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EQUIVALENT STATEMENTS OF A MORE ACCURATE EXTENDED MULHOLLAND'S INEQUALITY WITH A BEST POSSIBLE CONSTANT FACTOR

BICHENG YANG, MEIFA HUANG AND YANRU ZHONG*

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Abstract. By the use of the weight functions, the idea of introduced parameters and Hermite-Hadamard's inequality, a more accurate extended Mulholland's inequality and its equivalent form are given. A few equivalent statements of the best possible constant factor related to some parameters, some particular cases and the operator expressions are considered.

1. Introduction

Assuming that $0 < \sum_{m=1}^{\infty} a_m^2 < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^2 < \infty$, we have the following Hilbert's inequality with the best possible constant factor π (cf. [1], Theorem 315):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \pi\left(\sum_{m=1}^{\infty} a_m^2 \sum_{n=1}^{\infty} b_n^2\right)^{\frac{1}{2}}. \tag{1}$$

We still have the following Mulholland's inequality with the same best possible constant factor π (cf. [1], Theorem 343):

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{a_m b_n}{\ln mn} < \pi\left(\sum_{m=2}^{\infty} m a_m^2 \sum_{n=2}^{\infty} n b_n^2\right)^{\frac{1}{2}}, \tag{2}$$

If $0 < \int_0^{\infty} f^2(x)dx < \infty$ and $0 < \int_0^{\infty} g^2(y)dy < \infty$, then we have the following Hilbert's integral inequality:

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{x+y}dxdy < \pi\left(\int_0^{\infty} f^2(x)dx \int_0^{\infty} g^2(y)dy\right)^{\frac{1}{2}}, \tag{3}$$

where, the constant factor π is the best possible (cf. [1], Theorem 316).

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* Corresponding author.

Inequalities (1), (2), (3) and their extensions with the conjugate exponents (p,q) $(p>1,\frac{1}{p}+\frac{1}{q}=1)$ are important in analysis and its applications (cf. [2]-[16]).

The following half-discrete Hilbert-type inequality was provided by [1] (Theorem 351): If $K(x)(x>0)$ is decreasing, $p>1,\frac{1}{p}+\frac{1}{q}=1, 0<\phi(s)=\int_0^\infty K(x)x^{s-1}dx<\infty$, then

$$\int_0^\infty x^{p-2}(\sum_{n=1}^\infty K(nx)a_n)^p dx < \phi^p(\frac{1}{p})\sum_{n=1}^\infty a_n^p. \tag{4}$$

Some new extensions of (4) were made by [17]-[22].

In 2016, by the use of the technique of real analysis, Hong [23] considered some equivalent statements of the extensions of (1) with the best possible constant factor related to a few parameters. The other similar works on the extensions of (3) and (4) were given by [24]-[27].

In this paper, following the way of [23], by the use of the weight functions, the idea of introduced parameters and Hermite-Hadamard's inequality, a more accurate extended Mulholland's inequality and its equivalent form are given by Lemma 2 and Theorem 1. The equivalent statements of the best possible constant factor related to a few parameters are given by Theorem 2. Some particular cases and the operator expressions are obtained by Remark 3 and Theorem 3.

2. Some Lemmas

In what follows, we suppose that $p>1,\frac{1}{p}+\frac{1}{q}=1,\xi,\eta\in(0,\frac{1}{2}],\lambda>0,\lambda_i,\lambda-\lambda_i\in(0,1](i=1,2),\ a_m,b_n\geqslant 0\ \ (m,n\in\mathbf{N}\backslash\{1\}=\{2,3,\cdots\})$, such that

$$0<\sum_{m=2}^\infty \frac{\ln^{p[1-\alpha(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}(m-\xi)}{(m-\xi)^{1-p}}a_m^p<\infty, \text{ and}$$
$$0<\sum_{n=2}^\infty \frac{\ln^{q[1-\beta(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q})]-1}(n-\eta)}{(n-\eta)^{1-q}}b_n^q<\infty. \tag{5}$$

LEMMA 1. *The weight functions is defined as follows:*

$$\omega_\lambda(\lambda_2,m) := \ln^{\lambda-\lambda_2}(m-\xi)\sum_{n=2}^\infty \frac{\ln^{\lambda_2-1}(n-\eta)}{\ln^\lambda(m-\xi)(n-\eta)}\frac{1}{n-\eta}\ (m\in\mathbf{N}\backslash\{1\}), \tag{6}$$
$$\varpi_\lambda(\lambda_1,n) := \ln^{\lambda-\lambda_1}(n-\eta)\sum_{m=2}^\infty \frac{\ln^{\lambda_1-1}(m-\xi)}{\ln^\lambda(m-\xi)(n-\eta)}\frac{1}{m-\xi}\ (n\in\mathbf{N}\backslash\{1\}). \tag{7}$$

