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MIXED INVEX EQUILIBRIUM PROBLEMS WITH GENERALIZED RELAXED MONOTONE AND RELAXED INVARIANT PSEUDOMONOTONE MAPPINGS

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Abstract. In this paper, we introduce generalized relaxed monotone mappings and relaxed invariant pseudomonotone mappings for bi-functions. By using KKM technique, we establish certain existence results for mixed invex equilibrium problems with the generalized relaxed monotone mappings and some of the results for invex equilibrium problems with the relaxed invariant pseudomonotone mappings in Banach spaces.

1. Introduction

The equilibrium problems (EP), which was first introduced by Blum and Oettli [5] in 1994 has now found applications in various branches of mathematics. It is a very important tool to solve many typical problems in mathematics like optimization, variational inequalities and complementarity problems. It includes many mathematical problems as special cases, such as mathematical programming problems, smooth and non-smooth optimization problems, etc. see [5, 12, 14].

In recent years many researchers extended the concept of monotonicity in various directions such as Verma defined $P-$ monotonicity [17], Bianchi et al. discussed quasi-monotonicity and strict pseudomonotonicity [4] in 2004, Bai et al. have talked about the concept of relaxed $\eta-\alpha$ pseudomonotonicity [3] in 2006. These monotonicity concepts were used to prove the equilibrium problems and variational inequalities problems in a number of directions [2, 6, 8, 15]. Liu [9] in 2016 studied invex equilibrium problem under relaxed $\eta-\alpha$ pseudomonotonicity.

In 2014, Arunchai et al. [1] introduced relaxed $\eta-\alpha$ pseudomonotonicity: let X be a real reflexive Banach space with its dual X' and $\langle\cdot,\cdot\rangle$ be the pairing between X^* and X. Let K be a nonempty subset of X and $\eta : K\times K \to X$ and $\alpha : X \to \mathbb{R}$ be the

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mappings with $\limsup_{t\to 0^+} \frac{\alpha(t\eta(x,y))}{t} = 0$, $\forall (x,y) \in K\times K$. A mapping $F : K \to X^*$ is said to be relaxed $\eta-\alpha$ pseudomonotone if for every distinct points $x,y \in K$,

$$\langle Fy, \eta(x,y)\rangle \geqslant 0 \implies \langle Fx, \eta(x,y)\rangle \geqslant \alpha(\eta(x,y)).$$

They have proved the following results for the variational-like inequality (VLI(K,F)): find a vector $x \in K$ such that

$$\langle Fx, \eta(y,x)\rangle \geqslant 0,\ \forall y \in K.$$

THEOREM A. [[1], Theorem 3.1] *Let K be a nonempty closed and convex subset of a real reflexive Banach space X. Let $F : K \to X^*$ and $\eta : K\times K \to X$ be mappings. Assume that:*
(i) F is η-hemicontinuous and relaxed $\eta-\alpha$ pseudomonotone;
(ii) $\eta(x,x) = 0$, $\forall x \in K$;
(iii) $\eta(tx+(1-t)z,y) = t\eta(x,y)+(1-t)\eta(z,y)$, $\forall x,y,z \in K$, $t \in [0,1]$.
Then $x \in K$ is a solution of VLI(K,F) if and only if

$$\langle Fy, \eta(y,x)\rangle \geqslant \alpha(\eta(y,x)),\ \forall y \in K.$$

THEOREM B. [[1], Theorem 3.2] *Let X be a real reflexive Banach space and K be a nonempty closed convex subset of X. Let $T : K \to X^*$ and $\eta : K\times K \to X$ be mappings. Assume that*
(i) T is relaxed $\eta-\alpha$ pseudomonotone and $\eta-$hemicontinuous;
(ii) $\eta(x,x) = 0$, for all $x \in K$;
(iii) $\eta(tx+(1-t)z,y) = t\eta(x,y)+(1-t)\eta(z,y)$, $\forall x,y,z \in K$, $t \in [0,1]$ and η is lower semicontinuous;
(iv) $\alpha : X \to R$ is lower semicontinuous.
Then the following statements are equivalent:

(a) There exists a reference point $x^{ref} \in K$ such that the set

$$L_<(T,x^{ref}) := \{x \in K : \langle Tx, \eta(x,x^{ref})\rangle < \alpha(\eta(x,x^{ref}))\},$$

is bounded.

(b) The variational-like inequality (VLI(K,T)) has a solution.

Moreover, if there exists a vector $x^{ref} \in K$ such that the set

$$L_\leqslant(T,x^{ref}) := \{x \in K : \langle Tx, \eta(x,x^{ref})\rangle \leqslant \alpha(\eta(x,x^{ref}))\},$$

is bounded and $\eta(x,y)+\eta(y,x) = 0$, $\forall x,y \in K$, then the solution set of variational-like inequality (VLI(K,T)) is nonempty and bounded.

Mahato and Nahak [11] in 2012 consider $(\rho-\theta)$ pseudomonotone operator with respect to θ, which is defined as follows: A function $f : K\times K \to R$ is said to be

$(\rho-\theta)$- pseudomonotone with respect to θ if for any pair of distnct points $x,y \in K$, one has

$$f(x,y) \geqslant 0 \text{ implies } f(y,x) \leqslant \rho\|\theta(x,y)\|^2.$$

They have used the above pseudomonotonicity and proved the following results for the equilibrium problem (EP): Find a vector $x^* \in K$ such that

$$f(x^*,y) \geqslant 0,\ \forall y \in K.$$

THEOREM C. [[11], Theorem 4.8] *Let K be a nonempty bounded convex subset of a real reflexive Banach space X. Suppose $f : K \times K \to \mathbb{R}$ is $(\rho-\theta)$- pseudomonotone with respect to θ and hemicontinuous in the first argument with the following conditions:*
(i) $f(x,y) = 0$, for all $x \in K$;
(ii) for fixed $z \in K$, the mapping $x \mapsto f(z,x)$ is convex and lower semicontinuous;
(iii) $\theta(x,y)+\theta(y,x) = 0$, for all $x,y \in K$;
(iv) θ is convex in first argument, concave in second argument and lower semicontinuous in the first argument.
Then the equilibrium problem (EP) has a solution.

