left|2\psi^{(m-1)}(1)\right|^{1/m} \left(\frac{d_m}{c_0}\right)^{1/m} < \left|2\psi^{(n-1)}(1)\right|^{1/n} \left(\frac{d_m}{c_0}\right)^{1/n}$$

holds for $m,n \in \mathbb{N}$ with $2 \leqslant m < n$, which follows from the known integral inequality

$$\left(\frac{\int_0^\infty \left(t^{-1} h_0(t)\right) t^m dt}{\int_0^\infty t^{-1} h_0(t)\, dt}\right)^{1/m} < \left(\frac{\int_0^\infty \left(t^{-1} h_0(t)\right) t^n dt}{\int_0^\infty t^{-1} h_0(t)\, dt}\right)^{1/n};$$

for another example, by the Scharwz inequality, we have

$$\frac{c_1^2}{c_0 d_2} = \frac{2\psi'(1)}{4} \frac{\left(\int_0^\infty h_0(t)\, dt\right)^2}{\left(\int_0^\infty t^{-1} h_0(t)\, dt\right)\left(\int_0^\infty t h_0(t)\, dt\right)} < \frac{\pi^2}{12},$$

which is equivalent to

$$\left(\ln \frac{Z_{n_0}(a;w)}{A_{n_0}(a;w)}\right)^2 < \frac{\pi^2}{12} \left(\frac{M_{n_0}^{[2]}(a;w)^2}{A_{n_0}(a;w)^2} - 1\right) \ln \frac{A_{n_0}(a;w)}{G_{n_0}(a;w)}.$$

Applying the relations (23) with $m = 0, 1$ to the integral representation (21), we get another integral representation concerning $\ln \mathscr{R}(x)$.

THEOREM 2. *Let $x \mapsto \mathscr{R}(x)$ be defined on $(0, \infty)$ by (3). Then $L(x) = \ln \mathscr{R}(x)$ can also be expressed as*

$$L(x) = \ln \mathscr{R}(x) = \frac{1}{2} \int_0^{\infty} \frac{h_0(t)}{t} \left(1 + tx + e^{-xt}\right) dt, \tag{25}$$

where $h_0(t)$ is defined by (15).

3.2. Asymptotic expansion

THEOREM 3. *Let $\mathscr{R}(x)$ be defined on $(0,\infty)$ by (3). Then the asymptotic expansion*

$$L(x) = \ln\mathscr{R}(x) \sim \frac{1}{2}c_0 + c_1 x + \sum_{k=1}^{\infty} \frac{d_{1-2k}B_{2k}}{2k(2k-1)} \frac{1}{x^{2k-1}} \quad \text{as } x \to \infty \tag{26}$$

holds, where B_{2k} is the Bernoulli number and d_r is defined by (17).

Proof. Using the asymptotic expansion in [1]

$$\ln\Gamma(x) \sim \frac{1}{2}\ln(2\pi) + \left(x - \frac{1}{2}\right)\ln x - x + \sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k-1)x^{2k-1}}, \tag{27}$$

we have

$$\ln\mathscr{R}(x) = \sum_{j=1}^{n_0} w_j \ln\Gamma(a_j x) - \ln\Gamma(\bar{a}x)$$

$$\begin{aligned}
&= \sum_{j=1}^{n_0} w_j \left[\frac{1}{2}\ln(2\pi) + \left(a_j x - \frac{1}{2}\right)\ln(a_j x) - a_j x + \sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k-1)(a_j x)^{2k-1}}\right] \\
&\quad - \left[\frac{1}{2}\ln(2\pi) + \left(\bar{a}x - \frac{1}{2}\right)\ln(\bar{a}x) - \bar{a}x + \sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k-1)(\bar{a}x)^{2k-1}}\right] \\
&= \frac{1}{2}\left(\ln\bar{a} - \sum_{j=1}^{n_0} w_j \ln a_j\right) + x\left(\sum_{j=1}^{n_0} w_j a_j \ln a_j - \bar{a}\ln\bar{a}\right) \\
&\quad + \sum_{k=1}^{\infty}\left(\sum_{j=1}^{n_0} \frac{w_j}{a_j^{2k-1}} - \frac{1}{\bar{a}^{2k-1}}\right) \frac{B_{2k}}{2k(2k-1)x^{2k-1}} \\
&= \frac{1}{2}c_0 + c_1 x + \sum_{j=1}^{\infty} \frac{d_{1-2k}B_{2k}}{2k(2k-1)x^{2k-1}},
\end{aligned}$$

which completes the proof.

