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ON STEVIĆ–SHARMA OPERATORS FROM HARDY SPACES TO STEVIĆ WEIGHTED SPACES

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Abstract. In this paper, we investigate the boundedness and compactness of Stević-Sharma operator $T_{\psi_1,\psi_2,\varphi}$ from Hardy space H^p to Stević weighted space $\mathscr{W}_\mu^{(n)}$ on the unit disk, and estimate the norm of $T_{\psi_1,\psi_2,\varphi}$ when it is bounded.

1. Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$, $\mathscr{H}(\mathbb{D})$ the space of all holomorphic functions on $\mathbb{D}$, $\mathbb{N}$ the set of all positive integers.

The Hardy space $H^p = H^p(\mathbb{D})$, $0 < p < \infty$, consists of all $f \in \mathscr{H}(\mathbb{D})$ such that

$$\|f\|_{H^p}^p = \sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} < \infty.$$

With this norm H^p is a Banach space when $1 \leqslant p < \infty$, while for $0 < p < 1$ it is a topological vector space with the translation invariant metric $d(f,g) = \|f-g\|_{H^p}^p$, $f,g \in H^p$, which is not locally convex. For more information about the H^p spaces, one may consult [2].

A positive continuous function μ on [0,1) is called normal if there exist two positive numbers s and t with $0 < s < t$, and $\delta \in [0,1)$ such that (see [15])

$$\frac{\mu(r)}{(1-r)^s} \text{ is decreasing on } [\delta,1), \ \lim_{r\to 1} \frac{\mu(r)}{(1-r)^s} = 0;$$
$$\frac{\mu(r)}{(1-r)^t} \text{ is increasing on } [\delta,1), \ \lim_{r\to 1} \frac{\mu(r)}{(1-r)^t} = \infty.$$

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Let $\mu(z)=\mu(|z|)$ be a normal function. The Stević weighted space on $\mathbb{D}$, denoted by $\mathscr{W}_\mu^{(n)}=\mathscr{W}_\mu^{(n)}(\mathbb{D})$, was introduced by Stević in [16] (it was called the nth weighted space there; see also [19, 22]) and consisted of all $f\in\mathscr{H}(\mathbb{D})$ such that

$$\|f\|_\mu=\sup_{z\in\mathbb{D}}\mu(z)|f^{(n)}(z)|<\infty,\quad n\in\mathbb{N}.$$

For $n=0$ the space becomes the weighted-type space H_μ^∞, for $n=1$ the Bloch-type space $\mathscr{B}_\mu$ and for $n=2$ the Zygmund-type space $\mathscr{Z}_\mu$ (the notation was essentially introduced in [6]). For some results on the space, their generalizations, and operators on them see, for example, [5, 6, 7, 8, 9, 10, 17, 20, 24, 29]. $\mathscr{W}_\mu^{(n)}$ becomes a Banach space normed by

$$\|f\|_{\mathscr{W}_\mu^{(n)}}=\sum_{j=0}^{n-1}|f^{(j)}(0)|+\|f\|_\mu.$$

It is well known that the differentiation operator D is defined by

$$(Df)(z)=f'(z),\quad f\in\mathscr{H}(\mathbb{D}).$$

Let $u\in\mathscr{H}(\mathbb{D})$, then the multiplication operator M_u is defined by

$$(M_uf)(z)=u(z)f(z),\quad f\in\mathscr{H}(\mathbb{D}).$$

Recently there has been some interest in product-type operators (see, for example, [5, 7, 8, 10, 11, 13, 17, 18, 20, 21, 22, 23, 25, 26, 27, 28] and the related references therein).

Let φ be an analytic self-map of $\mathbb{D}$. The composition operator C_φ is defined by

$$(C_\varphi f)(z)=f\big(\varphi(z)\big),\quad f\in\mathscr{H}(\mathbb{D}).$$

In [14], Sharma defined six product-type operators as follows:

$$\begin{aligned}
&(M_uC_\varphi Df)(z)=u(z)f'\big(\varphi(z)\big),\\
&(M_uDC_\varphi f)(z)=u(z)\varphi'(z)f'\big(\varphi(z)\big),\\
&(C_\varphi M_uDf)(z)=u\big(\varphi(z)\big)f'\big(\varphi(z)\big),\\
&(DM_uC_\varphi f)(z)=u'(z)f\big(\varphi(z)\big)+u(z)\varphi'(z)f'\big(\varphi(z)\big),\\
&(C_\varphi DM_uf)(z)=u'\big(\varphi(z)\big)f\big(\varphi(z)\big)+u\big(\varphi(z)\big)f'\big(\varphi(z)\big),\\
&(DC_\varphi M_uf)(z)=u'\big(\varphi(z)\big)\varphi'(z)f\big(\varphi(z)\big)+u\big(\varphi(z)\big)\varphi'(z)f'\big(\varphi(z)\big)
\end{aligned}$$

for $z\in\mathbb{D}$ and $f\in\mathscr{H}(\mathbb{D})$.