THEOREM D. [[11], Theorem 4.9] *Let K be a nonempty unbounded closed convex subset of a real reflexive Banach space X. Suppose $f : K \times K \to \mathbb{R}$ is $(\rho-\theta)$- pseudomonotone with respect to θ and hemicontinuous in the first argument and satisfy the following assumptions:*
(i) $f(x,y) = 0$, for all $x \in K$;
(ii) for fixed $z \in K$, the mapping $x \mapsto f(z,x)$ is convex and lower semicontinuous;
(iii) $\theta(x,y)+\theta(y,x) = 0$, for all $x,y \in K$;
(iv) θ is convex in first argument, concave in second argument and lower semicontinuous in the first argument;
(v) f is weakly coercive, that is there exists $x_0 \in K$ such that $f(x,x_0) < 0$, whenever $\|x\| \to +\infty$ and $x \in K$.
Then the equilibrium problem (EP) has a solution.

Again in 2014, Mahato and Nahak [10] have defined generalized relaxed α-monotonicity and proved some results on mixed equilibrium problems: Let K be a nonempty closed convex subset of a real reflexive Banach space X and $\alpha : K \times K \to \mathbb{R}$ be a real valued function. The bi-function $f : K \times K \to \mathbb{R}$ is said to be generalized relaxed α-monotone if

$$f(x,y)+f(y,x) \leqslant \alpha(y,x),\ \forall x,y \in K,$$

where $\lim_{t\to 0} \frac{\alpha(ty+(1-t)x,x)}{t} = 0$. The mixed equilibrium problem (MEP) considered by Mahato and Nahak [10] is to find a vector $\bar{x} \in K$ such that

$$f(\bar{x},y)+\phi(y)-\phi(\bar{x}) \geqslant 0,\ \forall y \in K,$$

where $\phi : K \to \mathbb{R}$ is a real valued function and $f : K \times K \to \mathbb{R}$ is an equilibrium bi-function. They proved the following existence results for the mixed equilibrium problem (MEP).

THEOREM E. [[10], Theorem 3.2] *Let K be a nonempty bounded closed convex subset of a real reflexive Banach space X. Suppose $f : K\times K \to \mathbb{R}$ with $f(x,x) = 0$, $\forall x \in K$ is generalized relaxed α-monotone and hemicontinuous in the first argument. Let $\phi : K \to \mathbb{R}$ be a convex and lower semicontinuous function. Also assume that:*
(i) for fixed $z \in K$, the mapping $x \mapsto f(z,x)$ is convex and lower semicontinuous;
(ii) $\alpha : X\times X \to \mathbb{R}$ is weakly upper semicontinuous in the second argument.
Then the mixed equilibrium problem (MEP) has a solution.

THEOREM F. [[10], Theorem 3.3] *Let K be a nonempty unbounded closed convex subset of a real reflexive Banach space X. Suppose $f : K\times K \to \mathbb{R}$ with $f(x,x) = 0$, $\forall x \in K$ is generalized relaxed α-monotone and hemicontinuous in the first argument; let $\phi : K \to \mathbb{R}$ be a convex and lower semicontinuous function. Assume that:*
(i) for fixed $z \in K$, the mapping $x \mapsto f(z,x)$ is convex and lower semicontinuous;
(ii) $\alpha : X\times X \to \mathbb{R}$ is weakly upper semicontinuous in the second argument;
(iii) f satisfies the weakly coercivity condition: that is there exists a point $x_0 \in K$ such that $f(x,x_0)+\phi(x_0)-\phi(x) < 0$, whenever $\|x\| \to +\infty$ and $x \in K$.
Then the mixed equilibrium problem (MEP) has a solution.

Verma [18] also defined the following strongly pseudomonotone operator and proved the following result for nonlinear variational inequality problem. Let X be a real non-reflexive Banach space with its dual X' and X'' be the dual of X'. A mapping T from subset K of X'' into X' is said to be strongly pseudomonotone if there exists a constant $r > 0$ such that

$$\langle Ty, x-y\rangle \geqslant 0 \implies \langle Tx, x-y\rangle \geqslant r\|x-y\|^2,\ \forall x,y \in K.$$

THEOREM G. [[18], Theorem 2.2] *Let K be a nonempty bounded closed and convex subset of X'' and $T : K \to X'$ be a strongly pseudomonotone operator. If T is continuous on finite dimensional space, then there exists a unique element x_0 in K such that*

$$\langle Tx_0, x-x_0\rangle \geqslant 0,\ \forall x \in K,$$

Recently, Gayatri et al. [13] defined generalized weakly relaxed α- monotonicity: Let K be a nonempty compact and convex subset of a real reflexive Banach space E with the dual E^* and $\alpha : E\times E \to \mathbb{R}$ be a function. The function $\phi : K\times K\times K \to \mathbb{R}$ is said to be generalized weakly relaxed α- monotone if

$$\phi(y,v,w)+\phi(y,w,v) \leqslant \alpha(v,w),$$

with $\lim_{t\to 0} \frac{d}{dt}\alpha(tv+(1-t)w,w) = 0$. They have used the generalized weakly relaxed $\alpha-$ monotonicity and proved some existence results of the following mixed equilibrium problem. Let $N : E\times E \to E^*$, $b : E\times E \to \mathbb{R}$ and $\eta : K\times K \to E$ be the functions. If $\phi : K\times K\times K \to \mathbb{R}$ is a function defined by $\phi(y,v,w) = \langle N(v,y), \eta(w,v)\rangle$, then the mixed equilibrium problem considered by Gayatri et al. [13] is to find a vector $\bar{w}$, such that

$$\phi(y,v,\bar{w})+b(\bar{w},v)-b(\bar{w},\bar{w}) \geqslant 0,\ \forall v \in K.$$

Inspired and motivated by these works we proved the results of this paper assuming that the published results and methods introduced by Arunchai [1] are correct. We introduce generalized relaxed $\eta-\alpha$ monotone mappings and relaxed $\rho-\theta$ invariant pseudomonotone mappings for bi-functions. We also introduce a class of equilibrium problems named as mixed invex equilibrium problem (MIEP). We then prove some existence results for MIEP with generalized relaxed $\eta-\alpha$ monotone mappings and some results for IEP with relaxed $\rho-\theta$ invariant pseudomonotone mappings by using KKM technique in reflexive Banach spaces. The results we establish here extend and generalize the corresponding results in [1, 9, 11, 10, 16].