3.3. Maclaurin series

THEOREM 4. *Let $\mathscr{R}(x)$ be defined on $[0,\infty)$ by (3). Then $\ln\mathscr{R}(x)$ can be expressed by the following Maclaurin series*

$$L(x) = \ln\mathscr{R}(x) = c_0 + \sum_{k=2}^{\infty} \frac{d_k \psi^{(k-1)}(1)}{k!} x^k, \tag{28}$$

where d_k is defined by (17).

Proof. Using the integral representation (25) and expanding e^{-xt} in power series yield

$$L(x) = \ln \mathscr{R}(x) = \frac{1}{2}\int_0^\infty \frac{h_0(t)}{t}\left(1+tx+e^{-xt}\right)dt$$
$$= \frac{1}{2}\int_0^\infty \frac{h_0(t)}{t}\left(1+tx+\sum_{k=0}^\infty \frac{(-1)^k t^k}{k!}x^k\right)dt$$

$$= \frac{1}{2}\int_0^\infty \frac{h_0(t)}{t}\left(2+\sum_{k=2}^\infty \frac{(-1)^k t^k}{k!}x^k\right)dt$$
$$= \int_0^\infty \frac{h_0(t)}{t}dt + \frac{1}{2}\sum_{k=2}^\infty\left(\int_0^\infty t^{k-1}h_0(t)\,dt\right)\frac{(-1)^k x^k}{k!}.$$

By the relation (23), the desired Maclaurin series expansion follows.

4. Complete monotonicity and inequalities

THEOREM 5. *Let $L(x) = \ln \mathscr{R}(x)$, where $\mathscr{R}(x)$ is defined on $(0,\infty)$ by (3). Then $x \mapsto L''(x)$ is a completely monotonic function on $(0,\infty)$, while $x \mapsto L'(x)$ is the Bernstein function on $(0,\infty)$. Consequently, the inequalities*

$$0 < \frac{\sum_{j=1}^{n_0} w_j a_j \psi(xa_j)}{\sum_{j=1}^{n_0} a_j w_j} - \psi\left(x\sum_{j=1}^{n_0} w_j a_j\right) < \ln \frac{Z_{n_0}(a;w)}{A_{n_0}(a;w)} \tag{29}$$

hold for $x > 0$.

Proof. Using the integral representation (25) and Lemma 5 we have

$$L'(x) = \frac{1}{2}\int_0^\infty h_0(t)\left(1-e^{-xt}\right)dt > 0, \tag{30}$$
$$(-1)^m L^{(m+2)}(x) = \frac{1}{2}\int_0^\infty t^{m+1}h_0(t)\,e^{-xt}dt > 0 \tag{31}$$

for $x > 0$ and $m = 0,1,\ldots$. By Lemma 5 and Bernstein theorem, the function L'' is a completely monotonic function, and L' is a Bernstein function. Using the increasing property of $L'(x)$ on $(0,\infty)$ with $L'(0) = 0$ and

$$L'(\infty) = \frac{1}{2}\int_0^\infty h_0(t)\,dt = c_1,$$

we derive that

$$0 < L'(x) = \sum_{j=1}^{n_0} w_j a_j \psi(a_j x) - \left(\sum_{j=1}^{n_0} a_j w_j\right)\psi\left(x\sum_{j=1}^{n_0} a_j w_j\right) < c_1,$$

which is equivalent to (29). This completes the proof.

THEOREM 6. *Let $m \geqslant 0$ be an integer. Then the function*

$$F_m(x) = \begin{cases} \ln \mathscr{R}(x) - \frac{1}{2}c_0 - c_1 x & \text{if } m = 0, \\ \ln \mathscr{R}(x) - \frac{1}{2}c_0 - c_1 x - \sum_{k=1}^{m} \frac{d_{1-2k}B_{2k}}{2k(2k-1)} \frac{1}{x^{2k-1}} & \text{if } m \geqslant 1 \end{cases} \tag{32}$$

is completely monotonic on $(0,\infty)$ if m is even, and so is $-F_m(x)$ if m is odd.

