Mathematical
Inequalities
& Applications
Volume 23, Number 1 (2020), 257–266 doi:10.7153/mia-2020-23-20

THE OPTIMAL CONSTANT IN GENERALIZED HARDY'S INEQUALITY

YING LI AND YONG-HUA MAO*

(*Communicated by S. Varošanec*)

Abstract. We obtain the sharp factor of the two-sides estimates of the optimal constant in generalized Hardy's inequality with two general Borel measures on $\mathbb{R}$, which generalizes and unifies the known continuous and discrete cases.

1. Introduction

Hardy's inequality is a powerful technical tool not only in advanced theoretical studies of the spectrum of non-negative self-adjoint differential operators such as elliptic operators [6, 20], but also in the study of probability such as the stability of diffusion processes or birth-death processes [4, Chapter 6]. Our motivation is to study the stability of generalized diffusion processes. However, we shall deal with this problem in separate papers.

For $p > 1$ and any non-negative number sequence $\{a_n\}_{n\geqslant 1}$ such that $\sum_{n=1}^{+\infty} a_n^p < +\infty$, Hardy's inequality was given by

$$\sum_{n=1}^{+\infty}\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^p \leqslant \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{+\infty} a_n^p \tag{1}$$

in [8], the optimal constant $\left(\frac{p}{p-1}\right)^p$ was fixed by Landau, Schur and Hardy in [12].

The continuous analogue of Hardy's inequality (1) was introduced in [8] as

$$\int_0^{+\infty}\left[\frac{1}{x}\int_0^x f(t)\mathrm{d}t\right]^p \mathrm{d}x \leqslant \left(\frac{p}{p-1}\right)^p \int_0^{+\infty} f(x)^p \mathrm{d}x \tag{2}$$

for $p > 1$ and $f \geqslant 0$ such that $f \in L^p(\mathbb{R}^+)$, the optimal constant $\left(\frac{p}{p-1}\right)^p$ was fixed by Hardy in [9].

Afterwards, Hardy's inequality has been generalized in various direction. In [19], Prokhorov gave necessary and sufficient conditions for validity of Hardy's inequality

Mathematics subject classification (2010): 26D10, 26D15, 26A30.
Keywords and phrases: Hardy's inequality, optimal constant, Cantor set.
We would like to gratefully thank the referee for very valuable suggestions which lead to the improvement of the new version.
This work is supported in part by NSFC (Grant No. 11771047) and Hu Xiang Gao Ceng Ci Ren Cai Ju Jiao Gong Cheng-Chuang Xin Ren Cai (Grant No. 2019RS1057).
* Corresponding author.

with three measures. He also claimed that the Hardy's inequality with three measures can be reduced to the following case with two measures. Let $1 < p \leqslant q < +\infty$, μ, ν be σ-finite Borel measures on $\mathbb{R}$, consider

$$\left[\int_{\mathbb{R}} \left(\int_{(-\infty,x)} f \mathrm{d}\nu\right)^q \mathrm{d}\mu(x)\right]^{1/q} \leqslant A \left(\int_{\mathbb{R}} f^p \mathrm{d}\nu\right)^{1/p}. \tag{3}$$

A two-sided estimate for the best constant A can be given as

$$B \leqslant A \leqslant k(q,p)B, \tag{4}$$

where the constant $k(q,p)$ can be taken as $p^{1/q}(p^*)^{1/p^*}$ and B is defined in (6) below. This findings generalize many existing estimates. For example, please refer to [3, 17] for both μ and ν absolutely continuous with respect to Lebesgue measure and refer to [14, 16] for both μ and ν discrete measures.

When μ and ν are both absolutely continuous with respect to Lebesgue measure, Maz'ya ([15]) presented the factor $k(q,p)$ as $(q^*)^{1/p^*}q^{1/q}$ for $1 < p < q < +\infty$, Opic and Kufner ([18]) improved it to $(1+q/p^*)^{1/q}(1+p^*/q)^{1/p^*}$ for $1 < p \leqslant q < +\infty$. When $p = q$, the factor $p^{1/q}(p^*)^{1/p^*}$, $(q^*)^{1/p^*}q^{1/q}$ and $(1+q/p^*)^{1/q}(1+p^*/q)^{1/p^*}$ are the same and [10, Theorem 326 and 327] indicates the factor is sharp. For $1 < p < q < +\infty$, Chen ([5]) obtained a sharp factor as

$$k_{q,p} = \left(\frac{r}{B(1/r,(q-1)/r)}\right)^{1/p-1/q}, \tag{5}$$

where $B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\mathrm{d}x$ and $r = q/p - 1$.