Stević and Sharma introduced the following so-called Stević-Sharma operator to treat the operators above in a unified manner:

$$(T_{\psi_1,\psi_2,\varphi}f)(z)=\psi_1(z)f\big(\varphi(z)\big)+\psi_2(z)f'\big(\varphi(z)\big),\quad f\in\mathscr{H}(\mathbb{D}),$$

where $\psi_1,\ \psi_2 \in \mathscr{H}(\mathbb{D})$ and φ is an analytic self-map of $\mathbb{D}$ (see, for example, [25] and [26]).

By taking some specific choices of the involving symbols, we can obtain the above mentioned six product-type operators:

$$M_uC_\varphi D = T_{0,u,\varphi}, \quad M_uDC_\varphi = T_{0,u\varphi',\varphi}, \quad C_\varphi M_uD = T_{0,u\circ\varphi,\varphi},$$
$$DM_uC_\varphi = T_{u',u\varphi',\varphi}, \quad C_\varphi DM_u = T_{u',u\varphi',\varphi}, \quad DC_\varphi M_u = T_{\varphi'u'\circ\varphi,\varphi'u\circ\varphi,\varphi}.$$

Quite recently, many authors considered Stević-Sharma operator $T_{\psi_1,\psi_2,\varphi}$ and characterized the boundedness and compactness between various spaces. For instance, Jiang in [3] studied the boundedness and compactness of $T_{\psi_1,\psi_2,\varphi}$ from Zygmund space to Bloch-Orlicz space. Liu and Yu in [12] completely described the boundedness and compactness of $T_{\psi_1,\psi_2,\varphi}$ from the Besov space B_p $(1<p<\infty)$ into the (little) weighted-type space. Yu and Liu in [28] investigated the boundedness and compactness of the operator $T_{\psi_1,\psi_2,\varphi}$ from H^∞ space to the logarithmic Bloch space. Zhang and Zeng in [30] characterized the boundedness and compactness of the weighted differentiation composition operator from weighted Bergman space to Stević weighted space. Stević in [19] studied the composition operator from Hardy space to Stević weighted space on the unit disk: $C_\varphi : H^p \to \mathscr{W}_\mu^{(n)}$ is bounded if and only if

$$\sup_{z\in\mathbb{D}} \frac{\mu(z)\left|\sum \frac{n!}{k_1!\cdots k_n!}\prod_{j=1}^n \left(\frac{\varphi^{(j)}(z)}{j!}\right)^{k_j}\right|}{\left(1-|\varphi(z)|^2\right)^{k+\frac{1}{p}}} < \infty, \quad k=1,2,\cdots,n,$$

where for each fixed $k\in\{1,2,\cdots,n\}$, the sum is over all non-negative integers $k_1,k_2,\cdots,k_n$ such that $k=k_1+k_2+\cdots+k_n$ and $k_1+2k_2+\cdots+nk_n=n$. Zhang and Liu in [29] studied the boundedness and compactness of Stević-Sharma operator $T_{\psi_1,\psi_2,\varphi}$ from Hardy space to Zygmund-type space on the unit disk. Recall that Zygmund-type space is a special Stević weighted space for $n=2$. It is of some interest to extend the results for the case of Stević weighted space $\mathscr{W}_\mu^{(n)}$. For this purpose, we first present a formula for the n-th-order derivative of $T_{\psi_1,\psi_2,\varphi}f$, which is a simple consequence of a formula in [21] (see also [22]), and is based on the classical Faàdi Bruno's formula (see, e.g., [4]). To prove our main results on the boundedness and compactness of the operator from Hardy space to Stević weighted space, we follow the methods and ideas, for example, in [16, 19, 21, 22].

In what follows, we use the letter C to denote a positive constant whose value may change at each occurrence. The notation $a \preceq b$ means that there is a positive constant C such that $a \leqslant Cb$. Moreover, if both $a \preceq b$ and $b \preceq a$ hold, then one says that $a \asymp b$.

2. Preliminaries

In this section we formulate some auxiliary results which will be used in the proof of the main results. The following two lemmas are folklore (see, for example, [19]).

LEMMA 1. *Assume that* $0<p<\infty$, $f\in H^p$ *and* $n\in\mathbb{N}_0$. *Then there is a positive constant* C *independent of* f *such that*

$$\left|f^{(n)}(z)\right| \leqslant C\frac{\|f\|_{H^p}}{\left(1-|z|^2\right)^{\frac{1}{p}+n}}, \quad z\in\mathbb{D}.$$

LEMMA 2. *Let* $0<p<\infty$, $j\in\mathbb{N}$. *For a fixed* $\omega\in\mathbb{D}$, *set*

$$h_{\omega,j}(z)=\frac{\left(1-|\omega|^2\right)^j}{\left(1-\overline{\omega}z\right)^{\frac{1}{p}+j}}, \quad z\in\mathbb{D},$$

then there is a positive constant C_j *such that* $h_{\omega,j}\in H^p$ *and* $\sup\limits_{\omega\in\mathbb{D}}\|h_{\omega,j}\|_{H^p}\leqslant C_j$.