Proof. We first write $F_m(x)$ in the following form

$$F_m(x) = \frac{1}{2}\int_0^\infty \frac{h_m(t)}{t} e^{-xt} dt,$$

where $h_m(t)$ is defined by (16). Using the integral representation (21) and the following formula

$$\frac{1}{x^s} = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-xt} dt,$$

we obtain that for $m \geqslant 1$,

$$\begin{aligned} F_m(x) &= \frac{1}{2}\int_0^\infty \frac{h_0(t)}{t} e^{-xt} dt - \sum_{k=1}^{m} \frac{d_{1-2k}B_{2k}}{2k(2k-1)} \frac{1}{(2k-2)!}\int_0^\infty t^{2k-2}e^{-xt} dt \\ &= \frac{1}{2}\int_0^\infty \frac{h_0(t)}{t} e^{-xt} dt - \int_0^\infty \left(\sum_{k=1}^{m} \frac{d_{1-2k}B_{2k}}{(2k)!} t^{2k-2}\right) e^{-xt} dt. \end{aligned}$$

Since $d_1 = 0$, we see that $\sum_{k=1}^{m}(\cdot) = \sum_{k=0}^{m}(\cdot)$ in the second integrand above. This in combination with (16) yields

$$F_m(x) = \frac{1}{2}\int_0^\infty \left(\frac{h_0(t)}{t} - \sum_{k=0}^{m} \frac{2d_{1-2k}B_{2k}}{(2k)!} t^{2k-2}\right) e^{-xt} dt = \frac{1}{2}\int_0^\infty \frac{h_m(t)}{t} e^{-xt} dt.$$

Evidently, it is also true for $m = 0$. By Lemma 5 and Bernstein theorem the desired result follows.

As a direct consequence of Theorem 6 we immediately get the following

COROLLARY 1. *For $n \in \mathbb{N}$, the following inequalities*

$$\frac{1}{2}c_0 + c_1 x < \ln \mathscr{R}(x) < c_0 + c_1 x, \tag{33}$$

$$\sum_{k=1}^{2n} \frac{d_{1-2k}B_{2k}}{2k(2k-1)} \frac{1}{x^{2k-1}} < \ln \mathscr{R}(x) - \left(\frac{1}{2}c_0 + c_1 x\right) < \sum_{k=1}^{2n-1} \frac{d_{1-2k}B_{2k}}{2k(2k-1)} \frac{1}{x^{2k-1}} \tag{34}$$

hold for $x > 0$.

Proof. The double inequality (33) follows from the monotonicity of

$$F_0(x) = \frac{1}{2}\int_0^\infty \frac{h_0(t)}{t} e^{-xt}\,dt$$

with $F_0(0) = c_0/2$ and $F_0(\infty) = 0$. The double inequality (34) follows from $F_{2n}(x) > 0$ and $-F_{2n-1}(x) > 0$.

REMARK 4. Note that $F_m(x)$ is in fact the remainder of the asymptotic expansion (26). By Theorem 6 and Corollary 1 we see that the remainder satisfies $(-1)^m F_m(x) \in \mathscr{C}[(0,\infty)]$ and

$$|F_m(x)| < \frac{d_{-2m-1}B_{2m+2}}{(2m+1)(2m+2)} \frac{1}{x^{2m+1}}.$$

REMARK 5. Letting $w = (w_1, w_2)$ with $w_1 + w_2 = 1$ and $a = (a_1, a_2)$ with $a_1 \neq a_2$ in inequalities (33) we arrive at

$$\sqrt{\frac{w_1a_1 + w_2a_2}{a_1^{w_1}a_2^{w_2}}}\left(\frac{a_1^{w_1a_1}a_2^{w_2a_2}}{(w_1a_1 + w_2a_2)^{(w_1a_1+w_2a_2)}}\right)^x$$
$$< \frac{\Gamma(a_1x)^{w_1}\Gamma(a_2x)^{w_2}}{\Gamma(w_1a_1x + w_2a_2x)} < \frac{w_1a_1 + w_2a_2}{a_1^{w_1}a_2^{w_2}}\left(\frac{a_1^{w_1a_1}a_2^{w_2a_2}}{(w_1a_1 + w_2a_2)^{w_1a_1+w_2a_2}}\right)^x$$

for $x > 0$. In particular, putting $(w_1, w_2) = (1/2, 1/2)$ and $x = 1$ we have

$$\frac{a_1^{a_1-1/2}a_2^{a_2-1/2}}{((a_1+a_2)/2)^{a_1+a_2-1}} < T(a_1, a_2) < \frac{a_1^{a_1-1}a_2^{a_2-1}}{((a_1+a_2)/2)^{a_1+a_2-2}}. \tag{35}$$