2. Preliminaries

In this section we first define the mixed invex equilibrium problem (MIEP) and then describe how our problem contains some problems in the literature as special cases. After doing that, we recall some definitions and results that will be required for the proof of our results. Unless mentioned otherwise, we assume X to be a real reflexive Banach space and K to be a nonempty subset of X.

DEFINITION 1. Let $\phi : K \to R$ and $\eta : K\times K \to X$ be the mappings. If $f : K\times K \to R$ is an equilibrium bi-function, i.e., $f(x,x)=0,\ \forall x\in K$. Then the mixed invex equilibrium problem (MIEP) is to find a vector $x^*\in K$ such that

$$f(x^*,\eta(y,x^*))+\phi(y)-\phi(x^*)\geqslant 0,\forall y\in K. \tag{1}$$

REMARK 1. Now we give some special cases below:

1. If we take $\phi\equiv 0$, then the problem (1) reduces to invex equilibrium problems (IEP) given by Liu [9] in 2016, that is find a vector $x^*\in K$ such that

$$f(x^*,\eta(y,x^*))\geqslant 0,\ \forall y\in K. \tag{2}$$

2. If we take $\eta(y,x^*)=y$, then the problem (1) reduces to mixed equilibrium problem (MEP) given by Mahato [10]: Find $x^*\in K$ such that

$$f(x^*,y)+\phi(y)-\phi(x^*)\geqslant 0,\ \forall y\in K. \tag{3}$$

3. If we consider $\eta(y,x^*)=y$ and $\phi\equiv 0$, then the mixed invex equilibrium problem (1) reduces to classical equilibrium problem (EP) established by Blum and Oettli [5], that is find a vector $x^*\in K$ such that

$$f(x^*,y)\geqslant 0,\ \forall y\in K. \tag{4}$$

A real valued function f defined on a convex subset K of X is said to be hemicontinuous if $\lim\limits_{t\to 0^+} f(tx+(1-t)y)=f(y)$ for all $x,y\in K$ and is said to be positively homogeneous if $f(\lambda x)=\lambda f(x)$ for $\lambda>0$. A mapping $f : X\to R$ is said to be upper semicontinuous at $x\in X$ if for any sequence $\{x_n\}\in X$ converging to x, we have

$\limsup_{n\to\infty} f(x_n) \leqslant f(x)$ and is said to be weakly lower semicontinuous at $x \in X$ if for any sequence $\{x_n\} \in X$ converging weakly to x, we have $f(x) \leqslant \liminf_{n\to\infty} f(x_n)$. If $y_1, y_2, ..., y_n$ are n elements of K, then the convex hull of $y_1, y_2, ..., y_n$ is denoted by $co\{y_1, y_2, ..., y_n\}$.

DEFINITION 2. [7] The set-valued mapping $f : K \to 2^X$ is said to be a KKM mapping if for any finite subset $\{y_1, y_2, ..., y_n\}$ of K we have $co\{y_1, y_2, ..., y_n\} \subset \bigcup_{i=1}^{n} f(y_i)$.

LEMMA 1. ([7]) *Let M be a nonempty subset of a Hausdorff topological vector space X and $f : K \to 2^X$ be a KKM mapping. If $f(y)$ is closed in X for all $y \in M$ and compact for some $y \in M$, then*

$$\bigcap_{y\in M} f(y) \neq \phi.$$

3. MIEP with generalized relaxed $\eta - \alpha$ monotonicity

In this section we define generalized relaxed $\eta - \alpha$ monotone mappings for bifunctions and establish some existence results for mixed invex equilibrium problems (1) in reflexive Banach spaces by using KKM technique.

DEFINITION 3. A function $f : K \times K \to R$ is said to be generalized relaxed $\eta - \alpha$ monotone if there exists a function $\eta : K \times K \to X$ and a function $\alpha : K \times K \to R$ with $\lim_{t\to 0^+} \frac{\alpha(t\eta(y,x),x)}{t} = 0$, $\forall (y,x) \in K \times K$ such that, for any $x, y \in K$, we have

$$f(y, \eta(y,x)) - f(x, \eta(y,x)) \geqslant \alpha(\eta(y,x),x). \tag{5}$$

REMARK 2. If $f(x,y) = \langle Fx, y\rangle$, $\eta(y,x) = y - x$ and $\alpha(x,y) = \beta(x)$, then the generalized relaxed $\eta - \alpha$ monotone mapping f reduces to relaxed α monotone mapping $F : K \to X^*$, where X^* is the dual of X.

We begin by proving our first result:

THEOREM 1. *Let K be a nonempty closed and convex subset of a real reflexive Banach space X and $\phi : K \to R$ be a convex function. Suppose the mapping $f : K \times K \to R$ with $f(x,x) = 0$, $\forall x \in R$ is generalized relaxed $\eta - \alpha$ monotone and hemicontinuous in the first argument. Assume the following conditions:*
(i) for any fixed y, z, the mapping $x \to f(z, \eta(x,y))$ is convex;
(ii) $\eta(x,x) = 0$, $\forall x \in K$;
(iii) $\eta(tx + (1-t)y, z) = t\eta(x,z) + (1-t)\eta(y,z)$, $\forall x, y, z \in K$, $t \in [0,1]$.
Then the mixed invex equilibrium problem (1) and the following problem (6) are equivalent: Find a vector $x^* \in K$ such that

$$f(y, \eta(y,x^*)) + \phi(y) - \phi(x^*) \geqslant \alpha(\eta(y,x^*),x^*), \ \forall y \in K. \tag{6}$$

Proof. Suppose the mixed invex equilibrium problem (1) has a solution, then $\exists\, x^* \in K$ such that $f(x^*, \eta(y,x^*)) + \phi(y) - \phi(x^*) \geqslant 0,\ \forall y \in K$. Since f is generalized relaxed $\eta - \alpha$ monotone, we have $f(y, \eta(y,x^*)) + \phi(y) - \phi(x^*) \geqslant \alpha(\eta(y,x^*),x^*) + f(x^*, \eta(y,x^*)) + \phi(y) - \phi(x^*) \geqslant \alpha(\eta(y,x^*),x^*)$, for all $y \in K$. Thus $x^* \in K$ is a solution of the problem (6).
Conversely, suppose the problem (6) has a solution. Then, let $y \in K$ be any point and $x_t = ty + (1-t)x^*$, $t \in (0,1]$. Since K is convex, $x_t \in K$ and hence $f(x_t, \eta(x_t,x^*)) + \phi(x_t) - \phi(x^*) \geqslant \alpha(\eta(x_t,x^*),x^*)$. Which gives $tf(x_t, \eta(y,x^*)) + t(\phi(y) - \phi(x^*)) \geqslant \alpha(t\eta(y,x^*),x^*),\ \forall y \in K$. Thus $f(x_t, \eta(y,x^*)) + \phi(y) - \phi(x^*) \geqslant \frac{\alpha(t\eta(y,x^*),x^*)}{t}, \forall y \in K$. Since f is hemicontinuous in the first argument, we have $f(x^*, \eta(y,x^*)) + \phi(y) - \phi(x^*) \geqslant 0,\ \forall y \in K$.