LEMMA 3. *Let* $a>0$ *and*

$$D_{n+2}(a)=\begin{vmatrix} 1 & 1 & \cdots & 1\\ a & a+1 & \cdots & a+n+1\\ \vdots & \vdots & & \vdots\\ \prod_{j=0}^{n}(a+j) & \prod_{j=0}^{n}(a+j+1) & \cdots & \prod_{j=0}^{n}(a+j+n+1)\end{vmatrix}.$$

Then $D_{n+2}(a)=\prod_{j=1}^{n+1}j!$.

Proof. Replacing n by $n+2$ in [16, Lemma 2.3], the lemma easily follows. □

LEMMA 4. [21, Lemma 4] *Assume that* $n\in\mathbb{N}$, $u,f\in\mathscr{H}(\mathbb{D})$ *and* φ *is an analytic self-map of* $\mathbb{D}$. *Then*

$$\left(u(z)f\left(\varphi(z)\right)\right)^{(n)}=\sum_{k=0}^{n}f^{(k)}\left(\varphi(z)\right)\sum_{l=k}^{n}C_n^l u^{(n-l)}(z)B_{l,k}\left(\varphi'(z),\varphi''(z),\cdots,\varphi^{(l-k+1)}(z)\right),$$

where

$$B_{l,k}\left(\varphi'(z),\varphi''(z),\cdots,\varphi^{(l-k+1)}(z)\right)=\sum_{k_1,\cdots,k_l}\frac{l!}{k_1!\cdots k_l!}\prod_{j=1}^{l}\left(\frac{\varphi^{(j)}(z)}{j!}\right)^{k_j},$$

and the sum is overall non-negative integers $k_1,\cdots,k_l$ *satisfying* $k_1+k_2+\cdots+k_l=k$ *and* $k_1+2k_2+\cdots+lk_l=l$.

By using Lemma 4, we can get the following lemma.

LEMMA 5. *Assume that* $n\in\mathbb{N}$, $\psi_1,\psi_2,f\in\mathscr{H}(\mathbb{D})$ *and* φ *is an analytic self-map of* $\mathbb{D}$. *Then*

$$\left(T_{\psi_1,\psi_2,\varphi}f(z)\right)^{(n)}=\sum_{k=0}^{n+1}f^{(k)}\left(\varphi(z)\right)\Omega_k(z),$$

where

$$\Omega_k(z) = \begin{cases} \psi_1^{(n)}(z), & k=0, \\ \sum_{l=k}^n C_n^l \psi_1^{(n-l)}(z) B_{l,k}\big(\varphi'(z),\cdots,\varphi^{(l-k+1)}(z)\big) & \\ +\sum_{l=k-1}^n C_n^l \psi_2^{(n-l)}(z) B_{l,k-1}\big(\varphi'(z),\cdots,\varphi^{(l-k+2)}(z)\big), & k=1,2,\cdots,n, \\ \psi_2(z)\varphi'(z)^n, & k=n+1. \end{cases}$$

Proof. By a direct calculation, we have

$$\begin{aligned}
&\big(T_{\psi_1,\psi_2,\varphi} f(z)\big)^{(n)} \\
=&\big(\psi_1(z) f(\varphi(z))\big)^{(n)} + \big(\psi_2(z) f'(\varphi(z))\big)^{(n)} \\
=&\sum_{k=0}^n f^{(k)}(\varphi(z)) \sum_{l=k}^n C_n^l \psi_1^{(n-l)}(z) B_{l,k}\big(\varphi'(z),\varphi''(z),\cdots,\varphi^{(l-k+1)}(z)\big) \\
&+\sum_{k=0}^n f^{(k+1)}(\varphi(z)) \sum_{l=k}^n C_n^l \psi_2^{(n-l)}(z) B_{l,k}\big(\varphi'(z),\varphi''(z),\cdots,\varphi^{(l-k+1)}(z)\big) \\
=&\psi_1^{(n)}(z) f(\varphi(z)) + \sum_{k=1}^n f^{(k)}(\varphi(z)) \sum_{l=k}^n C_n^l \psi_1^{(n-l)}(z) B_{l,k}\big(\varphi'(z),\cdots,\varphi^{(l-k+1)}(z)\big) \\
&+\sum_{k=1}^{n+1} f^{(k)}(\varphi(z)) \sum_{l=k-1}^n C_n^l \psi_2^{(n-l)}(z) B_{l,k-1}\big(\varphi'(z),\cdots,\varphi^{(l-k+2)}(z)\big) \\
=&\psi_1^{(n)}(z) f(\varphi(z)) + \sum_{k=1}^n f^{(k)}(\varphi(z)) \sum_{l=k}^n C_n^l \psi_1^{(n-l)}(z) B_{l,k}\big(\varphi'(z),\cdots,\varphi^{(l-k+1)}(z)\big) \\
&+\sum_{k=1}^{n} f^{(k)}(\varphi(z)) \sum_{l=k-1}^n C_n^l \psi_2^{(n-l)}(z) B_{l,k-1}\big(\varphi'(z),\cdots,\varphi^{(l-k+2)}(z)\big) \\
&+f^{(n+1)}(\varphi(z))\psi_2(z)\varphi'(z)^n.
\end{aligned}$$

Therefore, the lemma is established. □

The following lemma characterizes the compactness in terms of sequential convergence.