Multiplying the second inequality of (35) by $4a_1a_2/(a_1+a_2)^2$ yields

$$T(a_1+1, a_2+1) < \frac{a_1^{a_1}a_2^{a_2}}{((a_1+a_2)/2)^{a_1+a_2}},$$

which was established by Kečkić and Vasić [13]. The first inequality of (35) was proved in [17, Theorem 1] by Merkle. Furthermore, taking $n = 1$ and $x = 1$ in the right hand side inequality of (34) yields

$$\ln \mathscr{R}(1) < \frac{1}{2}c_0 + c_1 + \frac{d_{-1}}{12},$$

which is equivalent to

$$T(a_1, a_2) < \frac{a_1^{a_1-1/2}a_2^{a_2-1/2}}{((a_1+a_2)/2)^{a_1+a_2-1}} \exp\frac{(a_1-a_2)^2}{12a_1a_2(a_1+a_2)}.$$

This is due to Merkle [17, Theorem 1].

THEOREM 7. *Let $n \in \mathbb{N}$. The double inequality*

$$c_0 + \sum_{k=2}^{2n+1} \frac{d_k \psi^{(k-1)}(1)}{k!} x^k < \ln \mathscr{R}(x) < c_0 + \sum_{k=2}^{2n} \frac{d_k \psi^{(k-1)}(1)}{k!} x^k \tag{36}$$

holds for all $x > 0$.

Proof. By the double inequality (20), we have

$$2 + \sum_{k=2}^{2n+1} \frac{(-1)^k t^k}{k!} x^k < 1 + tx + e^{-tx} < 2 + \sum_{k=2}^{2n} \frac{(-1)^k t^k}{k! x^k}.$$

Applying the second inequality of the above double inequality to the integral representation (25) yields

$$\begin{aligned} \ln \mathscr{R}(x) &= \frac{1}{2}\int_0^\infty \frac{h_0(t)}{t}\left(1 + tx + e^{-tx}\right) dt < \frac{1}{2}\int_0^\infty \frac{h_0(t)}{t}\left(2 + \sum_{k=2}^{2n} \frac{(-1)^k t^k}{k!} x^k\right) dt \\ &= \int_0^\infty \frac{h_0(t)}{t} dt + \frac{1}{2}\sum_{k=2}^{2n} (-1)^k \left(\int_0^\infty t^{k-1} h_0(t)\, dt\right) \frac{x^k}{k!}, \end{aligned}$$

which, by means of Proposition 1, gives the right hand side inequality of (36). The left hand side one of (36) can be proved in a similar way.

REMARK 6. Theorem 7 tells us that the remainder of the Maclaurin series (28) satisfies

$$|R_m(x)| = \left| \ln \mathscr{R}(x) - c_0 - \sum_{k=2}^{m} d_k \psi^{(k-1)}(1) \frac{x^k}{k!} \right| < \frac{d_{m+1} \left|\psi^{(m)}(1)\right|}{(m+1)!} x^{m+1}.$$

Letting $w = (1/2, 1/2)$, $a = (a_1, a_2) = 1$ and $n, x = 1$ in Theorem 7 we have

$$c_0 + \frac{1}{2} d_2 \psi'(1) + \frac{1}{6} d_3 \psi''(1) < \ln \mathscr{R}(1) < c_0 + \frac{1}{2} d_2 \psi'(1)$$

with

$$d_2 = \frac{(a_1 - a_2)^2}{4}, d_3 = \frac{3}{8}(a_1 + a_2)(a_1 - a_2)^2.$$

From this we derive the following corollary.

COROLLARY 2. *The double inequality*

$$\begin{aligned} &\frac{(a_1 + a_2)^2}{4a_1 a_2} \exp\left(\frac{\pi^2}{24}(a_1 - a_2)^2 - \frac{1}{4}\zeta(3)(a_1 + a_2)(a_1 - a_2)^2\right) < \\ &\qquad T(a_1, a_2) < \frac{(a_1 + a_2)^2}{4a_1 a_2} \exp\left(\frac{\pi^2}{24}(a_1 - a_2)^2\right) \end{aligned} \tag{37}$$

holds for $a_1, a_2 > 0$ with $a_1 \neq a_2$.