We have the following inequalities:

$$\omega_\lambda(\lambda_2,m) < B(\lambda_2,\lambda-\lambda_2)\ (\lambda_2\leqslant 1; m\in\mathbf{N}\backslash\{1\}), \tag{8}$$
$$B(\lambda_1,\lambda-\lambda_1)-\frac{1}{\lambda_1}\left[\frac{\ln(2-\xi)}{\ln(n-\eta)}\right]^{\lambda_1}$$
$$<\varpi_\lambda(\lambda_1,n)<B(\lambda_1,\lambda-\lambda_1)\ (\lambda_1\leqslant 1; n\in\mathbf{N}\backslash\{1\}), \tag{9}$$

where, $B(u,v)(u,v>0)$ *is the beta function.*

Proof. Sine $m,n\in\mathbf{N}\backslash\{1\},\lambda>0,\ln^{\lambda}(m-\xi)(t-\eta)=[\ln(m-\xi)+\ln(t-\eta)]^{\lambda}>0$ $(t>\frac{3}{2})$, we find

$$\frac{d}{dt}\left\{\frac{1}{[\ln(m-\xi)+\ln(t-\eta)]^{\lambda}}\right\} = \frac{-\lambda}{[\ln(m-\xi)+\ln(t-\eta)]^{\lambda+1}}\frac{1}{t-\eta}<0,$$

$$\frac{d^2}{dt^2}\left\{\frac{1}{[\ln(m-\xi)+\ln(t-\eta)]^{\lambda}}\right\} = \left\{\frac{\lambda(\lambda+1)}{\ln^{\lambda+2}(m-\xi)(t-\eta)}+\frac{\lambda}{\ln^{\lambda+1}(m-\xi)(t-\eta)}\right\}\frac{1}{(t-\eta)^2}>0.$$

For $\lambda_2\in(0,1]$, $t>\frac{3}{2}$, we still have

$$(-1)^i\frac{d^i}{dt^i}\left\{\frac{1}{\ln^{\lambda}(m-\xi)(t-\eta)}\frac{1}{\ln^{1-\lambda_2}(t-\eta)}\frac{1}{t-\eta}\right\}>0\ (i=1,2),$$

and then by Hemite-Hadamard's inequality (cf. [28]), we obtain

$$\begin{aligned}\omega_{\lambda}(\lambda_2,m) &< \ln^{\lambda-\lambda_2}(m-\xi)\int_{\frac{3}{2}}^{\infty}\frac{\ln^{\lambda_2-1}(t-\eta)}{\ln^{\lambda}(m-\xi)(t-\eta)}\frac{1}{t-\eta}dt\\ &= \int_{\frac{\ln(\frac{3}{2}-\eta)}{\ln(m-\xi)}}^{\infty}\frac{u^{\lambda_2-1}}{(1+u)^{\lambda}}du \leqslant \int_0^{\infty}\frac{u^{\lambda_2-1}}{(1+u)^{\lambda}}du\\ &= B(\lambda_2,\lambda-\lambda_2).\end{aligned}$$

Hence, we have (8).

Similarly, for $\lambda_1\in(0,1]$, we have

$$\varpi_{\lambda}(\lambda_1,n)<\int_0^{\infty}\frac{u^{\lambda_1-1}du}{(1+u)^{\lambda}}=B(\lambda_1,\lambda-\lambda_1).$$

By the decreasingness property, we still obtain

$$\begin{aligned}\varpi_{\lambda}(\lambda_1,n) &\geqslant \ln^{\lambda-\lambda_1}(n-\eta)\int_2^{\infty}\frac{\ln^{\lambda_1-1}(t-\xi)}{\ln^{\lambda}(t-\xi)(n-\eta)}\frac{1}{t-\xi}dt\\ &= \int_{\frac{\ln(2-\xi)}{\ln(n-\eta)}}^{\infty}\frac{u^{\lambda_1-1}du}{(1+u)^{\lambda}} = B(\lambda_1,\lambda-\lambda_1)-\int_0^{\frac{\ln(2-\xi)}{\ln(n-\eta)}}\frac{u^{\lambda_1-1}du}{(1+u)^{\lambda}}\\ &> B(\lambda_1,\lambda-\lambda_1)-\int_0^{\frac{\ln(2-\xi)}{\ln(n-\eta)}}u^{\lambda_1-1}du\\ &= B(\lambda_1,\lambda-\lambda_1)-\frac{1}{\lambda_1}\left[\frac{\ln(2-\xi)}{\ln(n-\eta)}\right]^{\lambda_1},\end{aligned}$$

and then inequality (9) follows. Thus, Lemma 1 is proved. □

LEMMA 2. *Let* $k_\lambda(\gamma) := B(\gamma, \lambda-\gamma)$ $(\gamma = \lambda_1, \lambda_2)$. *We have the following inequality:*

$$\begin{aligned} I &:= \sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{a_m b_n}{\ln^{\lambda}(m-\xi)(n-\eta)} \\ &< k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)\left\{\sum_{m=2}^{\infty}\frac{\ln^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}(m-\xi)}{(m-\xi)^{1-p}}a_m^p\right\}^{\frac{1}{p}} \\ &\times\left\{\sum_{n=2}^{\infty}\frac{\ln^{q[1-(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q})]-1}(n-\eta)}{(n-\eta)^{1-q}}b_n^q\right\}^{\frac{1}{q}}. \end{aligned} \tag{10}$$