LEMMA 6. *Suppose that $0<p<\infty$, $\psi_1,\psi_2 \in \mathscr{H}(\mathbb{D})$, φ is an analytic self-map of $\mathbb{D}$. Then $T_{\psi_1,\psi_2,\varphi}: H^p \to \mathscr{W}_\mu^{(n)}$ is compact if and only if $T_{\psi_1,\psi_2,\varphi}: H^p \to \mathscr{W}_\mu^{(n)}$ is bounded and for any bounded sequence $\{f_i\}_{i\in\mathbb{N}}$ in H^p which converges to zero uniformly on compact subsets of $\mathbb{D}$ as $i\to\infty$, we have $\|T_{\psi_1,\psi_2,\varphi} f_i\|_{\mathscr{W}_\mu^{(n)}} \to 0$ as $i\to\infty$.*

Proof. The proof is inspired by the classical argument as in [1, Proposition 3.11]. Here we outline the proof for completeness. Since $\mu(z)$ is a normal function on $\mathbb{D}$, similar to the inequality (2.6) in [20], we have that

$$|f^{(n-1)}(z)| \leqslant C\|f\|_{\mathscr{W}_\mu^{(n)}}\left(1+\int_0^{|z|}\frac{dt}{\mu(t)}\right)$$

for every $z \in \mathbb{D}$. If K is compact, then it belongs to a closed disk $\overline{r\mathbb{D}} \subset \mathbb{D}$, $r \in [0,1)$, so that

$$\max_{z\in K} |f^{(n-1)}(z)| \leqslant C_K \|f\|_{\mathscr{W}_\mu^{(n)}},$$

where $C_K = C(1 + \int_0^r \frac{dt}{\mu(t)})$.

From this and since

$$|f^{(n-2)}(z)| \leqslant |f^{(n-2)}(0)| + \int_0^1 |f^{(n-1)}(tz)||z|dt$$

for every $z \in \mathbb{D}$, it follows that

$$\max_{z\in K} |f^{(n-2)}(z)| \leqslant (1 + C_K)\|f\|_{\mathscr{W}_\mu^{(n)}}.$$

By repeating use of the procedure we get that there is a constant C_K' such that

$$|f(z)| \leqslant C_K' \|f\|_{\mathscr{W}_\mu^{(n)}} \tag{1}$$

for all $z \in K$.

Now we suppose that $T_{\psi_1,\psi_2,\varphi} : H^p \to \mathscr{W}_\mu^{(n)}$ is compact, then it is clear that $T_{\psi_1,\psi_2,\varphi} : H^p \to \mathscr{W}_\mu^{(n)}$ is bounded. Let $\{f_i\}_{i\in\mathbb{N}}$ be a bounded sequence which converges to zero uniformly on compact subsets of $\mathbb{D}$ as $i \to \infty$. We need to show that $\left\|T_{\psi_1,\psi_2,\varphi} f_i\right\|_{\mathscr{W}_\mu^{(n)}} \to 0$ as $i \to \infty$. If the conclusion is false, then there exists an $\varepsilon > 0$ and a subsequence $i_1 < i_2 < \cdots$ such that

$$\|T_{\psi_1,\psi_2,\varphi} f_{i_j}\|_{\mathscr{W}_\mu^{(n)}} \geqslant \varepsilon \tag{2}$$

for all $j = 1, 2, \cdots$. Since $\{f_i\}$ is a bounded sequence and $T_{\psi_1,\psi_2,\varphi}$ is a compact operator we can find a further subsequence $i_{j_1} < i_{j_2} < \cdots$ and $f \in \mathscr{W}_\mu^{(n)}$ such that

$$\|T_{\psi_1,\psi_2,\varphi} f_{i_{j_k}} - f\|_{\mathscr{W}_\mu^{(n)}} \to 0 \tag{3}$$

as $k \to \infty$. By (1),

$$|T_{\psi_1,\psi_2,\varphi} f_{i_{j_k}}(z) - f(z)| \leqslant C\|T_{\psi_1,\psi_2,\varphi} f_{i_{j_k}} - f\|_{\mathscr{W}_\mu^{(n)}}. \tag{4}$$