REMARK 7. Inequalities (37) seem to offer a new type of bound for the Gurland ratio $T(a_1,a_2)$, which cannot be compared with corresponding bounds given in (35). Moreover, after replacing (a_1,a_2) by (a_1+1,a_2+1), inequalities (37) can be written as

$$\exp\left(\frac{\pi^2}{24}(a_1-a_2)^2-\frac{1}{4}\zeta(3)(a_1+a_2+2)(a_1-a_2)^2\right)<$$

$$T(a_1,a_2)<\exp\left(\frac{\pi^2}{24}(a_1-a_2)^2\right)$$

for $a_1,a_2>1$ with $a_1\neq a_2$.

5. Approximations for the inverse of the function $x\mapsto \ln\mathscr{R}(1/x)$

Let

$$\mathscr{L}(x)=L\left(\frac{1}{x}\right)=\ln\mathscr{R}\left(\frac{1}{x}\right),\quad x\in(0,\infty).$$

By (30) we see that

$$\mathscr{L}'(x)=-\frac{1}{x^2}L'\left(\frac{1}{x}\right)<0\ \text{ for } x\in(0,\infty),$$

and

$$\lim_{x\to 0^+}\mathscr{L}(x)=\lim_{x\to 0^+}\ln\mathscr{R}\left(\frac{1}{x}\right)=\lim_{x\to\infty}\ln\mathscr{R}(x)=\infty,$$
$$\lim_{x\to\infty}\mathscr{L}(x)=\lim_{x\to\infty}\ln\mathscr{R}\left(\frac{1}{x}\right)=\lim_{x\to 0^+}\ln\mathscr{R}(x)=d_0.$$

Therefore, the inverse of $\mathscr{L}$ exists and is decreasing from (d_0,∞) onto $(0,\infty)$. In this section, we are devote to present some approximations for $\mathscr{L}^{-1}$ by simple elementary functions, as applications of our results.

First, from the double inequality (33) we immediately get

$$\frac{1}{2}c_0+\frac{c_1}{x}<\mathscr{L}(x)=\ln\mathscr{R}\left(\frac{1}{x}\right)<c_0+\frac{c_1}{x},$$

which, by replacing $(x,\mathscr{L}(x))$ with $\left(\mathscr{L}^{-1}(x),x\right)$, yields

$$\frac{c_1}{x-c_0/2}<\mathscr{L}^{-1}(x)<\frac{c_1}{x-c_0} \tag{38}$$

for $x>c_0$. This implies that some combinations of $c_1/(x-\lambda_i)$ with $\lambda_i\in[c_0/2,c_0]$ ($i=1,\ 2,\ \ldots$) are better approximations for $\mathscr{L}^{-1}(x)$ than $c_1/(x-c_0/2)$ and $c_1/(x-c_0)$, for example,

$$\frac{1}{2}\left(\frac{c_1}{x-c_0/2}+\frac{c_1}{x-c_0}\right),\quad \frac{c_1}{\sqrt{(x-c_0/2)(x-c_0)}}.$$

Second, replacing x by $1/x$ and letting $n=1$ in the double inequality (34), we obtain

$$\frac{1}{2}c_0+\frac{c_1}{x}+\frac{d_{-1}}{12}x-\frac{d_{-3}}{360}x^3<\mathscr{L}(x)=\ln\mathscr{R}\left(\frac{1}{x}\right)<\frac{1}{2}c_0+\frac{c_1}{x}+\frac{d_{-1}}{12}x.$$

This gives a better approximation for $\mathscr{L}(x)$:

$$\mathscr{L}(x)\approx\frac{1}{2}c_0+\frac{c_1}{x}+\frac{d_{-1}}{12}x,$$

whose absolute error is less than $d_{-3}x^3/360$. Clearly, it is suitable for the case of $x\in(0,1)$. Solving the approximate equation above for x gives

$$x\approx\frac{6}{d_{-1}}\left(\mathscr{L}-c_0/2\pm\sqrt{(\mathscr{L}-c_0/2)^2-d_{-1}c_1/3}\right)$$

if $(\mathscr{L}-c_0/2)^2-d_{-1}c_1/3\geqslant 0$, that is, $\mathscr{L}>c_0/2+\sqrt{d_{-1}c_1/3}$. Since $\left(\mathscr{L}^{-1}(x)\right)'<0$ and

$$\frac{d_{-1}}{6}\frac{d}{dx}\left(x-c_0/2-\sqrt{(x-c_0/2)^2-d_{-1}c_1/3}\right)=1-\frac{x-c_0/2}{\sqrt{(x-c_0/2)^2-d_{-1}c_1/3}}<0,$$

we choose the smaller root as an approximation for $\mathscr{L}^{-1}(x)$.