Proof. By Hölder's inequality with weight (cf. [28]), we obtain

$$\begin{aligned} I &= \sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{1}{\ln^{\lambda}(m-\xi)(n-\eta)}\left[\frac{\ln^{(\lambda_2-1)/p}(n-\eta)}{(n-\eta)^{1/p}}\frac{\ln^{(1-\lambda_1)/q}(m-\xi)}{(m-\xi)^{-1/q}}a_m\right] \\ &\times\left[\frac{\ln^{(\lambda_1-1)/q}(m-\xi)}{(m-\xi)^{1/q}}\frac{\ln^{(1-\lambda_2)/p}(n-\eta)}{(n-\eta)^{-1/p}}b_n\right] \\ &\leqslant \left\{\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{\ln^{\lambda}(m-\xi)(n-\eta)}\frac{\ln^{\lambda_2-1}(n-\eta)}{(n-\eta)}\frac{\ln^{(p-1)(1-\lambda_1)}(m-\xi)}{(m-\xi)^{1-p}}a_m^p\right\}^{\frac{1}{p}} \\ &\times\left\{\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{1}{\ln^{\lambda}(m-\xi)(n-\eta)}\frac{\ln^{\lambda_1-1}(m-\xi)}{(m-\xi)}\frac{\ln^{(q-1)(1-\lambda_2)}(n-\eta)}{(n-\eta)^{1-q}}b_n^q\right\}^{\frac{1}{q}} \\ &= \left\{\sum_{m=2}^{\infty}\omega_\lambda(\lambda_2,m)\frac{\ln^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}(m-\xi)}{(m-\xi)^{1-p}}a_m^p\right\}^{\frac{1}{p}} \\ &\times\left\{\sum_{n=2}^{\infty}\varpi_\lambda(\lambda_1,n)\frac{\ln^{q[1-(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q})]-1}(n-\eta)}{(n-\eta)^{1-q}}b_n^q\right\}^{\frac{1}{q}}. \end{aligned}$$

Then, in view of (8), (9) and (5), we have (10). □

REMARK 1. According to inequalities (5) and (10), for $\lambda_1+\lambda_2=\lambda$, we find

$$0 < \sum_{m=2}^{\infty}\frac{\ln^{p(1-\lambda_1)-1}(m-\xi)}{(m-\xi)^{1-p}}a_m^p < \infty \text{ and } 0 < \sum_{n=2}^{\infty}\frac{\ln^{q(1-\lambda_2)-1}(n-\eta)}{(n-\eta)^{1-q}}b_n^q < \infty,$$

and the following inequality:

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{a_m b_n}{\ln^{\lambda}(m-\xi)(n-\eta)} < B(\lambda_1,\lambda_2)\left[\sum_{m=2}^{\infty}\frac{\ln^{p(1-\lambda_1)-1}(m-\xi)}{(m-\xi)^{1-p}}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=2}^{\infty}\frac{\ln^{q(1-\lambda_2)-1}(n-\eta)}{(n-\eta)^{1-q}}b_n^q\right]^{\frac{1}{q}}. \quad (11)$$

LEMMA 3. *The constant factor $B(\lambda_1,\lambda_2)$ in (11) is the best possible.*

Proof. For any $0<\varepsilon<\lambda_1 p$, we set

$$\widetilde{a}_m := \frac{\ln^{\alpha(\lambda_1-\frac{\varepsilon}{p})-1}(m-\xi)}{(m-\xi)}, \widetilde{b}_n := \frac{\ln^{\beta(\lambda_2-\frac{\varepsilon}{q})-1}(n-\eta)}{(n-\eta)} \ (m,n\in\mathbf{N}\backslash\{1\}).$$

If there exists a positive constant M $(M\leqslant B(\lambda_1,\lambda_2))$, such that (11) is valid when replacing $B(\lambda_1,\lambda_2)$ by M, then in particular, for $a_m=\widetilde{a}_m, b_n=\widetilde{b}_n,$ we have

$$\widetilde{I} := \sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{\widetilde{a}_m\widetilde{b}_n}{\ln^{\lambda}(m-\xi)(n-\eta)} < M\left[\sum_{m=2}^{\infty}\frac{\ln^{p(1-\lambda_1)-1}(m-\xi)}{(m-\xi)^{1-p}}\widetilde{a}_m^p\right]^{\frac{1}{p}}\left[\sum_{n=2}^{\infty}\frac{\ln^{q(1-\lambda_2)-1}(n-\eta)}{(n-\eta)^{1-q}}\widetilde{b}_n^q\right]^{\frac{1}{q}}.$$