COROLLARY 1. *Theorem 1 generalizes Theorem A of variational-like inequality problems to equilibrium problems.*

THEOREM 2. *Let K be a nonempty closed, convex and bounded subset of a real reflexive Banach space X and $\phi : K \to R$ be a convex and lower semicontinuous mapping. Suppose the mapping $f : K \times K \to R$ with $f(x,x) = 0,\ \forall x \in R$ is generalized relaxed $\eta - \alpha$ monotone, hemicontinuous in first argument and positively homogeneous in second argument. Assume the following conditions:*
(i) for any fixed y, z, the mapping $x \to f(z, \eta(x,y))$ is convex;
(ii) for any fixed y, z, the mapping $x \to f(z, \eta(y,x))$ is upper semicontinuous;
(iii) for each $y \in X$, the function $x \to \alpha(\eta(y,x),x)$ is weakly lower semicontinuous;
(iv) $\eta(x,x) = 0,\ \forall x \in K$;
(v) $\eta(tx + (1-t)y, z) = t\eta(x,z) + (1-t)\eta(y,z),\ \forall x,y,z \in K,\ t \in [0,1]$.
Then the mixed invex equilibrium problem (1) has a solution.

Proof. Consider the set valued mapping $F : K \to 2^X$ defined by

$$F(y) = \{x \in K : f(x, \eta(y,x)) + \phi(y) - \phi(x) \geqslant 0\},\ \forall y \in K.$$

We claim that F is a KKM mapping. Suppose F is not a KKM mapping, then there exists a subset $\{x_1, x_2, \ldots, x_n\}$ of K, such that $co\{x_1, x_2, \ldots, x_n\} \not\subseteq \bigcup_{i=1}^n F(x_i)$. That is there exists $x_0 \in co\{x_1, x_2, \ldots, x_n\}$, $x_0 = \sum_{i=1}^n t_i x_i$, where $t_i \geqslant 0$, $i = 1,2,\ldots,n$, $\sum_{i=1}^n t_i = 1$, but $x_0 \notin \bigcup_{i=1}^n F(x_i)$. From the definition of F, we have $f(x_0, \eta(x_i,x_0)) + \phi(x_i) - \phi(x_0) < 0,\ \forall i = 1,2,\ldots,n$. Since $\sum_{i=1}^n t_i = 1$, for $t_i \geqslant 0$, $i = 1,2,\ldots,n$, we have

$$\sum_{i=1}^n t_i(f(x_0, \eta(x_i,x_0)) + \phi(x_i) - \phi(x_0)) < 0. \tag{7}$$

Since f is positively homogeneous, by using convexity of ϕ, we get $0 = f(x_0, 0) = f(x_0, \eta(x_0,x_0)) = f(x_0, \eta(\sum_{i=1}^n t_i x_i, x_0)) \leqslant \sum_{i=1}^n t_i f(x_0, \eta(x_i,x_0))$
$< \sum_{i=1}^n t_i(\phi(x_0) - \phi(x_i)) = \phi(x_0) - \sum_{i=1}^n t_i \phi(x_i) \leqslant \phi(x_0) - \phi(x_0) = 0$, which is a contradiction. Thus F is a KKM mapping. If $G : K \to 2^X$ is another set valued mapping such that

$$G(y) = \{x \in K : f(y, \eta(y,x)) + \phi(y) - \phi(x) \geqslant \alpha(\eta(y,x),x),\ \forall y \in K\},$$

then $F(y) \subseteq G(y)$, $\forall y \in K$. For given $y \in K$, let $x \in F(y)$, then $f(x,\eta(y,x)) + \phi(y) - \phi(x) \geqslant 0$. As f is generalized relaxed $\eta - \alpha$ monotone mapping, we have $f(y,\eta(y,x)) + \phi(y) - \phi(x) \geqslant \alpha(\eta(y,x),x) + f(x,\eta(y,x)) + \phi(y) - \phi(x) \geqslant \alpha(\eta(y,x),x)$. Hence $x \in G(y) \implies F(y) \subseteq G(y)$, $\forall y \in K$. As F is a KKM mapping, so is G.

From the definition of G, it is clear that $G(y)$ is weakly closed for all $y \in K$ and since K is closed, bounded and convex, we get $G(y)$ is weakly compact in K for each $y \in K$. Therefore from Lemma 1 and Theorem 1 we have

$$\bigcap_{y \in K} F(y) = \bigcap_{y \in K} G(y) \neq \phi.$$

Thus there exists $x^* \in K$ such that $f(x^*,\eta(y,x^*)) + \phi(y) - \phi(x^*) \geqslant 0$, $\forall y \in K$. Hence x^* is a solution of mixed invex equilibrium problem (1).

COROLLARY 2. *If we take $\eta(y,x^*) = y$, then we get the solution of MEP, which was proved by Mahato and Nahak in Theorem E and Sintunavarat in Theorem 14 of* [16].