When μ and ν are both discrete measures, Liao ([13]) gave the factor $k(q,p)$ as $k_{q,p}$ in (5) for $1 < p \leqslant q < +\infty$.

A natural question is whether one can also improve the factor $k(q,p)$ to the sharp $k_{q,p}$ for the above Hardy's inequality (3) concerning two general σ-finite Borel measure? In the present paper, we will give an affirmative answer to this question as follows.

THEOREM 1. *Let $1 < p \leqslant q < +\infty$, μ and ν be two σ-finite Borel measures on $\mathbb{R}$. Set*

$$B = \sup_{x\in\mathbb{R}} \nu((-\infty,x])^{1/p^*}\mu([x,+\infty))^{1/q}. \tag{6}$$

If A is the optimal constant such that for all $f: \mathbb{R} \to \mathbb{R}$,

$$\left[\int_{\mathbb{R}} \left|\int_{(-\infty,x)} f(y)\nu(\mathrm{d}y)\right|^q \mu(\mathrm{d}x)\right]^{1/q} \leqslant A \left[\int_{\mathbb{R}} |f(x)|^p \nu(\mathrm{d}x)\right]^{1/p}, \tag{7}$$

then

$$B \leqslant A \leqslant k_{q,p}B$$

with $k_{q,p}$ defined in (5).

REMARK 1. (1) Theorem 1 does not include Liao's result ([13]) since the Hardy operator with integral over $(-\infty,x)$ is different from the Hardy operator with integral over $(-\infty,x]$, when the inner measure has atoms. In fact, (7) is weaker than the classical Hardy's inequalities when both μ and ν are discrete.

(2) According to [13, p.809], when $p\to q$, the factor $k_{q,p}=p^{1/p}{p^*}^{1/p^*}$, which is consistent with the result in [5, 13, 14, 16, 17, 18, 19].

(3) By substituting the interval $(x,+\infty)$ to $(-\infty,x)$ in the left side of (7), we can get a dual form of Theorem 1.

(4) We can also present the sharp factor of the two-side estimate of the optimal constant in the Hardy's inequality with three measures just as in [19].

To obtain the sharp factor in (5), we use the integral transform theorem to explore a new version of Bliss's lemma (see Lemma 2). Both this new version of Bliss's lemma and its proof are novel as far as we know.

Now, we give some typical examples as applications of the generalized Hardy's inequality in Theorem 1. In these applications, μ and ν can be discrete measures, continuous measures (absolutely continuous w.r.t. Lebesgue measure), and even Cantor measures which are neither continuous nor discrete (see section 3). Additionally, we give the analogue forms as in (1) and (2) when $p=q$.

COROLLARY 1. *Let λ denote the standard Bernoulli measure on the Cantor set in $[0,+\infty)$. For any non-negative function f and $p>1$, we have*

$$\int_0^{+\infty}\left(\frac{1}{\lambda([0,x])}\int_0^x f(t)\lambda(\mathrm{d}t)\right)^p\lambda(\mathrm{d}x)\leqslant\left(\frac{p}{p-1}\right)^p\int_0^{+\infty}f(x)^p\lambda(\mathrm{d}x). \tag{8}$$

Additionally, the factor $\left(\frac{p}{p-1}\right)^p$ is sharp. However, neither the inequality

$$\left[\int_0^{+\infty}\left(\frac{1}{\lambda([0,x])}\int_0^x f(t)\lambda(\mathrm{d}t)\right)^q\lambda(\mathrm{d}x)\right]^{1/q}\leqslant A\left[\int_0^{+\infty}f(x)^p\lambda(\mathrm{d}x)\right]^{1/p} \tag{9}$$

for $1<p<q<+\infty$ nor the inequality

$$\left[\int_0^{+\infty}\left(\frac{1}{x}\int_0^x f(t)\mathrm{d}t\right)^q\lambda(\mathrm{d}x)\right]^{1/q}\leqslant A\left[\int_0^{+\infty}f(x)^p\mathrm{d}x\right]^{1/p} \tag{10}$$

for $1<p\leqslant q<+\infty$ holds for any finite A.