Further, we can obtain

$$\begin{aligned}
\widetilde{I} &< M\left[\sum_{m=2}^{\infty}\frac{\ln^{p(1-\lambda_1)-1}(m-\xi)}{(m-\xi)^{1-p}}\frac{\ln^{p(\lambda_1-\frac{\varepsilon}{p})-p}(m-\xi)}{(m-\xi)^p}\right]^{\frac{1}{p}} \\
&\quad\times\left[\sum_{n=2}^{\infty}\frac{\ln^{q(1-\lambda_2)-1}(n-\eta)}{(n-\eta)^{1-q}}\frac{\ln^{q(\lambda_2-\frac{\varepsilon}{q})-q}(n-\eta)}{(n-\eta)^q}\right]^{\frac{1}{q}} \\
&= M\left[\frac{\ln^{-\varepsilon-1}(2-\xi)}{2-\xi}+\sum_{m=3}^{\infty}\frac{\ln^{-\varepsilon-1}(m-\xi)}{m-\xi}\right]^{\frac{1}{p}} \\
&\quad\times\left[\frac{\ln^{-\varepsilon-1}(2-\eta)}{2-\eta}+\sum_{n=3}^{\infty}\frac{\ln^{-\varepsilon-1}(n-\eta)}{n-\eta}\right]^{\frac{1}{q}} \\
&< M\left[\frac{\ln^{-\varepsilon-1}(2-\xi)}{2-\xi}+\int_2^{\infty}\frac{\ln^{-\varepsilon-1}(t-\xi)}{t-\xi}dt\right]^{\frac{1}{p}} \\
&\quad\times\left[\frac{\ln^{-\varepsilon-1}(2-\eta)}{2-\eta}+\int_2^{\infty}\frac{\ln^{-\varepsilon-1}(t-\eta)}{t-\eta}dt\right]^{\frac{1}{q}}
\end{aligned}$$

$$= \frac{M}{\varepsilon}\left[\frac{\varepsilon \ln^{-\varepsilon-1}(2-\xi)}{2-\xi}+\ln^{-\varepsilon}(2-\xi)\right]^{\frac{1}{p}} \times\left[\frac{\varepsilon \ln^{-\varepsilon-1}(2-\eta)}{2-\eta}+\ln^{-\varepsilon}(2-\eta)\right]^{\frac{1}{q}}.$$

In view of (9), for $0<\lambda_1-\frac{\varepsilon}{p}$, we find

$$\begin{aligned}
\widetilde{I} &= \sum_{n=2}^{\infty}\left[\ln^{(\lambda_2+\frac{\varepsilon}{p})}(n-\eta)\sum_{m=2}^{\infty}\frac{\ln^{(\lambda_1-\frac{\varepsilon}{p})-1}(m-\xi)}{\ln^{\lambda}(m-\xi)(n-\eta)}\frac{1}{m-\xi}\right]\frac{\ln^{-\varepsilon-1}(n-\eta)}{n-\eta}\\
&= \sum_{n=2}^{\infty}\varpi_{\lambda}(\lambda_1-\frac{\varepsilon}{p},n)\frac{\ln^{-\varepsilon-1}(n-\eta)}{n-\eta}\\
&\geqslant B(\lambda_1-\frac{\varepsilon}{p},\lambda_2+\frac{\varepsilon}{p})\sum_{n=2}^{\infty}\frac{\ln^{-\varepsilon-1}(n-\eta)}{n-\eta}\\
&\quad -\frac{\ln^{\lambda_1-\frac{\varepsilon}{p}}(2-\xi)}{\lambda_1-\frac{\varepsilon}{p}}\left[\frac{\ln^{-\lambda_1-\frac{\varepsilon}{q}-1}(2-\eta)}{2-\eta}+\sum_{n=3}^{\infty}\frac{\ln^{-\lambda_1-\frac{\varepsilon}{q}-1}(n-\eta)}{n-\eta}\right]\\
&> B(\lambda_1-\frac{\varepsilon}{p},\lambda_2+\frac{\varepsilon}{p})\int_2^{\infty}\frac{\ln^{-\varepsilon-1}(t-\eta)}{t-\eta}dt\\
&\quad -\frac{\ln^{\lambda_1-\frac{\varepsilon}{p}}(2-\xi)}{\lambda_1-\frac{\varepsilon}{p}}\left[\frac{\ln^{-\lambda_1-\frac{\varepsilon}{q}-1}(2-\eta)}{2-\eta}+\int_2^{\infty}\frac{\ln^{-\lambda_1-\frac{\varepsilon}{q}-1}(t-\eta)}{t-\eta}dt\right]\\
&= \frac{1}{\varepsilon}\left\{B(\lambda_1-\frac{\varepsilon}{p},\lambda_2+\frac{\varepsilon}{p})\ln^{-\varepsilon}(2-\eta)\right.\\
&\quad \left.-\frac{\varepsilon\ln^{\lambda_1-\frac{\varepsilon}{p}}(2-\xi)}{\lambda_1-\frac{\varepsilon}{p}}\left[\frac{\ln^{-\lambda_1-\frac{\varepsilon}{q}-1}(2-\eta)}{2-\eta}+\frac{\ln^{-\lambda_1-\frac{\varepsilon}{q}}(2-\eta)}{\lambda_1+\frac{\varepsilon}{q}}\right]\right\}.
\end{aligned}$$