THEOREM 3. *Let K be a nonempty closed, convex and unbounded subset of a real reflexive Banach space X and $\phi : K \to R$ be a convex and lower semicontinuous function. Suppose the mapping $f : K \times K \to R$ with $f(x,x) = 0$, $\forall x \in R$ is generalized relaxed $\eta - \alpha$ monotone, hemicontinuous in first argument and positively homogeneous in second argument. Assume the following conditions:*
(i) for any fixed y, z, the mapping $x \to f(z,\eta(x,y))$ is convex;
(ii) for any fixed y, z, the mapping $x \to f(z,\eta(y,x))$ is upper semicontinuous;
(iii) for each $y \in X$, the function $x \to \alpha(\eta(y,x),x)$ is weakly lower semicontinuous;
(iv) $\eta(x,x) = 0$, $\forall x \in K$;
(v) $\eta(tx + (1-t)y, z) = t\eta(x,z) + (1-t)\eta(y,z)$, $\forall x,y,z \in K$, $t \in [0,1]$;
(vi) f is weakly coercive, that is there exists $x_0 \in K$ such that $f(x,\eta(x_0,x)) + \phi(x_0) - \phi(x) < 0$, whenever $\|x\| \to +\infty$ and $x \in K$.
Then MIEP (1) has a solution.

Proof. For $r > 0$, let $B_r = \{y \in K : \|y\| \leqslant r\}$. Consider the problem: find $x_r \in K \cap B_r$ such that

$$f(x_r,\eta(y,x_r)) + \phi(y) - \phi(x_r) \geqslant 0,\ \forall y \in K \cap B_r. \tag{8}$$

By Theorem 2, the problem (8) has at least one solution $x_r \in K \cap B_r$. Choose $\|x_0\| < r$ with x_0 as in condition (vi). Then $x_0 \in K \cap B_r$ and

$$f(x_r,\eta(x_0,x_r)) + \phi(x_0) - \phi(x_r) \geqslant 0. \tag{9}$$

If $\|x_r\| = r$, $\forall r$, we may choose r large enough, so that $f(x_r,\eta(x_0,x_r)) + \phi(x_0) - \phi(x_r) < 0$, which contradicts (9). Therefore there exists an r, such that $||x_r|| < r$. For any $y \in K$, we can choose $0 < t < 1$ small enough such that $x_r + t(y - x_r) \in K \cap B_r$. From equation (8), we have $0 \leqslant f(x_r,\eta(x_r + t(y-x_r),x_r)) + \phi(x_r + t(y-x_r)) - \phi(x_r) =$

$f(x_r, t\eta(y,x_r)) + \phi(x_r + t(y-x_r)) - \phi(x_r) \leqslant f(x_r, t\eta(y,x_r)) + t\phi(y) + (1-t)\phi(x_r) - \phi(x_r) = tf(x_r, \eta(y,x_r)) + t\phi(y) - t\phi(x_r) = t[f(x_r, \eta(y,x_r)) + \phi(y) - \phi(x_r)]$. Therefore $f(x_r, \eta(y,x_r)) + \phi(y) - \phi(x_r) \geqslant 0,\ \forall y \in K$. Hence MIEP (1) has a solution.

COROLLARY 3. *By taking $\eta(y,x^*) = y$, we get the solution of MEP, that Mahato and Nahak proved in Theorem F and Sintunavarat proved in Theorem 16 of* [16].

THEOREM 4. *Let K be a nonempty closed and convex subset of a real reflexive Banach space X and $\phi : K \to R$ be a convex and lower semicontinuous function. Suppose the function $f : K \times K \to R$ with $f(x,x) = 0,\ \forall x \in R$ is a generalized relaxed $\eta - \alpha$ monotone, hemicontinuous in first argument and positively homogeneous in second argument. Assume the following conditions:*
(i) for any fixed $y,\ z$, the mapping $x \to f(z, \eta(x,y))$ is convex;
(ii) for any fixed $y,\ z$, the mapping $x \to f(z, \eta(y,x))$ is upper semicontinuous;
(iii) for each $y \in X$, the function $x \to \alpha(\eta(y,x),x)$ is weakly lower semicontinuous;
(iv) $\eta(x,x) = 0,\ \forall x \in K$;
(v) $\eta(tx + (1-t)y, z) = t\eta(x,z) + (1-t)\eta(y,z),\ \forall x,y,z \in K,\ t \in [0,1]$.
Then MIEP (1) and the following problem (10) are equivalent:
Find a vector $x^* \in K$ such that the set

$$B^0_{x^*} = \{y \in K : f(y, \eta(y,x^*)) + \phi(y) - \phi(x^*) < \alpha(\eta(y,x^*),x^*)\}, \tag{10}$$

is bounded.

Proof. Suppose the mixed invex equilibrium problem (1) has a solution, then $\exists x^* \in K$ such that $f(x^*, \eta(y,x^*)) + \phi(y) - \phi(x^*) \geqslant 0,\ \forall\, y \in K$. Since f is generalized relaxed $\eta - \alpha$ monotone, we have $f(y, \eta(y,x^*)) + \phi(y) - \phi(x^*) \geqslant \alpha(\eta(y,x^*),x^*) + f(x^*, \eta(y,x^*)) + \phi(y) - \phi(x^*) \geqslant \alpha(\eta(y,x^*),x^*)$, for all $y \in K$. Thus, $B^0_{x^*}$ is empty and hence bounded.
Conversely, suppose $B^0_{x^*}$ is bounded, then there is an open ball Ω such that $B^0_{x^*} \cup \{x^*\} \subset \Omega$. But as $\partial\Omega \cap B^0_{x^*} = \phi$, we get $f(y, \eta(y,x^*)) + \phi(y) - \phi(x^*) \geqslant \alpha(\eta(y,x^*),x^*),\ \forall y \in K \cap \partial\Omega$. Define the set valued mapping $F : K \to 2^X$ by

$$F(y) = \{x \in K \cap \bar{\Omega} : f(x, \eta(y,x)) + \phi(y) - \phi(x) \geqslant 0\},\ \forall y \in K.$$

We claim that F is a KKM mapping. Suppose F is not a KKM mapping, then there exists a subset $\{x_1, x_2, \ldots, x_n\}$ of K, such that $co\{x_1, x_2, \ldots, x_n\} \nsubseteq \bigcup_{i=1}^n F(x_i)$. That is there exists $x_0 \in co\{x_1, x_2, \ldots, x_n\}$, $x_0 = \sum_{i=1}^n t_i x_i$, where $t_i \geqslant 0,\ i = 1, 2, \ldots, n,\ \sum_{i=1}^n t_i = 1$, but $x_0 \notin \bigcup_{i=1}^n F(x_i)$. From the definition of F, we have $f(x_0, \eta(x_i,x_0)) + \phi(x_i) - \phi(x_0) < 0,\ \forall i = 1, 2, \ldots, n$. Since $\sum_{i=1}^n t_i = 1$, for $t_i \geqslant 0,\ i = 1, 2, \ldots, n$, we have

$$\sum_{i=1}^n t_i(f(x_0, \eta(x_i,x_0)) + \phi(x_i) - \phi(x_0)) < 0. \tag{11}$$