THEOREM 8. *Let c_0, c_1 and d_{-1} be given by (18), (22) and (17), respectively. Then the inequality*

$$\mathscr{L}^{-1}(x)\lessapprox\frac{6}{d_{-1}}\left(x-c_0/2-\sqrt{(x-c_0/2)^2-d_{-1}c_1/3}\right)=A_1(x) \tag{39}$$

holds for $x\geqslant c_0/2+\sqrt{d_{-1}c_1/3}$.

Now let $(w_1,w_2)=(1/2,1/2)$, $(a_1,a_2)=(1,3)$. Simple calculations yield

$$c_0=\ln\frac{a_1+a_2}{2}-\frac{1}{2}\ln(a_1a_2)=\frac{1}{2}\ln\frac{4}{3}\approx 0.143\,84,$$
$$c_1=\frac{1}{2}(a_1\ln a_1+a_2\ln a_2)-\frac{a_1+a_2}{2}\ln\frac{a_1+a_2}{2}=\frac{1}{2}\ln\frac{27}{16}\approx 0.261\,62,$$
$$d_{-1}=\frac{1/a_1+1/a_2}{2}-\frac{2}{a_1+a_2}=\frac{1}{6}.$$

Then we derive a concrete approximation formula for $\mathscr{L}^{-1}(x)$:

$$\mathscr{L}^{-1}(x)\lessapprox A_1(x)=36\left(x-\frac{1}{4}\ln\frac{4}{3}-\sqrt{\left(x-\frac{1}{4}\ln\frac{4}{3}\right)^2-\frac{1}{36}\ln\frac{27}{16}}\right) \tag{40}$$

for $x\geqslant\ln(4/3)/4+\sqrt{\ln(27/16)}/6\approx 0.192\,48$, where

$$x=L(y)=\frac{1}{2}\ln\Gamma(y)+\frac{1}{2}\ln\Gamma(3y)-\ln\Gamma(2y).$$

REMARK 8. From Table 1, we find that the maximum absolute error of the approximation formula (40) is less 0.04 for $x \geqslant 0.24895$. Clearly, it is suitable for the case of $\mathscr{L}^{-1}(x) \leqslant 5/3$, that is, the shape parameter in a generalized Gaussian distribute is not greater than $5/3$.

Table 1: The absolute error estimating $\mathscr{L}^{-1}(x)$ by $A_1(x)$

y	$\mathscr{L}^{-1}(x) = 1/y$	$x = L(y)$	$A_1(x)$	$A_1(x) - \mathscr{L}^{-1}(x)$
$7/20$	$20/7$	0.19300	3.95550	1.09840
$2/5$	$5/2$	0.20359	2.83440	0.33440
$1/2$	2	0.22579	2.09740	0.09740
$4/7$	$7/4$	0.23856	1.80018	0.05018
$3/5$	$5/3$	0.24895	1.70630	0.03963
$3/4$	$4/3$	0.28486	1.34700	0.01367
$4/5$	$5/4$	0.29705	1.26010	0.01010
$9/10$	$10/9$	0.32168	1.11690	0.00579
1	1	0.34657	1.00350	0.00350
$4/3$	$3/4$	0.43077	0.75089	0.00089
$3/2$	$2/3$	0.47333	0.66716	0.00049

Third, replacing x by $1/x$ and letting $n = 1$ in the double inequality (36), we obtain

$$c_0 + \frac{\pi^2 d_2}{12} \frac{1}{x^2} - \frac{d_3 \zeta(3)}{3} \frac{1}{x^3} < \mathscr{L}(x) = \ln \mathscr{R}\left(\frac{1}{x}\right) < c_0 + \frac{\pi^2 d_2}{12} \frac{1}{x^2}.$$

This gives another better approximation for $\mathscr{L}(x)$:

$$\mathscr{L}(x) \lessapprox c_0 + \frac{\pi^2 d_2}{12} \frac{1}{x^2}, \tag{41}$$

whose absolute error is less than $d_3 \zeta(3) x^{-3}/3$. Clearly, it is suitable for the case of $x \in (1, \infty)$.

