Mathematical
Inequalities
& Applications
Volume 23, Number 1 (2020), 267–286 doi:10.7153/mia-2020-23-21

WEIGHTED WEAK–TYPE INEQUALITIES FOR SQUARE FUNCTIONS

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(*Communicated by I. Pinelis*)

Abstract. The paper is devoted to weighted weak-type inequalities for square functions of continuous-path martingales and identifies the optimal dependence of the weak norm on the characteristic of the weight. The proof rests on Bellman function technique: the estimates are deduced from the existence of special functions enjoying appropriate size conditions and concavity.

1. Introduction

In [2], the authors used the Bellman function approach to give new proofs of weighted L^2-norm inequalities for martingales and Littlewood-Paley square functions with the optimal dependence on the A_2 characteristics $[w]_{A_2}$ of the weight w and further explicit constants, and in [3] improved the results for the full range $1 < p < \infty$. This paper is a continuation of these works and contains a complete description of the corresponding weak-type estimates in the martingale setting.

Let us introduce the necessary probabilistic background and formulate our main results. Assume that $(\Omega, \mathscr{F}, \mathbb{P})$ is a complete probability space, filtered by $(\mathscr{F}_t)_{t\geqslant 0}$, a nondecreasing right-continuous sequence of sub-σ-algebras of $\mathscr{F}$. Suppose in addition that $\mathscr{F}_0$ contains all the events of probability 0 and all $(\mathscr{F}_t)_{t\geqslant 0}$-adapted martingales have continuous paths (for instance, this holds for Brownian filtration). Assume further that $X = (X_t)_{t\geqslant 0}$ is an adapted, uniformly integrable martingale (with no risk of confusion, we will often identify X with its terminal variable X_∞) and let $\langle X\rangle = (\langle X_t\rangle)_{t\geqslant 0}$ denote its quadratic covariance process (square function). See e.g. Dellacherie and Meyer [5] for more information on the subject.

The inequalities between X and $\langle X\rangle$ are of fundamental importance to the theory of stochastic integration, and the principal purpose of this paper is to study certain class of such bounds in the weighted context. In what follows, the word 'weight' will refer to a uniformly integrable martingale $W = (W_t)_{t\geqslant 0}$; as in the case of X, we will usually identify W with W_∞. Any weight W gives rise to the corresponding L^p and weak L^p spaces, $0 < p < \infty$, given by

$$L^p(W) = \{f : \|f\|_{L^p(W)} := (\mathbb{E}|f|^p W)^{1/p} < \infty\}$$

Mathematics subject classification (2010): 46E30, 60G42.
Keywords and phrases: Square function, weight, Bellman function.

Paper MIA-23-21

and

$$L^{p,\infty}(W) = \left\{ f : \|f\|_{L^{p,\infty}(W)} := \sup_{\lambda>0} \left(\lambda^p W(|f|>\lambda)\right)^{1/p} < \infty \right\},$$

where $W(A) := \mathbb{E}1_A W$. Let $1 < p < \infty$ be fixed. Motivated by the classical Burkholder-Davis-Gundy inequalities, one can ask about the characterization of weights W for which the estimate

$$\|\langle X\rangle^{1/2}\|_{L^p(W)} \leqslant C_{p,W}\|X\|_{L^p(W)} \tag{1}$$

holds true, with some finite constant $C_{p,W}$ depending only on the parameters indicated. The same question can be posed for the weak-type inequality

$$\|\langle X\rangle^{1/2}\|_{L^{p,\infty}(W)} \leqslant c_{p,W}\|X\|_{L^p(W)}. \tag{2}$$

It can be shown that both (1) and (2) hold true if and only if W satisfies the so-called Muckenhoupt's condition A_p. The class A_p was originally introduced in the analytic setting by Muckenhoupt [12] during the study of maximal operators on weighted spaces. In the probabilistic context, the appropriate definition was given by Izumisawa and Kazamaki [9]: we say that W satisfies Muckenhoupt's condition A_p, if there is a deterministic constant c such that

$$\mathbb{E}(W|\mathscr{F}_t)\mathbb{E}(W^{1/(1-p)}|\mathscr{F}_t)^{p-1} \leqslant c \tag{3}$$