Observing the proof of Corollary 1, we have that (8) holds for any σ-finite Borel measures, while (9) fails to hold for any σ-finite Borel measures such that $\Lambda(x):=\lambda([0,x])$ being a continuous increasing function.

By taking one measure discrete and another one absolutely continuous with respect to Lebesgue measure, we have the two following mixed forms of Hardy's inequalities.

COROLLARY 2. *For any non-negative function f and $p>1$, we have*

$$\sum_{n=1}^{+\infty}\left(\frac{1}{n}\int_1^n f(t)\mathrm{d}t\right)^p \leqslant \left(\frac{p}{p-1}\right)^p \int_1^{+\infty} f^p(x)\mathrm{d}x. \tag{11}$$

And for $1<p\leqslant q<+\infty$,

$$\left[\int_1^{+\infty}\left(\frac{1}{x}\sum_{1\leqslant n<x} f(n)\right)^q \mathrm{d}x\right]^{1/q} \leqslant A\left[\sum_{n=1}^{+\infty} f^p(n)\right]^{1/p} \tag{12}$$

holds with $(q-1)^{-1/q}\leqslant A\leqslant k_{q,p}(q-1)^{-1/q}$.

2. Proof of Theorem 1

In [5, 13], a key step in improving the factor to sharp is using the following Bliss lemma [2] directly or extending it to the case of discrete measures.

LEMMA 1. *Let $1<p<q<+\infty$ and f be a non-negative function on $[0,+\infty)$. Then we have*

$$\left[\int_0^{+\infty}\mathrm{d}(-x^{-q/p^*})\left(\int_0^x f(y)\mathrm{d}y\right)^q\right]^{1/q} \leqslant k_{q,p}\left[\int_0^{+\infty} f(x)^p\mathrm{d}x\right]^{1/p}.$$

Moreover, the optimal constant is attained when

$$f(x)=\gamma(\delta x^r+1)^{-(r+1)/r}$$

with $r=q/p-1$ and γ, δ being non-negative constants.

We will extend Bliss lemma to deal with general Borel measures on $\mathbb{R}$. First, let us recall some basic facts about any Borel measure ν on $\mathbb{R}$. Define its ‘cumulative distribution function’ and ‘inverse cumulative distribution function’ as:

$$S(x):=\nu((-\infty,x]), \qquad S^{-1}(y):=\inf\{x:S(x)\geqslant y\}.$$

Since S is right-continuous and increasing, it is well known that

$$\{y:S^{-1}(y)\leqslant x\}=\{y:y\leqslant S(x)\}, \qquad \{y:S^{-1}(y)>x\}=\{y:y>S(x)\}, \tag{13}$$

$$S(S^{-1}(y))\geqslant y, \qquad S(S^{-1}(y)-)\leqslant y. \tag{14}$$

In particular, if S is continuous, then $S(S^{-1}(y))=y$.

Let m denote the Lebesgue measure, for any $-\infty<a<b<+\infty$, we have from (13) that

$$\begin{aligned} m_{S^{-1}}((a,b]) &:= m(\{t:S^{-1}(t)\in(a,b]\})=m(\{t:t\in(S(a),S(b)]\}) \\ &= \int_{S(a)}^{S(b)}\mathrm{d}t = S(b)-S(a)=\nu((a,b]). \end{aligned}$$

Then the measure extension theorem implies that $m_{S^{-1}} = \nu$.

According to the integral transform theorem (see for example [7, Theorem 39.C.]), for any Borel set Γ and measurable function f, it follows that

$$\int_{\Gamma} f \mathrm{d}\nu = \int_{\{y: S^{-1}(y)\in\Gamma\}} f\circ S^{-1}(y)\mathrm{d}y. \tag{15}$$

Now, we state our generalized Bliss lemma.