From (3) and (4) it follows that

$$T_{\psi_1,\psi_2,\varphi} f_{i_{j_k}}(z) - f(z) \to 0 \tag{5}$$

uniformly on compact subsets of $\mathbb{D}$. Moreover, since $f_{i_{j_k}} \to 0$ uniformly on compact subsets of $\mathbb{D}$, by Cauchy's estimate, $f'_{i_{j_k}} \to 0$ uniformly on compact subsets of $\mathbb{D}$. Since $\{\varphi(z)\}$ is a compact set,

$$T_{\psi_1,\psi_2,\varphi} f_{i_{j_k}}(z) = \psi_1(z) f_{i_{j_k}}(\varphi(z)) + \psi_2(z) f'_{i_{j_k}}(\varphi(z)) \to 0$$

for each $z\in\mathbb{D}$. Thus by (5), $f=0$. Hence (3) yields $\|T_{\psi_1,\psi_2,\varphi}f_{i_{j_k}}\|_{\mathscr{W}_\mu^{(n)}}\to 0$ as $k\to\infty$, which contradicts (2). Therefore, we must have $\left\|T_{\psi_1,\psi_2,\varphi}f_i\right\|_{\mathscr{W}_\mu^{(n)}}\to 0$ as $i\to\infty$.

Conversely, suppose that $T_{\psi_1,\psi_2,\varphi}:H^p\to\mathscr{W}_\mu^{(n)}$ is bounded. Let $\{g_i\}$ be a bounded sequence in H^p. We can suppose without loss of generality that $\{g_i\}$ belongs to the unit ball $\mathfrak{B}$ of H^p, then by Lemma 1 we have

$$|g_i(z)|\leqslant C\frac{\|g_i\|_{H^p}}{(1-|z|^2)^{\frac{1}{p}}}\leqslant\frac{C}{(1-|z|^2)^{\frac{1}{p}}},\quad z\in\mathbb{D},$$

where C is a positive constant independent of g_i. Thus $\{g_i\}$ is uniformly bounded on compacts of $\mathbb{D}$ and consequently normal by Montel's theorem. Hence, we may extract a subsequence $\{g_{i_j}\}$ that converges uniformly on the compact subsets of $\mathbb{D}$ to some $g\in\mathscr{H}(\mathbb{D})$. By using Fatou's lemma we can obtain

$$\begin{aligned}\|g\|_{H^p}^p&=\sup_{0<r<1}\int_0^{2\pi}|g(re^{i\theta})|^p\frac{d\theta}{2\pi}\\&=\sup_{0<r<1}\int_0^{2\pi}|\lim_{j\to\infty}g_{i_j}(re^{i\theta})|^p\frac{d\theta}{2\pi}\\&\leqslant\liminf_{j\to\infty}\sup_{0<r<1}\int_0^{2\pi}|g_{i_j}(re^{i\theta})|^p\frac{d\theta}{2\pi}\leqslant 1,\end{aligned}$$

whence $g\in H^p$ and $\|g\|_{H^p}\leqslant 1$. Therefore, $\{g_{i_j}-g\}$ is a bounded sequence in H^p and converges to zero on the compact subsets of $\mathbb{D}$ as $j\to\infty$. By the hypotheses we have that $T_{\psi_1,\psi_2,\varphi}g_{i_j}\to T_{\psi_1,\psi_2,\varphi}g$ in $\mathscr{W}_\mu^{(n)}$ as $j\to\infty$. Thus the set $T_{\psi_1,\psi_2,\varphi}(\mathfrak{B})$ is relatively compact, which finishes the proof. □

3. Main Results

In this section, we characterize the boundedness and compactness of $T_{\psi_1,\psi_2,\varphi}:H^p\to\mathscr{W}_\mu^{(n)}$.

THEOREM 1. *Assume that $0<p<\infty$, $\psi_1,\psi_2\in\mathscr{H}(\mathbb{D})$, φ is an analytic self-map of $\mathbb{D}$. Then $T_{\psi_1,\psi_2,\varphi}:H^p\to\mathscr{W}_\mu^{(n)}$ is bounded if and only if*

$$I_k:=\sup_{z\in\mathbb{D}}\frac{\mu(z)\left|\Omega_k(z)\right|}{\left(1-|\varphi(z)|^2\right)^{\frac{1}{p}+k}}<\infty,\quad k=0,1,\cdots,n+1. \tag{6}$$

Moreover, if the operator $T_{\psi_1,\psi_2,\varphi}:H^p\to\mathscr{W}_\mu^{(n)}$ is nonzero and bounded, then

$$\|T_{\psi_1,\psi_2,\varphi}\|_{H^p\to\mathscr{W}_\mu^{(n)}}\asymp\sum_{k=0}^{n+1}I_k. \tag{7}$$

Proof. Suppose that (6) holds. For each $f\in H^p$, by Lemmas 1 and 5, we have

$$\begin{aligned}\mu(z)\left|(T_{\psi_1,\psi_2,\varphi}f)^{(n)}(z)\right| &\leqslant \mu(z)\sum_{k=0}^{n+1}\left|f^{(k)}\big(\varphi(z)\big)\right|\left|\Omega_k(z)\right|\\ &\leqslant C\mu(z)\sum_{k=0}^{n+1}\frac{\|f\|_{H^p}\left|\Omega_k(z)\right|}{\big(1-|\varphi(z)|^2\big)^{\frac{1}{p}+k}}\\ &\leqslant C\sum_{k=0}^{n+1}I_k\|f\|_{H^p}.\end{aligned}$$