Then we have

$$\begin{aligned}
&B(\lambda_1-\frac{\varepsilon}{p},\lambda_2+\frac{\varepsilon}{p})\ln^{-\varepsilon}(2-\eta)\\
&\quad -\frac{\varepsilon\ln^{\lambda_1-\frac{\varepsilon}{p}}(2-\xi)}{\lambda_1-\frac{\varepsilon}{p}}\left[\frac{\ln^{-\lambda_1-\frac{\varepsilon}{q}-1}(2-\eta)}{2-\eta}+\frac{\ln^{-\lambda_1-\frac{\varepsilon}{q}}(2-\eta)}{\lambda_1+\frac{\varepsilon}{q}}\right]\\
&\leqslant \varepsilon\widetilde{I}<M\left[\frac{\varepsilon\ln^{-\varepsilon-1}(2-\xi)}{2-\xi}+\ln^{-\varepsilon}(2-\xi)\right]^{\frac{1}{p}}\\
&\quad \times\left[\frac{\varepsilon\ln^{-\varepsilon-1}(2-\eta)}{2-\eta}+\ln^{-\varepsilon}(2-\eta)\right]^{\frac{1}{q}}.
\end{aligned}$$

For $\varepsilon\to 0^+$, in view of the continuous property of the beta function, we find $B(\lambda_1,\lambda_2)\leqslant M$. Hence, $M=B(\lambda_1,\lambda_2)$ is the best possible constant factor of (11). □

REMARK 2. Let $\widetilde{\lambda}_1 := \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q}$, $\widetilde{\lambda}_2 := \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p}$. We find $\widetilde{\lambda}_1 + \widetilde{\lambda}_2 = \lambda$, and for $\lambda_i, \lambda - \lambda_i \in (0,1]$ $(i = 1,2)$,

$$0 < \widetilde{\lambda}_1 = \lambda - \widetilde{\lambda}_2 \leqslant \frac{1}{p} + \frac{1}{q} = 1,$$
$$0 < \widetilde{\lambda}_2 = \lambda - \widetilde{\lambda}_1 \leqslant \frac{1}{q} + \frac{1}{p} = 1,$$

and then we can rewrite (10) as follows:

$$I < k_\lambda^{\frac{1}{p}}(\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1) \left[\sum_{m=2}^{\infty} \frac{\ln^{p(1-\widetilde{\lambda}_1)-1}(m-\xi)}{(m-\xi)^{1-p}} a_m^p \right]^{\frac{1}{p}} \times \left[\sum_{n=2}^{\infty} \frac{\ln^{q(1-\widetilde{\lambda}_2)-1}(n-\eta)}{(n-\eta)^{1-q}} b_n^q \right]^{\frac{1}{q}}. \tag{12}$$

LEMMA 4. *If the constant factor* $k_\lambda^{\frac{1}{p}}(\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1)$ *in (12) is the best possible, then we have* $\lambda_1 + \lambda_2 = \lambda$.

Proof. If the constant factor $k_\lambda^{\frac{1}{p}}(\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1)$ in (12) is the best possible, then by (11), the unique best possible constant factor must be $B(\widetilde{\lambda}_1, \widetilde{\lambda}_2)$ $(\in \mathbf{R}_+)$, namely,

$$B(\widetilde{\lambda}_1, \widetilde{\lambda}_2) = k_\lambda^{\frac{1}{p}}(\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1).$$

By Hölder's inequality, we find

$$\begin{aligned} B(\widetilde{\lambda}_1, \widetilde{\lambda}_2) &= k_\lambda(\widetilde{\lambda}_1) = k_\lambda\left(\frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q}\right) \\ &= \int_0^\infty \frac{1}{(1+u)^\lambda} u^{(\frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q})-1} du \\ &= \int_0^\infty \frac{1}{(1+u)^\lambda} (u^{\frac{\lambda-\lambda_2-1}{p}})(u^{\frac{\lambda_1-1}{q}}) du \\ &\leqslant \left[\int_0^\infty \frac{v^{\lambda_2-1}}{(1+v)^\lambda} dv \right]^{\frac{1}{p}} \left[\int_0^\infty \frac{u^{\lambda_1-1}}{(1+u)^\lambda} du \right]^{\frac{1}{q}} \\ &= k_\lambda^{\frac{1}{p}}(\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1). \end{aligned} \tag{13}$$

We observe that (13) keeps the form of equality if and only if there exist constants A and B, such that they are not all zero and (cf. [28])

$$Au^{\lambda-\lambda_2-1} = Bu^{\lambda_1-1} \text{ a.e. in } \mathbf{R}_+.$$

Assuming that $A \neq 0$ (otherwise, $B = A = 0$), it follows that $u^{\lambda-\lambda_2-\lambda_1} = \frac{B}{A}$ a.e. in $\mathbf{R}_+$, and then $\lambda - \lambda_2 - \lambda_1 = 0$, namely, $\lambda_1 + \lambda_2 = \lambda$. □

3. Main results

THEOREM 1. *Inequality (10) is equivalent to the following inequality:*

$$J := \left\{ \sum_{n=2}^{\infty} \frac{\ln^{p(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q})-1}(n-\eta)}{n-\eta} \left[\sum_{m=2}^{\infty} \frac{a_m}{\ln^{\lambda}(m-\xi)(n-\eta)} \right]^p \right\}^{\frac{1}{p}}$$

$$< k_{\lambda}^{\frac{1}{p}}(\lambda_2) k_{\lambda}^{\frac{1}{q}}(\lambda_1) \left\{ \sum_{m=2}^{\infty} \frac{\ln^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}(m-\xi)}{(m-\xi)^{1-p}} a_m^p \right\}^{\frac{1}{p}}. \tag{14}$$

If the constant factor in (10) is the best possible, then the constant factor in (14) is also the best possible.