LEMMA 2. *Suppose $S(+\infty) = +\infty$, the Borel measure $\widetilde{\nu}$ is defined by*

$$\widetilde{\nu}((x,+\infty)) := S(x)^{-q/p^*},\ \forall\, x\in\mathbb{R}.$$

Then for any non-negative real function f and $1 < p \leqslant q < +\infty$, we have

$$\left[\int_{\mathbb{R}} \widetilde{\nu}(\mathrm{d}x)\left(\int_{(-\infty,x)} f(y)\nu(\mathrm{d}y)\right)^q\right]^{1/q} \leqslant k_{q,p}\left[\int_{\mathbb{R}} f(y)^p\nu(\mathrm{d}y)\right]^{1/p}.$$

Different from Liao's case ([13]), here we take $\widetilde{\nu}((x,+\infty)) = \nu((-\infty,x])^{-q/p^*}$ instead of $\widetilde{\nu}([x,+\infty)) = \nu((-\infty,x])^{-q/p^*}$ since $\widetilde{\nu}([x,+\infty))$ is left-continuous with respect to x while $\nu((-\infty,x])^{-q/p^*}$ is right-continuous with respect to x, which lead to the integral over $(-\infty,x)$ in (7).

Now, let us prove Lemma 2.

Proof. In the case of $p = q$, the assertion holds as a result of Remark 1 (2) and [19, Theorem 1].

In the case of $p < q$, set $\widetilde{m}(\mathrm{d}x) = \mathrm{d}(-x^{-q/p^*})$. Since $S(+\infty) = +\infty$, we have that for any $x\in\mathbb{R}$,

$$\begin{aligned}
\widetilde{m}_{S^{-1}}((x,+\infty)) &:= \widetilde{m}(\{t : S^{-1}(t)\in(x,+\infty)\}) = \widetilde{m}(\{t : t\in(S(x),+\infty)\}) \\
&= \int_{S(x)}^{+\infty} \mathrm{d}(-t^{-q/p^*}) = S(x)^{-q/p^*} = \widetilde{\nu}((x,+\infty)).
\end{aligned}$$

Then we have $\widetilde{m}_{S^{-1}} = \widetilde{\nu}$ by measure extension theorem. Moreover, the integral transform formula implies that for any measurable function g,

$$\int_{\mathbb{R}} g(x)\widetilde{\nu}(\mathrm{d}x) = \int_0^{+\infty} g\circ S^{-1}(x)\mathrm{d}(-x^{-q/p^*}). \tag{16}$$

By (13), (15) and note that f is non-negative

$$\begin{aligned}
\int_{(-\infty,u)} f(y)\nu(\mathrm{d}y) &= \int_{\{y: S^{-1}(y)\in(-\infty,u)\}} f\circ S^{-1}(y)\mathrm{d}y \\
&\leqslant \int_{(0,S(u-)]} f\circ S^{-1}(y)\mathrm{d}y.
\end{aligned}$$

Furthermore, substituting $g(x) = \left(\int_{(-\infty,x)} f\mathrm{d}\nu\right)^q$ into (16), we obtain from (14) that

$$\begin{aligned}\int_{\mathbb{R}} \widetilde{\nu}(\mathrm{d}x)\left(\int_{(-\infty,x)} f(y)\nu(\mathrm{d}y)\right)^q &= \int_0^{+\infty} \mathrm{d}(-x^{-q/p^*})\left(\int_{(-\infty,S^{-1}(x))} f(y)\nu(\mathrm{d}y)\right)^q \\ &\leqslant \int_0^{+\infty} \mathrm{d}(-x^{-q/p^*})\left(\int_{(0,S(S^{-1}(x)-)]} f\circ S^{-1}(y)\mathrm{d}y\right)^q \\ &\leqslant \int_0^{+\infty} \mathrm{d}(-x^{-q/p^*})\left(\int_{(0,x]} f\circ S^{-1}(y)\mathrm{d}y\right)^q.\end{aligned}$$

According to Lemma 1 and (15), we have

$$\begin{aligned}\left[\int_0^{+\infty} \mathrm{d}(-x^{-q/p^*})\left(\int_0^x f\circ S^{-1}(y)\mathrm{d}y\right)^q\right]^{1/q} &\leqslant k_{q,p}\left(\int_0^{+\infty}\left(f\circ S^{-1}(x)\right)^p\mathrm{d}x\right)^{1/p} \\ &= k_{q,p}\left(\int_{\mathbb{R}} f(y)^p\nu(\mathrm{d}y)\right)^{1/p}.\end{aligned}$$

The next technical lemma shows that if one measure is dominated by another measure, then so are certain of their integrals.