Since f is positively homogeneous, by using convexity of ϕ, we get $0 = f(x_0, 0) = f(x_0, \eta(x_0,x_0)) = f(x_0, \eta(\sum_{i=1}^n t_i x_i, x_0)) \leqslant \sum_{i=1}^n t_i f(x_0, \eta(x_i,x_0)) < \sum_{i=1}^n t_i(\phi(x_0) - \phi(x_i))$

$= \phi(x_0) - \sum_{i=1}^{n} t_i \phi(x_i) \leqslant \phi(x_0) - \phi(x_0) = 0$, which is a contradiction. Thus F is a KKM mapping. If $G : K \to 2^X$ is another set valued mapping such that

$$G(y) = \{x \in K \cap \bar{\Omega} : f(y, \eta(y,x)) + \phi(y) - \phi(x) \geqslant \alpha(\eta(y,x), x),\ \forall y \in K,$$

then $F(y) \subseteq G(y)$, $\forall y \in K$. For given $y \in K$, let $x \in F(y)$, then $f(x, \eta(y,x)) + \phi(y) - \phi(x) \geqslant 0$. As f is generalized relaxed $\eta - \alpha$ monotone mapping, we have $f(y, \eta(y,x)) + \phi(y) - \phi(x) \geqslant \alpha(\eta(y,x),x) + f(x, \eta(y,x)) + \phi(y) - \phi(x) \geqslant \alpha(\eta(y,x),x)$. Hence $x \in G(y) \implies F(y) \subseteq G(y)$, $\forall y \in K$. As F is a KKM mapping, so is G.
From the definition of G, it is clear that $G(y)$ is weakly closed for all $y \in K \cap \bar{\Omega}$. Since K is closed and convex, $K \cap \bar{\Omega}$ is weakly compact and hence $G(y)$ is weakly compact in $K \cap \bar{\Omega}$ for each $y \in K$. Therefore from Lemma 1 and Theorem 1 we have

$$\bigcap_{y \in K} F(y) = \bigcap_{y \in K} G(y) \neq \phi.$$

Thus there exists $x^* \in K \cap \bar{\Omega}$ such that $f(x^*, \eta(y,x^*)) + \phi(y) - \phi(x^*) \geqslant 0$, $\forall y \in K$. Hence x^* is a solution of MIEP (1).

COROLLARY 4. *Theorem 4 generalizes the Theorem B of Arunchai [1] for variational inequality problem to the equilibrium problems in case of generalized relaxed $\eta - \alpha$ monotone mapping.*

4. IEP with relaxed $\rho - \theta$ invariant pseudomonotone mapping with respect to η

In this section we define relaxed $\rho - \theta$ invariant pseudomonotone mappings for bi-functions and prove some existence results of invex equilibrium problem (IEP) in reflexive Banach spaces by using a KKM technique.

DEFINITION 4. Let X be a real reflexive Banach space and K be a nonempty subset of X. Assume $\eta : K \times K \to R$ and $\theta : K \times K \to R$ are the functions and $\rho \in R$ is a constant. A function $f : K \times K \to R$ is said to be relaxed $\rho - \theta$ invariant pseudomonotone mapping with respect to η if for any $x, y \in K$, we have

$$f(x, \eta(y,x)) \geqslant 0 \implies f(y, \eta(y,x)) \geqslant \rho|\theta(y,x)|^2. \tag{12}$$

REMARK 3. If $f(x,y) = \langle F(x), y \rangle$, $\eta(y,x) = y - x$ and $\rho = 0$, then the relaxed $\rho - \theta$ invariant pseudomonotone mapping f reduces to pseudomonotone mapping $F : K \to X^*$, where X^* is the dual of X.

THEOREM 5. *Let K be a nonempty closed and convex subset of a real reflexive Banach space X. Suppose the mapping $f : K \times K \to R$ with $f(x,x) = 0$, $\forall x \in R$ is relaxed $\rho - \theta$ invariant pseudomonotone with respect to η and hemicontinuous in first argument. Assume the following conditions:*
(i) for any fixed y, z, the mapping $x \to f(z, \eta(x,y))$ is convex;

(ii) $\theta(x,y)+\theta(y,x)=0,\ \forall x,y\in K$*;*
(iii) $\theta(x,y)$ *is convex in second argument and concave in first argument.*
Then IEP (2) *and the following problem* (13) *are equivalent: Find a vector* $x^*\in K$ *such that*

$$f(y,\eta(y,x^*))\geqslant \rho|\theta(y,x^*)|^2,\ \forall y\in K. \tag{13}$$

Proof. Suppose the invex equilibrium problem (2) has a solution, then $\exists x^*\in K$ such that $f(x^*,\eta(y,x^*))\geqslant 0,\ \forall y\in K$. Since f is relaxed $\rho-\theta$ invariant pseudomonotone, we have $f(y,\eta(y,x^*))\geqslant \rho|\theta(y,x^*)|^2$, for all $y\in K$. Thus $x^*\in K$ is a solution of problem (13).
Conversely, suppose the problem (13) has a solution. Let $y\in K$ be any point and $x_t=ty+(1-t)x^*,\ t\in(0,1]$. Since K is convex, $x_t\in K$ and hence $f(x_t,\eta(x_t,x^*))\geqslant \rho|\theta(x_t,x^*)|^2$. Thus $tf(x_t,\eta(y,x^*))\geqslant \rho|\theta(x_t,x^*)|^2$. Now we have the following cases
Case I. For $\rho=0$.
$f(x_t,\eta(y,x^*))\geqslant 0$. Since f is hemicontinuous in the first argument, we get $f(x^*,\eta(y,x^*))\geqslant 0,\ \forall y\in K$.
Case II. For $\rho\leqslant 0$, let $\rho=-k^2$.
By convexity of θ in the second argument, we get $tf(x_t,\eta(y,x^*))\geqslant -k^2t^2|\theta(y,x^*)|^2 \Longrightarrow f(x_t,\eta(y,x^*))\geqslant -k^2t|\theta(y,x^*)|^2$. Since f is hemicontinuous in the first argument, we get $f(x^*,\eta(y,x^*))\geqslant 0,\ \forall y\in K$.
Case III. For $\rho\geqslant 0$, let $\rho=k^2$.
By concavity of θ in the first argument, we get $tf(x_t,\eta(y,x^*))\geqslant k^2t^2|\theta(y,x^*)|^2$. Which gives $f(x_t,\eta(y,x^*))\geqslant k^2t|\theta(y,x^*)|^2$. As f is hemicontinuous in the first argument, we have $f(x^*,\eta(y,x^*))\geqslant 0,\ \forall y\in K$. Combining the results of all the cases, the theorem is proved.