almost surely for all t. The smallest c with the above property is denoted by $[W]_{A_p}$ and called the A_p characteristic of the weight. Passing with $p \to 1$ or $p \to \infty$ in the above definition, one obtains the corresponding A_1 and A_∞ conditions. Namely, $[W]_{A_1}$ is the least constant c for which

$$\sup_{0\leqslant s\leqslant t} \mathbb{E}(W|\mathscr{F}_s) \leqslant c\mathbb{E}(W|\mathscr{F}_t)$$

almost surely for all t, while $[W]_\infty$ is the smallest constant such that

$$\mathbb{E}(W|\mathscr{F}_t)\exp\left(\mathbb{E}(-\log W|\mathscr{F}_t)\right) \leqslant c$$

with probability 1 for all t. Roughly speaking, the number $[W]_{A_p}$ measures the balance of W: the bigger $[W]_{A_p}$, the more 'unbalanced' the weight is. Note that $[W]_{A_p} \geqslant 1$, by Jensen's inequality; in addition, $[W]_{A_p} = 1$ if and only if W_∞ is a constant random variable. As we have mentioned above, the condition A_p characterizes the boundedness of square functions on the associated weighted spaces. It turns out that the classes A_p arise in the study of analogous boundedness problems for other operators, such as maximal functions, martingale transforms (stochastic integrals) and fractional operators: see [1, 10, 11, 17, 21] and cnsult references therein.

One can ask about the refinement of (1) and (2) which concerns the optimal dependence of the constants $C_{p,W}$ and $c_{p,W}$ on the characteristic $[W]_{A_p}$. The problem is the following. Given $1 < p < \infty$, find the least exponents κ_p, β_p such that $C_{p,W} \leqslant C_p[W]_{A_p}^{\kappa_p}$ and $c_{p,W} \leqslant c_p[W]_{A_p}^{\beta_p}$, where C_p and c_p depend only on p. Similar question can be asked also for other types of operators. Such problems, considered for various analytic operators, have gained a lot of interest in the literature: consult e.g. [1, 4, 7, 10, 11, 21]. The following weighted L^p bound for square functions was obtained in [2].

THEOREM 1. *Suppose that W is an A_p weight. Then for any $1<p<\infty$ and any X we have the estimate*

$$||\langle X\rangle_\infty^{1/2}||_{L^p(W)} \leqslant K_p[W]_{A_p}^{\max\{1/2,1/(p-1)\}}||X||_{L^p(W)}, \tag{4}$$

where $K_p = O((p-1)^{-1})$ as $p\to 1$ and $K_p = O(p^{1/2})$ as $p\to\infty$. The exponent $\max\{1/2,1/(p-1)\}$ *is the best possible.*

We will study the corresponding problem for weighted weak-type estimates. Here is our main result.

THEOREM 2. *Suppose that W is an A_p weight. Then for any $1\leqslant p<\infty$, $p\neq 2$, and any X we have the estimate*

$$||\langle X\rangle_\infty^{1/2}||_{L^{p,\infty}(W)} \leqslant K_p[W]_{A_p}^{\max\{1/p,1/2\}}||X||_{L^p(W)}, \tag{5}$$

where $K_p = O(p^{1/2})$ as $p\to\infty$. The exponent $\max\{1/p,1/2\}$ *is the best possible.*

Quite surprisingly, we do not know the appropriate sharp version of (5) for the case $p=2$. The same unexpected open problem arises in the context of square functions in harmonic analysis [6, 8]. In what follows, we will focus on the case $p>1$, the case $p=1$ has been already established by the author in [16].

The proof of Theorem 2 will exploit the so-called Bellman function method and will rest on the construction of a special function enjoying certain majorization and concavity-type properties. The precise description of our approach (i.e., the reduction to the existence of an appropriate special function) is presented in the next section. In Section 3 we present the proof of (5) in the case $1<p<2$; for these values of p we have found a very simple Bellman function. Section 4 is devoted to the study of the case $p>2$ and the final part of the paper addresses the optimality of the exponent $\max\{1/p,1/2\}$ involved in (5).

2. On the method of proof

In this section we show how to reduce the proof of our main inequality (5) to the construction of an appropriate special function of four variables. Given $1<p<\infty$ and $1\leqslant c<\infty$, consider the domain

$$D_{p,c} = \{(w,v)\in(0,\infty)\times(0,\infty)\,:\,1\leqslant wv^{p-1}\leqslant c\}.$$

Let K, $\kappa\geqslant 1$ and suppose that $U:[-1,1]\times[0,1]\times D_{p,c}\to\mathbb{R}$ is a function satisfying the following structural properties.