LEMMA 3. *Let μ_1 and μ_2 be two σ-finite Borel measures. If*

$$\mu_1((x,+\infty)) \leqslant \mu_2((x,+\infty)), \quad \forall\, x\in\mathbb{R},$$

then for any non-negative increasing function f, we have

$$\int_{\mathbb{R}} f(x)\mu_1(\mathrm{d}x) \leqslant \int_{\mathbb{R}} f(x)\mu_2(\mathrm{d}x).$$

Proof. According to Fubini's theorem, for any non-negative increasing function f and σ-finite measures $\mu_i\,(i=1,2)$,

$$\begin{aligned}\int f(x)\mu_i(\mathrm{d}x) &= \int_{\{x:f(x)>0\}} f(x)\mu_i(\mathrm{d}x) \\ &= \int_{\{x:f(x)>0\}} \mu_i(\mathrm{d}x)\int_0^{f(x)}\mathrm{d}t \\ &= \int_{\mathbb{R}} I_{\{x:f(x)>0\}}\mu_i(\mathrm{d}x)\int_0^{+\infty} I_{\{t:f(x)>t\}}\mathrm{d}t \\ &= \int_0^{+\infty}\mathrm{d}t\int_{\mathbb{R}} I_{\{x:f(x)>t\}}\mu_i(\mathrm{d}x) \\ &= \int_0^{+\infty}\mu_i(\{x:f(x)>t\})\mathrm{d}t.\end{aligned}$$

Since f is an increasing function, it is easy to check that for any given $t\geqslant 0$, the set $\{x:f(x)>t\}$ have the form of $(a,+\infty)$ or $[a,+\infty)$. Thus, it suffices to show that

$$\mu_1([x,+\infty)) \leqslant \mu_2([x,+\infty)),\ x\in\mathbb{R}. \tag{17}$$

Without loss of generality, suppose for any given $x \in \mathbb{R}$, $\mu_2((x,+\infty)) < +\infty$. Since μ_2 is Radon, namely locally finite, we have $\mu_2((x-1/n,x]) < +\infty$ for any $n \geqslant 1$. Furthermore,

$$\mu_1((x-1/n,+\infty)) \leqslant \mu_2((x-1/n,+\infty)) = \mu_2((x-1/n,x]) + \mu_2((x,+\infty)) < +\infty.$$

Then (17) holds by the upper continuity of $\mu_i (i=1,2)$.

Proof. [Proof of Theorem 1:] We divide the proof into two steps:

(i) First, we prove the first assertion provided $\nu(\mathbb{R}) = +\infty$. Note that $B = +\infty$ implies $A = +\infty$ by [19]. Assume $B < +\infty$. Let

$$S(x) = \nu((-\infty,x]), \ \ \widetilde{\nu}((x,+\infty)) = S(x)^{-q/p^*}.$$

By the definition of B, we have that for any $x \in \mathbb{R}$,

$$\mu((x,+\infty)) \leqslant \mu([x,+\infty)) \leqslant B^q \nu((-\infty,x])^{-q/p^*} = B^q S(x)^{-q/p^*} = B^q \widetilde{\nu}((x,+\infty)).$$

According to Lemma 3 and Lemma 2, for any non-negative function f, we have

$$\begin{aligned}\int_{\mathbb{R}} \mu(\mathrm{d}x) \left(\int_{(-\infty,x)} f(y)\nu(\mathrm{d}y) \right)^q &\leqslant B^q \int_{\mathbb{R}} \widetilde{\nu}(\mathrm{d}x) \left(\int_{(-\infty,x)} f(y)\nu(\mathrm{d}y) \right)^q \\ &\leqslant k_{q,p}^q B^q \left(\int_{\mathbb{R}} f(x)^p \nu(\mathrm{d}x) \right)^{q/p}.\end{aligned}$$

Thus, $A \leqslant k_{q,p} B$. In addition, we have $B \leqslant A$ according to [19, Theorem 1]. Hence, $B \leqslant A \leqslant k_{q,p} B$.