THEOREM 6. *Let* K *be a nonempty closed, convex and bounded subset of a real reflexive Banach space* X*. Suppose the mapping* $f:K\times K\to R$ *with* $f(x,x)=0,\ \forall x\in R$ *is relaxed* $\rho-\theta$ *invariant pseudomonotone with respect to* η*, hemicontinuous in the first argument and positively homogeneous in the second argument. Assume the following conditions:*
(i) for any fixed $y,\ z$*, the mapping* $x\to f(z,\eta(x,y))$ *is convex;*
(ii) for any fixed $y,\ z$*, the mapping* $x\to f(z,\eta(y,x))$ *is upper semicontinuous;*
(iii) $\theta(x,y)+\theta(y,x)=0,\ \forall x,y\in K$*;*
(iv) $\theta(x,y)$ *is convex in second argument, concave in first argument and lower semicontinuous with respect to second argument;*
(v) $\eta(x,x)=0,\ \forall x\in K$*.*
Then IEP (2) *has a solution.*

Proof. Consider the set valued mapping $F:K\to 2^X$ defined by

$$F(y)=\{x\in K: f(x,\eta(y,x))\geqslant 0\},\ \forall y\in K.$$

We claim that F is a KKM mapping. Suppose F is not a KKM mapping, then there exists a subset $\{x_1,x_2,\dots,x_n\}$ of K, such that $co\{x_1,x_2,\dots,x_n\}\not\subseteq\bigcup_{i=1}^n F(x_i)$. That is there exists $x_0 \in co\{x_1,x_2,...,x_n\}$, $x_0 = \sum_{i=1}^{n} t_i x_i$, where $t_i \geqslant 0,\ i=1,2,...,n,\ \sum_{i=1}^{n} t_i = 1$, but $x_0 \notin \bigcup_{i=1}^{n} F(x_i)$. From the definition of F, we have $f(x_0,\eta(x_i,x_0)) < 0,\ \forall i = 1,2,...,n$. Since $\sum_{i=1}^{n} t_i = 1$, for $t_i \geqslant 0,\ i=1,2,...,n$, we have

$$\sum_{i=1}^{n} t_i f(x_0,\eta(x_i,x_0)) < 0. \tag{14}$$

Since f is positively homogeneous, we get $0 = f(x_0,0) = f(x_0,\eta(x_0,x_0))$ $= f(x_0,\eta(\sum_{i=1}^{n} t_i x_i, x_0)) \leqslant \sum_{i=1}^{n} t_i f(x_0,\eta(x_i,x_0)) < 0$, which is a contradiction. Thus F is a KKM mapping. If $G : K \to 2^X$ is another set valued mapping such that

$$G(y) = \{x \in K : f(y,\eta(y,x)) \geqslant \rho|\theta(y,x)|^2,\ \forall y \in K,$$

then $F(y) \subseteq G(y),\ \forall y \in K$. For given $y \in K$, let $x \in F(y)$, then $f(x,\eta(y,x)) \geqslant 0$. As f is relaxed $\rho-\theta$ invariant pseudomonotone, we have $f(y,\eta(y,x)) \geqslant \rho|\theta(y,x)|^2,\ \forall y \in K$. Hence $x \in G(y) \implies F(y) \subseteq G(y),\ \forall y \in K$. As F is a KKM mapping, so is G. From the definition of G, it is clear that $G(y)$ is weakly closed for all $y \in K$ and since K is closed bounded and convex, $G(y)$ is weakly compact in K for each $y \in K$. Therefore from Lemma 1 and Theorem 5, we have

$$\bigcap_{y\in K} F(y) = \bigcap_{y\in K} G(y) \neq \phi.$$

Thus there exists $x^* \in K$ such that $f(x^*,\eta(y,x^*)) \geqslant 0,\ \forall y \in K$. Hence x^* is a solution of IEP (2).

COROLLARY 5. *Theorem 6 is a proper generalization the Theorem C from equilibrium problem (EP) to invex equilibrium problem (IEP).*

THEOREM 7. *Let K be a nonempty closed, convex and unbounded subset of a real reflexive Banach space X. Suppose $f : K \times K \to R$ with $f(x,x) = 0,\ \forall x \in R$ is a relaxed $\rho-\theta$ invariant pseudomonotone mapping with respect to η, hemicontinuous in the first argument and positively homogeneous in the second argument. Assume the following conditions:*
(i) for any fixed $y,\ z$, the mapping $x \to f(z,\eta(x,y))$ is convex;
(ii) for any fixed $y,\ z$, the mapping $x \to f(z,\eta(y,x))$ is upper semicontinuous;
(iii) $\theta(x,y) + \theta(y,x) = 0,\ \forall x,y \in K$;
(iv) $\theta(x,y)$ is convex in second argument, concave in first argument and lower semicontinuous with respect to second argument;
(v) $\eta(x,x) = 0,\ \forall x \in K$;
(vi) f is weakly coercive, that is there exists $x_0 \in K$ such that $f(x,\eta(x_0,x)) < 0$, whenever $\|x\| \to +\infty$ and $x \in K$.
Then IEP (2) has a solution.