1° U is continuous and of class C^2 in the interior of its domain.

2° We have $U(x,x^2,w,v)\leqslant 0$ for all $x\in[0,1]$ and $(w,v)\in D_{p,c}$.

3° For all $(x,y,w,v) \in [-1,1] \times [0,1] \times D_{p,c}$ we have

$$U(x,y,w,v) \geqslant w 1_{\{|x| \vee y \geqslant 1\}} - Kc^{\kappa}|x|^p v^{1-p}. \tag{6}$$

4° For any (x,y,w,v) lying in the interior of the domain of U, the matrix

$$\mathfrak{D}^2 U = \begin{bmatrix} U_{xx}+2U_y & U_{xw} & U_{xv} \\ U_{wx} & U_{ww} & U_{wv} \\ U_{vx} & U_{vw} & U_{vv} \end{bmatrix} \tag{7}$$

is nonpositive definite.

THEOREM 3. *If U satisfies 1°, 2°, 3° and 4°, then for any martingale X and any weight W satisfying $[W]_{A_p} \leqslant c$ we have*

$$\lambda^p W(\langle X \rangle_\infty^{1/2} > \lambda) \leqslant Kc^{\kappa}\mathbb{E}|X|^p W. \tag{8}$$

Proof. By homogeneity, we may assume that $\lambda = 1$. The reasoning rests on Itô's formula and appropriate stopping time arguments. Let

$$\tau = \inf\{t \geqslant 0 : |X_t| \vee \langle X \rangle_t \geqslant 1\},$$

with the usual convention $\inf \emptyset = \infty$. On the set $\tau = 0$ we have $|X_0|^2 = \langle X \rangle_0 \geqslant 1$ and hence

$$\begin{aligned} W(|X_\tau| \vee \langle X \rangle_\tau \geqslant 1,\, \tau = 0) &= W(\tau = 0) \\ &= \mathbb{E}1_{\{\tau=0\}} W_0 \\ &\leqslant c\mathbb{E}|X_0|^p V_0^{1-p} 1_{\{\tau=0\}} \\ &\leqslant c\mathbb{E}|X|^p V^{1-p} 1_{\{\tau=0\}} \\ &\leqslant Kc^{\kappa}\mathbb{E}|X|^p W 1_{\{\tau=0\}}, \end{aligned} \tag{9}$$

where in the third passage we have used the inequality $W_0 V_0^{p-1} \leqslant c$ (guaranteed by $[W]_{A_p} \leqslant c$), the fourth follows from the convexity of the function $(x,v) \mapsto |x|^p v^{1-p}$ on $\mathbb{R} \times (0,\infty)$, and the last inequality is due to the inequalities $K, \kappa \geqslant 1$ and the identity $W = V^{1-p}$. On the other hand, on the set $\{\tau > 0\}$, we apply Itô's formula to the composition of the C^2 function U (see 1°) and the process $Z_t = (X_t, \langle X \rangle_t, W_t, V_t)$ to obtain

$$U(Z_{\tau \wedge t}) = I_0 + I_1 + I_2/2.$$

Here

$$\begin{aligned} I_0 &= U(Z_0) = U(X_0, X_0^2, W_0, V_0), \\ I_1 &= \int_{0+}^{\tau \wedge t} U_x(Z_s)\mathrm{d}X_s + \int_{0+}^{\tau \wedge t} U_w(Z_s)\mathrm{d}W_s + \int_{0+}^{\tau \wedge t} U_v(Z_s)\mathrm{d}V_s, \\ I_2 &= \int_{0+}^{\tau \wedge t} \mathscr{D}^2 U(Z_s)\mathrm{d}\langle Z \rangle_s \end{aligned}$$

and the integral defining I_2 is the abbreviated sum of all second-order terms

$$I_2 = \int_{0+}^{\tau\wedge t} U_{xx}(Z_s)\mathrm{d}\langle X\rangle_s + 2\int_{0+}^{\tau\wedge t} U_{xw}(Z_s)\mathrm{d}\langle X,W\rangle_s + 2\int_{0+}^{\tau\wedge t} U_{xv}(Z_s)\mathrm{d}\langle X,V\rangle_s + \dots$$