THEOREM 9. *Let c_0 and d_r be given by (18) and (17), respectively. Then the inequality*

$$\mathscr{L}^{-1}(x) \leqslant \frac{\pi \sqrt{3 d_2}}{6} \frac{1}{\sqrt{x - c_0}} \tag{42}$$

holds for all $x > c_0$. Moreover, we have

$$\lim_{x \to c_0^+} \left(\mathscr{L}^{-1}(x) - \frac{\pi \sqrt{3 d_2}}{6} \frac{1}{\sqrt{x - c_0}} \right) = -\frac{2 \zeta(3)}{\pi^2} \frac{d_3}{d_2}.$$

Proof. Replacing $(x, \mathscr{L}(x))$ with $\left(\mathscr{L}^{-1}(x), x\right)$ in the inequality (41) gives (42). To obtain the desired limit value, we let $\mathscr{L}^{-1}(x) = 1/y$, $y \in (0, \infty)$. Then $x = \mathscr{L}(1/y) =$

$L(y)$, and then, by (30), $y' = 1/L'(y) > 0$ for $y > 0$. Using the Maclaurin series (28) yields

$$\mathscr{L}^{-1}(x) - \frac{\pi\sqrt{3d_2}}{6}\frac{1}{\sqrt{x-c_0}} = \frac{1}{y} - \frac{\pi\sqrt{3d_2}}{6}\frac{1}{\sqrt{L(y)-c_0}}$$

$$\sim \frac{1}{y} - \frac{\sqrt{c_2}}{\sqrt{c_2y^2+c_3y^3+c_4y^4}} = \frac{1}{y}\frac{\sqrt{c_2+c_3y+c_4y^2}-\sqrt{c_2}}{\sqrt{c_2+c_3y+c_4y^2}}$$
$$= \frac{1}{y}\frac{c_2+c_3y+c_4y^2-c_2}{\left(\sqrt{c_2+c_3y+c_4y^2}+\sqrt{c_2}\right)\sqrt{c_2+c_3y+c_4y^2}} \to \frac{c_3}{2c_2} \text{ as } y \to 0,$$

where $c_k = d_k\psi^{(k-1)}(1)/k!$ for $k \geqslant 2$. This ends the proof.

REMARK 9. Theorem 9 reminds us that

$$\mathscr{L}^{-1}(x) \approx \frac{\pi\sqrt{3d_2}}{6}\frac{1}{\sqrt{x-c_0}} - \frac{2\zeta(3)}{\pi^2}\frac{d_3}{d_2} = A_2(x)$$

as x is near to c_0. Unfortunately, numeric computations show that this approximation formula $A_2(x)$ is not accurate enough. This way for an improvement of accuracy is to extend $A_2(x)$ as

$$A_2^*(x) = A_2(x) + \sum_{k=1}^{n}\beta_k(x-c_0)^{p_k}$$

with $p_k > 0$ and $\beta_k \in \mathbb{R}$, which is still an open problem.

6. Conclusions

This paper is devoted to investigating properties of generalized Gaussian ratio $\mathscr{R}(x)$ defined by (3). By means of some lemmas, we established an integral, asymptotic expansion and Maclaurin series representations of the logarithm of generalized Gaussian ratio $\ln\mathscr{R}(x)$, and found that $(\ln\mathscr{R}(x))'$ and $(\ln\mathscr{R}(x))''$ are Bernstein and completely monotonic functions, respectively. More importantly, we showed a validity of Theorems 6 and 7. The former asserts that the function $(-1)^m F_m(x)$ defined by (32) is completely monotonic on $(0,\infty)$, which not only yields two interesting inequalities (33) and (34) for the function $\ln\mathscr{R}(x)$, but also provides an estimation of remainder of an asymptotic expansion for $\ln\mathscr{R}(x)$. While the latter gives a new double inequality (37) for the Gurland ratio and offers an estimation of remainder of the Maclaurin series for $\ln\mathscr{R}(x)$. Consequently, several approximate formulas for the inverse of $\ln\mathscr{R}(1/x)$ are obtained, which leads to an estimation of shape parameter for generalized Gaussian distribution.

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