Proof. Suppose that (14) is valid. By Hölder's inequality (cf. [28]), we find

$$I = \sum_{n=2}^{\infty} \left[\frac{\ln^{\frac{-1}{p}+(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q})}(n-\eta)}{(n-\eta)^{1/p}} \sum_{m=2}^{\infty} \frac{a_m}{\ln^{\lambda}(m-\xi)(n-\eta)} \right]$$

$$\times \left[\frac{\ln^{\frac{1}{p}-(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q})}(n-\eta)}{(n-\eta)^{-1/p}} b_n \right]$$

$$\leqslant J \left\{ \sum_{n=2}^{\infty} \frac{\ln^{q[1-(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q})]-1}(n-\eta)}{(n-\eta)^{1-q}} b_n^q \right\}^{\frac{1}{q}}. \tag{15}$$

Then by (14), we obtain (10).

On the other hand, assuming that (10) is valid, we set

$$b_n := \frac{\ln^{p(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q})-1}(n-\eta)}{n-\eta} \left[\sum_{m=2}^{\infty} \frac{a_m}{\ln^{\lambda}(m-\xi)(n-\eta)} \right]^{p-1}, n \in \mathbf{N} \backslash \{1\}.$$

If $J = 0$, then (14) is naturally valid; if $J = \infty$, then it is impossible to make (14) valid, namely, $J < \infty$. Suppose that $0 < J < \infty$. By (10), it follows that

$$\sum_{n=2}^{\infty} \frac{\ln^{q[1-(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q})]-1}(n-\eta)}{(n-\eta)^{1-q}} b_n^q = J^p = I$$

$$< k_{\lambda}^{\frac{1}{p}}(\lambda_2) k_{\lambda}^{\frac{1}{q}}(\lambda_1) \left\{ \sum_{m=2}^{\infty} \frac{\ln^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}(m-\xi)}{(m-\xi)^{1-p}} a_m^p \right\}^{\frac{1}{p}}$$

$$\times \left\{ \sum_{n=2}^{\infty} \frac{\ln^{q[1-(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q})]-1}(n-\eta)}{(n-\eta)^{1-q}} b_n^q \right\}^{\frac{1}{q}},$$

$$J = \left\{ \sum_{n=2}^{\infty} \frac{\ln^{q[1-(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q})]-1}(n-\eta)}{(n-\eta)^{1-q}} b_n^q \right\}^{\frac{1}{p}}$$

$$< k_\lambda^{\frac{1}{p}}(\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1) \left\{ \sum_{m=2}^{\infty} \frac{\ln^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}(m-\xi)}{(m-\xi)^{1-p}} a_m^p \right\}^{\frac{1}{p}},$$

namely, (14) follows, which is equivalent to (10).

If the constant factor in (10) is the best possible, then so is constant factor in (14). Otherwise, by (15), we would reach a contradiction that the constant factor in (10) is not the best possible. □

THEOREM 2. *The following statements (i), (ii), (iii) and (iv) are equivalent:*

(i) $k_\lambda^{\frac{1}{p}}(\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1)$ is independent of p, q;
(ii) $k_\lambda^{\frac{1}{p}}(\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1)$ is expressible as a single integral;
(iii) $k_\lambda^{\frac{1}{p}}(\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1)$ is the best possible constant factor of (10);
(iv) $\lambda_1 + \lambda_2 = \lambda$.

If the statement (iv) follows, namely, $\lambda_1 + \lambda_2 = \lambda$, then we have (11) and the following equivalent inequality with the best possible constant factor $B(\lambda_1, \lambda_2)$:

$$\left\{ \sum_{n=2}^{\infty} \frac{\ln^{p\lambda_2-1}(n-\eta)}{n-\eta} \left[\sum_{m=2}^{\infty} \frac{a_m}{\ln^{\lambda}(m-\xi)(n-\eta)} \right]^p \right\}^{\frac{1}{p}}$$

$$< B(\lambda_1, \lambda_2) \left[\sum_{m=2}^{\infty} \frac{\ln^{p(1-\lambda_1)-1}(m-\xi)}{(m-\xi)^{1-p}} a_m^p \right]^{\frac{1}{p}}. \tag{16}$$

Proof. $(i) => (ii)$. By (i) , we find

$$k_\lambda^{\frac{1}{p}}(\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1) = \lim_{p\to\infty} \lim_{q\to 1^+} k_\lambda^{\frac{1}{p}}(\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1) = k_\lambda(\lambda_1),$$

namely, $k_\lambda^{\frac{1}{p}}(\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1)$ is expressible as a single integral

$$k_\lambda(\lambda_1) = \int_0^\infty \frac{1}{(1+u)^\lambda} u^{\lambda_1 - 1} du.$$

$(ii) => (iv)$. In (13), if $k_\lambda^{\frac{1}{p}}(\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1)$ is expressible as a single integral, then (13) keeps the form of equality. In view of the proof of Lemma 4, it follows that $\lambda_1 + \lambda_2 = \lambda$.