We also have that

$$\begin{aligned}\sum_{j=0}^{n-1}\left|(T_{\psi_1,\psi_2,\varphi}f)^{(j)}(0)\right| &\leqslant \sum_{j=0}^{n-1}\sum_{k=0}^{j+1}\left|f^{(k)}(\varphi(0))\right|\left|\Omega_k(0)\right|\\ &\leqslant \sum_{j=0}^{n-1}\sum_{k=0}^{j+1}C_k\frac{\|f\|_{H^p}\left|\Omega_k(0)\right|}{\big(1-|\varphi(0)|^2\big)^{\frac{1}{p}+k}}.\end{aligned}$$

It follows that $\|T_{\psi_1,\psi_2,\varphi}f\|_{\mathscr{W}_\mu^{(n)}}\leqslant C\|f\|_{H^p}\sum_{k=0}^{n+1}I_k$. Thus $T_{\psi_1,\psi_2,\varphi}:H^p\to\mathscr{W}_\mu^{(n)}$ is bounded and

$$\|T_{\psi_1,\psi_2,\varphi}\|_{H^p\to\mathscr{W}_\mu^{(n)}}\preceq\sum_{k=0}^{n+1}I_k. \tag{8}$$

On the other hand, suppose that $T_{\psi_1,\psi_2,\varphi}:H^p\to\mathscr{W}_\mu^{(n)}$ is bounded. For a fixed $\omega\in\mathbb{D}$, and constants $c_0,c_1,\cdots,c_{n+1}$, set

$$f_\omega(z)=\sum_{j=0}^{n+1}c_j\frac{\big(1-|\omega|^2\big)^{j+1}}{\big(1-\overline{\omega}z\big)^{\frac{1}{p}+j+1}}. \tag{9}$$

By Lemma 2, we have that $f_\omega\in H^p$, $\sup_{\omega\in\mathbb{D}}\|f_\omega\|\leqslant C$, and

$$f_\omega(\omega)=\frac{1}{\big(1-|\omega|^2\big)^{\frac{1}{p}}}\sum_{j=0}^{n+1}c_j, \tag{10}$$

$$f_\omega^{(l)}(\omega)=\frac{\overline{\omega}^l}{\big(1-|\omega|^2\big)^{\frac{1}{p}+l}}\sum_{j=0}^{n+1}c_j\prod_{r=0}^{l-1}(\frac{1}{p}+j+1+r),\quad l=1,2,\cdots,n+1. \tag{11}$$

We claim that for each $k\in\{0,1,\cdots,n+1\}$, there are constants $c_0,c_1,\cdots,c_{n+1}$ such that

$$f_\omega^{(k)}(\omega)=\frac{\overline{\omega}^k}{\big(1-|\omega|^2\big)^{\frac{1}{p}+k}},\quad f_\omega^{(t)}(\omega)=0,\quad t\in\{0,1,\cdots,n+1\}\backslash\{k\}. \tag{12}$$

In fact, from (10) and (11) it follows that (12) is equivalent to the following system of liner equations

$$\begin{cases} c_0+c_1+\cdots+c_{n+1}=0, \\ c_0(\frac{1}{p}+1)+c_1(\frac{1}{p}+2)+\cdots+c_{n+1}(\frac{1}{p}+n+2)=0, \\ \cdots\cdots \\ c_0\prod_{r=0}^{k-1}(\frac{1}{p}+1+r)+c_1\prod_{r=0}^{k-1}(\frac{1}{p}+2+r)+\cdots+c_n\prod_{r=0}^{k-1}(\frac{1}{p}+n+2+r)=1, \\ \cdots\cdots \\ c_0\prod_{r=0}^{n}(\frac{1}{p}+1+r)+c_1\prod_{r=0}^{n}(\frac{1}{p}+2+r)+\cdots+c_n\prod_{r=0}^{n}(\frac{1}{p}+n+2+r)=0. \end{cases} \tag{13}$$

Applying Lemma 3 with $a=\frac{1}{p}+1$, we have that the determinant of system (13) is different from zero, from which the claim follows. For each $k\in\{0,1,\cdots,n+1\}$, we choose the corresponding family of functions that satisfies (12) and denote it by $f_{\omega,k}$. Thus, by using Lemma 5 and the boundedness of $T_{\psi_1,\psi_2,\varphi}:H^p\to\mathscr{W}_\mu^{(n)}$, for $\omega\in\mathbb{D}$ such that $|\varphi(\omega)|>\frac{1}{2}$,

$$\sup_{|\varphi(\omega)|>\frac{1}{2}}\frac{\mu(\omega)|\Omega_k(\omega)|}{(1-|\varphi(\omega)|^2)^{\frac{1}{p}+k}}\leqslant C\|T_{\psi_1,\psi_2,\varphi}f_{\varphi(\omega),k}\|_{\mathscr{W}_\mu^{(n)}}\leqslant C\|T_{\psi_1,\psi_2,\varphi}\|_{H^p\to\mathscr{W}_\mu^{(n)}}. \tag{14}$$