Proof. For $r > 0$, let $B_r = \{y \in K : \|y\| \leqslant r\}$. Consider the problem: find $x_r \in K \cap B_r$ such that

$$f(x_r,\eta(y,x_r)) \geqslant 0,\ \forall y \in K \cap B_r. \tag{15}$$

By Theorem 6, problem (15) has at least one solution $x_r \in K \cap B_r$. Choose $\|x_0\| < r$ with x_0 as in condition (vi). Then $x_0 \in K \cap B_r$ and

$$f(x_r, \eta(x_0, x_r)) \geqslant 0. \tag{16}$$

If $\|x_r\| = r$, $\forall r$, we may choose r large enough, so that $f(x_r, \eta(x_0, x_r)) < 0$, which contradicts (16). Thus there exists an r such that $||x_r|| < r$. For any $y \in K$, we can choose $0 < t < 1$ small enough such that $x_r + t(y - x_r) \in K \cap B_r$. From equation (15), we get $f(x_r, \eta(x_r + t(y - x_r), x_r)) \geqslant 0 \implies 0 \leqslant t f(x_r, \eta(y, x_r)) + (1-t) f(x_r, \eta(x_r, x_r)) = t[f(x_r, \eta(y, x_r))]$. Therefore $f(x_r, \eta(y, x_r)) \geqslant 0$, $\forall y \in K$ and hence IEP (2) has a solution.

COROLLARY 6. *Theorem 7 generalizes Theorem D from equilibrium problem (EP) to invex equilibrium problem (IEP).*

THEOREM 8. *Let K be a nonempty closed convex subset of a real reflexive Banach space X. Suppose $f : K \times K \to R$ with $f(x,x) = 0$, $\forall x \in R$ is a relaxed $\rho - \theta$ invariant pseudomonotone mapping with respect to η, hemicontinuous in the first argument and positively homogeneous in the second argument. Assume the following conditions:*
(i) for any fixed y, z, the mapping $x \to f(z, \eta(x,y))$ is convex;
(ii) for any fixed y, z, the mapping $x \to f(z, \eta(y,x))$ is upper semicontinuous;
(iii) $\theta(x,y) + \theta(y,x) = 0$, $\forall x, y \in K$;
(iv) $\theta(x,y)$ is convex in second argument, concave in first argument and lower semicontinuous with respect to second argument;
(v) $\eta(x,x) = 0$, $\forall x \in K$.
Then IEP (2) and the following problem (17) are equivalent:
Find a vector $x^* \in K$, such that the set

$$B^0_{x^*} = \{y \in K : f(y, \eta(y, x^*)) < \rho|\theta(y, x^*)|^2\}, \tag{17}$$

is bounded.

Proof. Suppose the invex equilibrium problem (2) has a solution, then $\exists x^* \in K$ such that $f(x^*, \eta(y, x^*)) \geqslant 0$, $\forall y \in K$. Since f is relaxed $\rho - \theta$ invariant pseudomonotone, we have $f(y, \eta(y, x^*)) \geqslant \rho|\theta(y, x^*)|^2$, $\forall y \in K$. Thus $B^0_{x^*}$ is empty and hence bounded.
Conversely, suppose the set $B^0_{x^*}$ is bounded, then there is an open ball Ω such that $B^0_{x^*} \cup \{x^*\} \subset \Omega$. As $\partial\Omega \cap B^0_{x^*} = \phi$, we get $f(y, \eta(y, x^*)) \geqslant \rho|\theta(y, x^*)|^2$, $\forall y \in K \cap \partial\Omega$. Consider the set valued mapping $F : K \to 2^X$ defined by

$$F(y) = \{x \in K \cap \bar{\Omega} : f(x, \eta(y, x)) \geqslant 0\}, \ \forall y \in K.$$

We claim that F is a KKM mapping. Suppose F is not a KKM mapping, then there exists a subset $\{x_1, x_2, ..., x_n\}$ of K, such that $co\{x_1, x_2, ..., x_n\} \not\subseteq \bigcup_{i=1}^n F(x_i)$. That is there exists $x_0 \in co\{x_1, x_2, ..., x_n\}$, $x_0 = \sum_{i=1}^n t_i x_i$, where $t_i \geqslant 0$, $i = 1, 2, ..., n$, $\sum_{i=1}^n t_i =$

1, but $x_0 \notin \bigcup_{i=1}^{n} F(x_i)$. From the definition of F, we have $f(x_0, \eta(x_i, x_0)) < 0$, $\forall i = 1,2,\dots,n$. Since $\sum_{i=1}^{n} t_i = 1$, for $t_i \geqslant 0$, $i = 1,2,\dots,n$, we have

$$\sum_{i=1}^{n} t_i f(x_0, \eta(x_i, x_0)) < 0. \tag{18}$$

Since f is positively homogeneous in the second argument, we get $0 = f(x_0, 0) = f(x_0, \eta(x_0, x_0)) = f(x_0, \eta(\sum_{i=1}^{n} t_i x_i, x_0)) \leqslant \sum_{i=1}^{n} t_i f(x_0, \eta(x_i, x_0)) < 0$, which is a contradiction. Thus F is a KKM mapping. If $G : K \to 2^X$ is another set valued mapping such that

$$G(y) = \{x \in K : f(y, \eta(y,x)) \geqslant \rho|\theta(y,x)|^2, \ \forall y \in K,$$

then $F(y) \subseteq G(y)$, $\forall y \in K$. For given $y \in K$, let $x \in F(y)$, then $f(x, \eta(y,x)) \geqslant 0$. As f is relaxed $\rho - \theta$ invariant pseudomonotone, we have $f(y, \eta(y,x)) \geqslant \rho|\theta(y,x)|^2$, $\forall y \in K$. Hence $x \in G(y) \implies F(y) \subseteq G(y)$, $\forall y \in K$. As F is a KKM mapping, so is G. From the definition of G, it is clear that $G(y)$ is weakly closed for all $y \in K \cap \bar{\Omega}$. Since K is closed and convex, we have $K \cap \bar{\Omega}$ is weakly compact and hence $G(y)$ is weakly compact in $K \cap \bar{\Omega}$ for each $y \in K$. Therefore from Lemma 1 and Theorem 5 we have

$$\bigcap_{y \in K} F(y) = \bigcap_{y \in K} G(y) \neq \phi.$$

Thus there exists $x^* \in K \cap \bar{\Omega}$, such that $f(x^*, \eta(y, x^*)) \geqslant 0$, $\forall y \in K$. Hence x^* is a solution of IEP (2).

REMARK 4. Theorem 8 is a generalization of Theorem B from variational inequality problem to the equilibrium problems in case of relaxed $\rho - \theta$ invariant pseudomonotone mapping with respect to η.

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