(ii) The next step is to remove the condition $\nu(\mathbb{R}) = +\infty$. This is easy to overcome by [5, Lemma 4.2].

3. Proof of Corollaries 1 and 2

First, we recall the standard Bernoulli measure on the Cantor set in $\mathbb{R}$. Let $\Omega_i = \{0,1\}$, $i = 0,1,\cdots$, and ρ_m be the uniform probability measure on $\Omega^m := \prod_{i=0}^m \Omega_i$, that is $\rho_m(\{x\}) = 2^{-(m+1)}$ for any $(x_0,x_1,\cdots,x_m) \in \Omega^m$. Consider the map $J : \Omega^m \to [0,1]$,

$$\forall x = (x_0,x_1,\cdots,x_m) \in \Omega^m, \ \ J(x) := a_0^m x_0 + a_1^m x_1 + \cdots + a_m^m x_m,$$

where $a_k^m = 3^{-m} b_k$, $b_0 = 1$, $b_k = 2 \cdot 3^{k-1}$.

Let $K_m = J(\Omega^m)$. Then the closure of $\cup_{m=0}^{+\infty} K_m$ is Cantor set in $[0,1]$, denoted by $\mathbb{K}$. Let $\lambda_m = \rho_m \circ J^{-1}$, then $\lambda_m(\{p\}) = 2^{-(m+1)}$, $\forall\, p \in K_m$.

Following [11], we know that there exists a unique probability measure λ on $\mathbb{K}$ such that $\lambda_m \Rightarrow \lambda$, that is, $\forall\, f \in C(\mathbb{K})$, $\lim_{m\to+\infty} \int_{K_m} f \mathrm{d}\lambda_m = \int_{\mathbb{K}} f \mathrm{d}\lambda$, thus λ is called the standard Bernoulli (probability) measure on $\mathbb{K}$. Let $\widetilde{\mathbb{K}} = \cup_{n=0}^{+\infty}(n+\mathbb{K})$ be Cantor set on $[0,+\infty)$ and denote again by λ the extended Bernoulli measure on $\widetilde{\mathbb{K}}$.

Under our settings, we can have an analogue of Hardy's inequality on Cantor set, see Corollary 1 in section 1. Now, we give the proof of these results.

Proof. [Proof of Corollary 1:] (i) (8) follows by adapting [19, Theorem 1] or [21, Theorem 1.1].

From [19, Theorem 1] or Theorem 1 in this paper, if we can prove $B=+\infty$, then (9) and (10) fail to hold.

(ii) Let $\Lambda(x)=\lambda([0,x])$, $x\in[0,+\infty)$. Then Λ is an increasing continuous function and $\Lambda(+\infty)=+\infty$. Define $\Lambda^{-1}(y)=\inf\{x:\Lambda(x)\geqslant y\}$, then $\Lambda(\Lambda^{-1}(y))=y$. The integral transform formula implies that for any Borel measurable function g

$$\int_x^{+\infty} g(\Lambda(t))\lambda(\mathrm{d}t)=\int_{\Lambda(x)}^{\Lambda(+\infty)} g(\Lambda(\Lambda^{-1}(t)))\mathrm{d}t=\int_{\Lambda(x)}^{+\infty} g(t)\mathrm{d}t. \tag{18}$$

For (9), set $\nu=\lambda$ and $\mu(\mathrm{d}x)=\lambda([0,x])^{-q}\lambda(\mathrm{d}x)$ on $[0,+\infty)$. Clearly,

$$B=\sup_{x\in[0,+\infty)}\lambda([0,x])^{1/p^*}\left(\int_x^{+\infty}\frac{\lambda(\mathrm{d}t)}{\lambda([0,t])^q}\right)^{1/q}.$$

Take $g(t)=t^{-q}$ in (18), we get

$$\begin{aligned}\int_x^{+\infty}\frac{\lambda(\mathrm{d}t)}{\lambda([0,t])^q}&=\int_x^{+\infty}\Lambda(t)^{-q}\lambda(\mathrm{d}t)=\int_{\Lambda(x)}^{+\infty}t^{-q}\mathrm{d}t\\&=(q-1)^{-1}\Lambda(x)^{1-q}=(q-1)^{-1}\lambda([0,x])^{1-q}.\end{aligned}$$