Taking the test functions $h_k(z)=z^k\in H^p$, $k=0,1,\cdots,n+1$, and applying Lemma 5 to $h_0(z)=1$, we can get

$$\big(T_{\psi_1,\psi_2,\varphi}h_0(z)\big)^{(n)}=\Omega_0(z)=\psi_1^{(n)}(z),$$

which along with the boundedness of $T_{\psi_1,\psi_2,\varphi}:H^p\to\mathscr{W}_\mu^{(n)}$ implies that

$$\sup_{z\in\mathbb{D}}\mu(z)\big|\Omega_0(z)\big|\leqslant C\|T_{\psi_1,\psi_2,\varphi}\|_{H^p\to\mathscr{W}_\mu^{(n)}}. \tag{15}$$

Now assume that we have proved the following inequalities

$$\sup_{z\in\mathbb{D}}\mu(z)\big|\Omega_i(z)\big|<\infty,\quad i\in\{1,2,\cdots,k-1\},\quad 2\leqslant k\leqslant n+1. \tag{16}$$

Applying Lemma 5 to $h_k(z)=z^k$, we have

$$\begin{aligned}&\big(T_{\psi_1,\psi_2,\varphi}h_k(z)\big)^{(n)}\\ &=\big(\varphi(z)\big)^k\Omega_0(z)+\sum_{s=1}^{k-1}k(k-1)\cdots(k-s+1)\big(\varphi(z)\big)^{k-s}\Omega_s(z)+k!\,\Omega_k(z),\end{aligned}$$

from which, along with the boundedness of $T_{\psi_1,\psi_2,\varphi}:H^p\to\mathscr{W}_\mu^{(n)}$, the fact that $\|\varphi\|_\infty\leqslant 1$, the triangle inequality, (15), and using hypothesis (16) we can obtain

$$\sup_{z\in\mathbb{D}}\mu(z)\big|\Omega_k(z)\big|\leqslant C\|T_{\psi_1,\psi_2,\varphi}\|_{H^p\to\mathscr{W}_\mu^{(n)}},\quad k\in\{0,1,\cdots,n+1\}. \tag{17}$$

Then

$$\sup_{|\varphi(\omega)|\leqslant\frac{1}{2}} \frac{\mu(\omega)\big|\Omega_k(\omega)\big|}{\big(1-|\varphi(\omega)|^2\big)^{\frac{1}{p}+k}} \leqslant C\sup_{\omega\in\mathbb{D}}\mu(\omega)\big|\Omega_k(\omega)\big| \leqslant C\|T_{\psi_1,\psi_2,\varphi}\|_{H^p\to\mathscr{W}_\mu^{(n)}}. \tag{18}$$

By using (14) and (18), we can get (6) and

$$\sum_{k=0}^{n+1} I_k \leqslant C\|T_{\psi_1,\psi_2,\varphi}\|_{H^p\to\mathscr{W}_\mu^{(n)}}. \tag{19}$$

From (8) and (19) it follows that the asymptotic expression (7) holds. □

THEOREM 2. *Assume that* $0<p<\infty$, $\psi_1,\psi_2\in\mathscr{H}(\mathbb{D})$, φ *is an analytic self-map of* $\mathbb{D}$. *Then* $T_{\psi_1,\psi_2,\varphi}: H^p\to\mathscr{W}_\mu^{(n)}$ *is compact if and only if* $T_{\psi_1,\psi_2,\varphi}: H^p\to\mathscr{W}_\mu^{(n)}$ *is bounded and for each* $k\in\{0,1,\cdots,n+1\}$,

$$\lim_{|\varphi(z)|\to1} \frac{\mu(z)\big|\Omega_k(z)\big|}{\big(1-|\varphi(z)|^2\big)^{\frac{1}{p}+k}} = 0. \tag{20}$$

Proof. Suppose that $T_{\psi_1,\psi_2,\varphi}: H^p\to\mathscr{W}_\mu^{(n)}$ is compact. It is clear that $T_{\psi_1,\psi_2,\varphi}: H^p\to\mathscr{W}_\mu^{(n)}$ is bounded. Let $\{z_i\}_{i\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_i)|\to1$ as $i\to\infty$. Let $f_{\varphi(z_i),k}$, $k\in\{0,1,\cdots,n+1\}$ be as defined in the proof of Theorem 1 that satisfies (12). Then the sequence $\{f_{\varphi(z_i),k}\}_{i\in\mathbb{N}}$ is bounded in H^p and converges to zero uniformly on compact subsets of $\mathbb{D}$ as $i\to\infty$. Since $T_{\psi_1,\psi_2,\varphi}: H^p\to\mathscr{W}_\mu^{(n)}$ is compact, by Lemma 6, we have that for each $k\in\{0,1,\cdots,n+1\}$,

$$\lim_{i\to\infty}\|T_{\psi_1,\psi_2,\varphi}f_{\varphi(z_i),k}\|_{\mathscr{W}_\mu^{(n)}} = 0. \tag{21}$$