$(iv) => (i)$. If $\lambda_1 + \lambda_2 = \lambda$, then $k_\lambda^{\frac{1}{p}}(\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1) = k_\lambda(\lambda_1)$, which is independent of p, q. Hence, we have $(i) <=> (ii) <=> (iv)$.

$(iii) => (iv)$. By Lemma 4, we have $\lambda_1 + \lambda_2 = \lambda$.

$(iv) => (iii)$. By Lemma 3, for $\lambda_1 + \lambda_2 = \lambda$, $k_\lambda^{\frac{1}{p}}(\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1) (= k_\lambda(\lambda_1))$ is the best possible constant factor of (10). Therefore, we have $(iii) <=> (iv)$.

Hence, the statements (i), (ii), (iii) and (iv) are equivalent. □

REMARK 3. (i) For $\lambda = 1, \lambda_1 = \frac{1}{q}, \lambda_2 = \frac{1}{p}$ in (11) and (16), we have the following equivalent inequalities with the best possible constant factor $\frac{\pi}{\sin(\pi/p)}$:

$$\sum_{n=2}^{\infty} \sum_{m=2}^{\infty} \frac{a_m b_n}{\ln(m-\xi)(n-\eta)} < \frac{\pi}{\sin(\pi/p)} \left[\sum_{m=2}^{\infty} \frac{a_m^p}{(m-\xi)^{1-p}} \right]^{\frac{1}{p}} \left[\sum_{n=2}^{\infty} \frac{b_n^q}{(n-\eta)^{1-q}} \right]^{\frac{1}{q}}, \tag{17}$$

$$\left\{ \sum_{n=2}^{\infty} \frac{1}{n-\eta} \left[\sum_{m=2}^{\infty} \frac{a_m}{\ln(m-\xi)(n-\eta)} \right]^p \right\}^{\frac{1}{p}} < \frac{\pi}{\sin(\pi/p)} \left[\sum_{m=2}^{\infty} \frac{a_m^p}{(m-\xi)^{1-p}} \right]^{\frac{1}{p}}. \tag{18}$$

For $p = q = 2, \xi = \eta = 0$, (17) reduces to (2), and (18) reduces to the following equivalent form of (2):

$$\left[\sum_{n=2}^{\infty} \frac{1}{n} \left(\sum_{m=2}^{\infty} \frac{a_m}{\ln mn} \right)^2 \right]^{\frac{1}{2}} < \pi \left(\sum_{m=2}^{\infty} m a_m^2 \right)^{\frac{1}{2}}. \tag{19}$$

Hence, (10) is a more accurate extension of (2).

(ii) For $\lambda = 1, \lambda_1 = \frac{1}{p}, \lambda_2 = \frac{1}{q}$ in (11) and (16), we have the following equivalent inequalities with the best possible constant factor $\frac{\pi}{\sin(\pi/p)}$:

$$\sum_{n=2}^{\infty} \sum_{m=2}^{\infty} \frac{a_m b_n}{\ln(m-\xi)(n-\eta)} < \frac{\pi}{\sin(\pi/p)} \left[\sum_{m=2}^{\infty} \frac{\ln^{p-2}(m-\xi)}{(m-\xi)^{1-p}} a_m^p \right]^{\frac{1}{p}} \left[\sum_{n=2}^{\infty} \frac{\ln^{q-2}(n-\eta)}{(n-\eta)^{1-q}} b_n^q \right]^{\frac{1}{q}}, \tag{20}$$

$$\left\{ \sum_{n=2}^{\infty} \frac{\ln^{p-2}(n-\eta)}{n-\eta} \left[\sum_{m=2}^{\infty} \frac{a_m}{\ln(m-\xi)(n-\eta)} \right]^p \right\}^{\frac{1}{p}} < \frac{\pi}{\sin(\pi/p)} \left[\sum_{m=2}^{\infty} \frac{\ln^{p-2}(m-\xi)}{(m-\xi)^{1-p}} a_m^p \right]^{\frac{1}{p}}. \tag{21}$$

(iii) For $\lambda = 1, \lambda_1 = \lambda_2 = \frac{1}{2}$, in (11) and (16), we have the following equivalent inequalities with the best possible constant factor π:

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{a_m b_n}{\ln(m-\xi)(n-\eta)}$$
$$< \pi\left[\sum_{m=2}^{\infty}\frac{\ln^{\frac{p}{2}-1}(m-\xi)}{(m-\xi)^{1-p}}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=2}^{\infty}\frac{\ln^{\frac{q}{2}-1}(n-\eta)}{(n-\eta)^{1-q}}b_n^q\right]^{\frac{1}{q}}, \tag{22}$$

$$\left\{\sum_{n=2}^{\infty}\frac{\ln^{\frac{p}{2}-1}(n-\eta)}{n-\eta}\left[\sum_{m=2}^{\infty}\frac{a_m}{\ln(m-\xi)(n-\eta)}\right]^p\right\}^{\frac{1}{p}}$$
$$< \pi\left[\sum_{m=2}^{\infty}\frac{\ln^{\frac{p}{2}-1}(m-\xi)}{(m-\xi)^{1-p}}a_m^p\right]^{\frac{1}{p}}. \tag{23}$$

4. Operator expressions

We set functions

$$\varphi(m) := \frac{\ln^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}(m-\xi)}{(m-\xi)^{1-p}}, \psi(n) := \frac{\ln^{q[1-(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q})]-1}(n-\eta)}{(n-\eta)^{1-q}},$$

from where,

$$\psi^{1-p}(n) = \frac{\ln^{p(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q})-1}(n-\eta)}{n-\eta}\ (m,n \in \mathbf{N}\backslash\{1\}).$$

Define the following real normed spaces:

$$l_{p,\varphi} := \left\{a = \{a_m\}_{m=2}^{\infty}; ||a||_{p,\varphi} := (\sum_{m=2}^{\infty}\varphi(m)|a_m|^p)^{\frac{1}{p}} < \infty\right\},$$
$$l_{q,\psi} := \left\{b = \{b_n\}_{n=2}^{\infty}; ||b||_{q,\psi} := (\sum_{n=2}^{\infty}\psi(n)|b_n|^q)^{\frac{1}{q}} < \infty\right\},$$
$$l_{p,\psi^{1-p}} := \left\{c = \{c_n\}_{n=2}^{\infty}; ||c||_{p,\psi^{1-p}} := (\sum_{n=2}^{\infty}\psi^{1-p}(n)|c_n|^p)^{\frac{1}{p}} < \infty\right\}.$$

Assuming that $a \in l_{p,\varphi}$, setting

$$c = \{c_n\}_{n=2}^{\infty}, c_n := \sum_{m=2}^{\infty}\frac{a_m}{\ln^{\lambda}(m-\xi)(n-\eta)}, n \in \mathbf{N}\backslash\{1\},$$

we can rewrite (14) as follows:

$$||c||_{p,\psi^{1-p}} < k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)||a||_{p,\varphi} < \infty,$$

namely, $c \in l_{p,\psi^{1-p}}$.

DEFINITION 1. Define an extended Mulholland's operator $T : l_{p,\varphi} \to l_{p,\psi^{1-p}}$ as follows: For any $a \in l_{p,\varphi}$, there exists a unique representation $c \in l_{p,\psi^{1-p}}$. Define the formal inner product of Ta and $b \in l_{q,\psi}$, and the norm of T as follows:

$$(Ta,b) := \sum_{n=2}^{\infty}\left[\sum_{m=2}^{\infty}\frac{a_m}{\ln^\lambda(m-\xi)(n-\eta)}\right]b_n = I,$$

$$||T|| := \sup_{a(\neq\theta)\in l_{p,\varphi}}\frac{||Ta||_{p,\psi^{1-p}}}{||a||_{p,\varphi}}.$$

By Theorem 1 and Theorem 2, we have

THEOREM 3. *If* $a \in l_{p,\varphi}, b \in l_{q,\psi}, ||a||_{p,\varphi}, ||b||_{q,\psi} > 0$, *then we have the following equivalent inequalities:*

$$(Ta,b) < k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)||a||_{p,\varphi}||b||_{q,\psi}, \tag{24}$$

$$||Ta||_{p,\psi^{1-p}} < k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)||a||_{p,\varphi}. \tag{25}$$

Moreover, $\lambda_1+\lambda_2=\lambda$ *if and only if the constant factor* $k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)$ *in (24) and (25) is the best possible, namely,*

$$||T|| = B(\lambda_1,\lambda_2). \tag{26}$$

5. Conclusions

In this paper, by the use of the weight functions, the idea of introducing parameters, and Hermite-Hadamard's inequality, a more accurate extended Mulholland's inequality and its equivalent form are given in Lemma 2 and Theorem 1. The equivalent statements of the best possible constant factor related to some parameters in Theorem 2. Some particular cases and the operator expressions are considered in Remark 3 and Theorem 3. The lemmas and theorems provide an extensive account of this type of inequalities.

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Bicheng Yang
Department of Mathematics
Guangdong University of Education
Guangzhou, Guangdong 510303, P. R. China
e-mail: bcyang@gdei.edu.cn

Meifa Huang
School of Mechanical Science and Electrical Engineering
Guilin University of Electronic Technology
Guilin, Guangxi 541004, P. R. China
e-mail: hmhmf@guet.edu.cn

Yanru Zhong
School of Computer Science and Information Security
Guilin University of Electronic Technology
Guilin, Guangxi 541004, P. R. China
e-mail: 18577399236@163.com