Since $p<q$, we have $p^*>q^*$. Hence,

$$B=(q-1)^{-1/q}\sup_{x\in[0,+\infty)}\lambda([0,x])^{1/p^*-1/q^*}=+\infty.$$

(iii) For (10), let $\mu(\mathrm{d}x)=x^{-q}\lambda(\mathrm{d}x)$ and ν be the Lebesgue measure on $[0,+\infty)$. It is obvious that

$$B=\sup_{x\in[0,+\infty)}x^{1/p^*}\left(\int_x^{+\infty}t^{-q}\lambda(\mathrm{d}t)\right)^{1/q}=\sup_{x\in\widetilde{\mathbb{K}}}x^{1/p^*}\left(\int_x^{+\infty}t^{-q}\lambda(\mathrm{d}t)\right)^{1/q}.$$

Take $x_m=2\cdot3^{-m}$ to derive

$$\begin{aligned}(x_m)^{1/p^*}\left(\int_{[x_m,+\infty)}t^{-q}\lambda(\mathrm{d}t)\right)^{1/q}&=\left(2\cdot3^{-m}\right)^{1/p^*}\left(\int_{[2\cdot3^{-m},+\infty)}t^{-q}\lambda(\mathrm{d}t)\right)^{1/q}\\&\geqslant\left(2\cdot3^{-m}\right)^{1/p^*}\left[\left(2\cdot3^{-m}\right)^{-q}\cdot2^{-(m+1)}\right]^{1/q}\\&=2^{-1/p-1/q}\cdot\left(\frac{3^{1/p}}{2^{1/q}}\right)^m.\end{aligned}$$

Since $1<p\leqslant q<+\infty$, we have $3^{1/p}\geqslant3^{1/q}>2^{1/q}$. Then we get

$$2^{-1/p-1/q}\cdot\left(\frac{3^{1/p}}{2^{1/q}}\right)^m\longrightarrow+\infty,\ \text{if } m\to+\infty.$$

Consequently, we have $B=+\infty$.

Proof. [Proof of Corollary 2:] To prove (11), let μ be the measure on $\mathbb{N}$ with $\mu_n = n^{-p}$ and $\nu(\mathrm{d}x)=\mathrm{d}x$ on $[1,+\infty)$. Clearly,

$$B=\sup_{x\geqslant 1}(x-1)^{1/p^*}\left(\sum_{k\geqslant x}k^{-p}\right)^{1/p}=\sup_{n\geqslant 2}(n-1)^{1/p^*}\left(\sum_{k\geqslant n}k^{-p}\right)^{1/p}.$$

For any $n\in\mathbb{Z}^+$, on the one hand,

$$\sum_{k\geqslant n}k^{-p}=\sum_{k\geqslant n}\int_{k-1}^{k}k^{-p}\mathrm{d}x\leqslant\sum_{k\geqslant n}\int_{k-1}^{k}x^{-p}\mathrm{d}x=\frac{(n-1)^{1-p}}{p-1}.$$

Then we have

$$B\leqslant(p-1)^{-1/p}\sup_{n\geqslant 2}(n-1)^{1/p^*-1/p^*}=(p-1)^{-1/p}.$$

On the other hand,

$$\sum_{k\geqslant n}k^{-p}=\sum_{k\geqslant n}\int_{k}^{k+1}k^{-p}\mathrm{d}x\geqslant\sum_{k\geqslant n}\int_{k}^{k+1}x^{-p}\mathrm{d}x=\frac{n^{1-p}}{p-1}.$$

Then we have

$$B\geqslant(p-1)^{-1/p}\sup_{n\geqslant 2}\left(\frac{n-1}{n}\right)^{1/p^*}=(p-1)^{-1/p}.$$

Consequently,

$$B=(p-1)^{-1/p}.$$

Then we have $k_{p,p}B=\frac{p}{p-1}$.