Then

$$\frac{\mu(z_i)\big|\varphi(z_i)\big|^k\big|\Omega_k(z_i)\big|}{\big(1-|\varphi(z_i)|^2\big)^{\frac{1}{p}+k}} \leqslant \|T_{\psi_1,\psi_2,\varphi}f_{\varphi(z_i),k}\|_{\mathscr{W}_\mu^{(n)}},$$

which along with $|\varphi(z_i)|\to1$ as $i\to\infty$ and (21) implies that

$$\lim_{|\varphi(z_i)|\to1} \frac{\mu(z_i)\big|\Omega_k(z_i)\big|}{\big(1-|\varphi(z_i)|^2\big)^{\frac{1}{p}+k}} = \lim_{i\to\infty} \frac{\mu(z_i)\big|\varphi(z_i)\big|^k\big|\Omega_k(z_i)\big|}{\big(1-|\varphi(z_i)|^2\big)^{\frac{1}{p}+k}} = 0$$

for each $k\in\{0,1,\cdots,n+1\}$, from which (20) holds.

On the other hand, assume that $T_{\psi_1,\psi_2,\varphi}: H^p\to\mathscr{W}_\mu^{(n)}$ is bounded and (20) holds. Let $\{f_i\}_{i\in\mathbb{N}}$ be a sequence in H^p such that $\sup_{i\in\mathbb{N}}\|f_i\|_{H^p}\leqslant L$ and f_i converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $i \to \infty$. By the assumption, for any $\varepsilon > 0$, there exists a $\delta \in (0,1)$ such that whenever $\delta < |\varphi(z)| < 1$,

$$\frac{\mu(z)\big|\Omega_k(z)\big|}{\big(1-|\varphi(z)|^2\big)^{\frac{1}{p}+k}} < \varepsilon, \quad k = 0,1,\cdots,n+1. \tag{22}$$

Then we have

$$\begin{aligned}
&\|T_{\psi_1,\psi_2,\varphi} f_i\|_{\mathscr{W}_\mu^{(n)}} \\
=&\sum_{j=0}^{n-1} \big|(T_{\psi_1,\psi_2,\varphi} f_i)^{(j)}(0)\big| + \sup_{z\in\mathbb{D}} \mu(z)\big|(T_{\psi_1,\psi_2,\varphi} f_i)^{(n)}(z)\big| \\
\leqslant& \sum_{j=0}^{n-1} \Big|\sum_{k=0}^{j+1} f_i^{(k)}\big(\varphi(0)\big)\Omega_k(0)\Big| \\
&+ \sup_{|\varphi(z)|\leqslant\delta} \mu(z)\Big|\sum_{k=0}^{n+1} f_i^{(k)}\big(\varphi(z)\big)\Omega_k(z)\Big| + \sup_{\delta<|\varphi(z)|<1} \mu(z)\Big|\sum_{k=0}^{n+1} f_i^{(k)}\big(\varphi(z)\big)\Omega_k(z)\Big| \\
=&J_1 + J_2 + J_3.
\end{aligned}$$

Now we estimate J_1, J_2 and J_3, by Cauchy's estimate we have

$$f_i^{(k)}\big(\varphi(0)\big) \to 0 \quad \text{and} \quad \sup_{|\varphi(z)|\leqslant\delta} f_i^{(k)}\big(\varphi(z)\big) \to 0. \tag{23}$$

By using (23) and (17) in Theorem 1, we can easily get that

$$J_1 = \sum_{j=0}^{n-1} \Big|\sum_{k=0}^{j+1} f_i^{(k)}(\varphi(0))\Omega_k(0)\Big| \to 0, \tag{24}$$

and

$$J_2 = \sup_{|\varphi(z)|\leqslant\delta} \mu(z)\Big|\sum_{k=0}^{n+1} f_i^{(k)}\big(\varphi(z)\big)\Omega_k(z)\Big| \to 0. \tag{25}$$

By Lemma 1 and (22), we have

$$\begin{aligned}
J_3 =& \sup_{\delta<|\varphi(z)|<1} \mu(z)\Big|\sum_{k=0}^{n+1} f_i^{(k)}\big(\varphi(z)\big)\Omega_k(z)\Big| \\
\leqslant& C\|f_i\|_{H^p} \sum_{k=0}^{n+1} \sup_{\delta<|\varphi(z)|<1} \frac{\mu(z)\big|\Omega_k(z)\big|}{\big(1-|\varphi(z)|^2\big)^{\frac{1}{p}+k}} \\
\leqslant& CL(n+2)\varepsilon.
\end{aligned} \tag{26}$$

From (24), (25) and (26) it follows that $\lim_{i\to\infty} \|T_{\psi_1,\psi_2,\varphi} f_i\|_{\mathscr{W}_\mu^{(n)}} = 0$. Applying Lemma 6 the implication follows. $\square$

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