To prove (12), let ν be the counting measure on $\mathbb{N}$ and $\mathrm{d}\mu(x)=x^{-q}\mathrm{d}x$ on $[1,+\infty)$, we get

$$B=\sup_{x\geqslant 1}[x]^{1/p^*}\left(\int_x^{+\infty}t^{-q}\mathrm{d}t\right)^{1/q}=(q-1)^{-1/q}\sup_{x\geqslant 1}[x]^{1/p^*}x^{-1/q^*}=(q-1)^{-1/q}.$$

REFERENCES

[1] W. BURNSIDE, *A rapidly convergent series for* $\log N!$, Messenger Math., **46**, 1 (1917), 157–159.

[2] G. A. BLISS, *An integral inequality*, Journal of the London Mathematical Society., **1**, 1 (1930), 40–46.

[3] J. BRADLEY, *Hardy inequalities with mixed norms*, Canadian Mathematical Bulletin., **21**, 4 (1978), 405–408.

[4] M. F. CHEN, *Eigenvalues, Inequalities, and Ergodic Theory*, Probability and its Applications, New York, 2005.

[5] M. F. CHEN, *The optimal constant in Hardy-type inequalities*, Acta Mathematica Sinica, English Series., **31**, 5 (2015), 731–754.
[6] E. B. DAVIES, *Spectral Theory and Differential Operators*, Cambridge Studis in Advanced Mathematics **42**, Cambridge University Press, Cambridge, 1995.
[7] P. R. HALMOS, *Measure Theory*, Van Nostrand Company, New York, 1950.
[8] G. H. HARDY, *Note on a theorem of Hilbert*, Mathematische Zeitschrif., **6**, 3 (1920), 314–317.
[9] G. H. HARDY, *Notes on some points in the integral calculus*, Messenger of Math., **54**, (1925), 150–156.
[10] G. H. HARDY, J. E. LITTLEWOOD AND G. PÓLYA, *Inequalities*, Cambridge University Press, Cambridge, 1988.
[11] J. KIGAMI AND M. L. LAPIDUS, *Weyl's problem for the spectral distribution of Laplacians on p.c.f. self-similar fractals*, Communications in Mathematical Physics., **158**, 1 (1993), 93–125.
[12] E. LANDAU, I. SCHUR AND G. H. HARDY, *A note on a theorem concerning series of positive terms*, Journal of the London Mathematical Society., **1**, 1 (1926), 38–39.
[13] Z. W. LIAO, *Discrete Hardy-type inequalities*, Advanced Nonlinear Studies., **15**, 4 (2015), 805–834.
[14] Y. H. MAO, *Nash inequalities for Markov processes in dimension one*, Acta Mathematica Sinica., **18**, 1 (2002), 147–156.
[15] V. MAZ'YA, *Sobolev Spaces with Applications to Elliptic Partial Differential Equations*, Springer, Second, revised and augmented edition **342**, Heidelber, 2011.
[16] L. MICLO, *An example of application of discrete Hardy's inequalities*, Markov Process. Related Fields., **5**, 3 (1999), 319–330.
[17] B. MUCKENHOUPT, *Hardy's inequality with weights*, Studia Mathematica., **44**, 1 (1972), 31–38.
[18] B. OPIC AND A. KUFNER, *Hardy-Type Inequalities*, Pitman Research Notes in Mathematics **219**, Essex, 1990.
[19] D. V. PROKHOROV, *Hardy's inequality with three measures*, Proceedings of the Steklov Institute of Mathematics., **255**, 1 (2006), 221–233.
[20] L. W. ROLLINS, *Criteria for discrete spectrum of singular selfadjoint differential operators*, Proceedings of the American Mathematical Society., **34**, 1 (1972), 195–200.
[21] G. SINNAMON, *Hardy's inequality and monotonicity*, Proceedings of Function Spaces, Differential Operators and Nonlinear Analysis, Academy of Sciences of the Czech Republic, Milovy., (2004), 292–310.

(Received December 31, 2018)

Ying Li
School of Mathematics and Computational Science
Xiangtan University
Xiangtan 411105, Hunan, People's Republic of China
e-mail: liying@xtu.edu.cn

Yong-Hua Mao
School of Mathematical Sciences, Beijing Normal University
Laboratory of Mathematics and Complex Systems, Ministry of Education
Beijing 100875, People's Republic of China
e-mail: maoyh@bnu.edu.cn