together with the integral

$$\int_{0+}^{\tau\wedge t} 2U_y(Z_s)\mathrm{d}\langle X\rangle_s.$$

Let us analyze the behavior of the terms I_1, I_2 and I_3. By 2°, we have $I_0 \leqslant 0$. The process I_1 is a mean-zero martingale as a function of t, and hence $\mathbb{E}I_1 1_{\{\tau>0\}} = 0$. Finally, the assumption 4° implies that $I_2 \leqslant 0$, which can be easily seen by approximating I_2 by discrete sums. Putting all the above facts together, we obtain $\mathbb{E}U(Z_{\tau\wedge t})1_{\{\tau>0\}} \leqslant 0$, so letting $t\to\infty$ gives

$$\mathbb{E}U(Z_\tau)1_{\{\tau>0\}} \leqslant 0,$$

by Lebesgue's dominated convergence theorem (U is bounded and Z has continuous paths). Consequently, by 3°,

$$\mathbb{E}W_\tau 1_{\{|X_\tau|\vee\langle X\rangle_\tau\geqslant 1\}}1_{\{\tau>0\}} \leqslant Kc^\kappa \mathbb{E}|X_\tau|^p V_\tau^{1-p} 1_{\{\tau>0\}} \tag{10}$$

and hence, using the identity $W_\tau = \mathbb{E}(W|\mathscr{F}_\tau)$ and the convexity of the function $(x,v)\mapsto |x|^p v^{1-p}$, we get

$$W(|X_\tau|\vee\langle X\rangle_\tau \geqslant 1,\, \tau>0) \leqslant Kc^\kappa\mathbb{E}|X|^pV^{1-p}1_{\{\tau>0\}} = Kc^\kappa\mathbb{E}|X|^pW1_{\{\tau>0\}}.$$

Adding this to (9), we obtain

$$W(|X_\tau|\vee\langle X\rangle_\tau \geqslant 1) \leqslant Kc^\kappa\mathbb{E}|X|^pW.$$

It remains to note that $\{\langle X\rangle_\infty > 1\} \subseteq \{|X_\tau|\vee\langle X\rangle_\tau\geqslant 1\}$ to get the claim.

REMARK 1. The above reasoning (see (9) and (10)) shows that under the assumptions of Theorem 3, we have the estimate

$$W(|X_\tau|\vee\langle X\rangle_\tau \geqslant 1) \leqslant Kc^\kappa\mathbb{E}|X_\tau|^pV_\tau^{1-p}.$$

The regularity condition 1° can be slightly relaxed: roughly speaking, if U is a minimum of several smooth functions satisfying 1°-4°, then (8) remains valid. The precise statement is as follows.

LEMMA 1. *Assume that $U_1,\ U_2,\ \dots,\ U_n : D\to\mathbb{R}$ are C^2 functions and define $U=\min\{U_1,U_2,\dots,U_n\}$. Suppose that U satisfies 2°, 3° and for each k we have $\mathfrak{D}U_k\leqslant 0$ on the set $\{U=U_k\}$. Then (8) holds true.*

The proof rests on a simple mollification argument; see e.g. Wang [20] for a similar reasoning. Alternatively, one can apply a variant of Itô's formula which allows the existence of non-differentiability points at the cost of certain local times: see [18].

When verifying 4°, it will sometimes be convenient to use the notation

$$Q''_U(x,y,w,v,d,r,s) = \left\langle \mathfrak{D}^2 U(x,y,w,v)(d,r,s),(d,r,s)\right\rangle.$$

Of course, the concavity condition 4° is equivalent to saying that $Q''_U(x,y,w,v,d,r,s)$ is nonpositive for all x, y, w, v, d, r and s. Usually, we will write Q''_U instead of $Q''_U(x,y,w,v,d,r,s)$ - in general, d, r, s will be arbitrary, and it will be clear from the context what the variables x, y, w and v are. We will frequently use the linearity $Q''_{aU_1+bU_2} = aQ''_{U_1} + bQ''_{U_2}$ and the fact that $Q''_U \leqslant 0$ if U is a concave function depending only on x, w and v.

3. A special function, $1<p<2$

In the range $1<p<2$ the Bellman function is very easy: let

$$U_p(x,y,w,v) = yw - \frac{pc}{2-p}x^2v^{1-p}.$$

LEMMA 2. *The function U_p satisfies the conditions 1°-4° with $\kappa = 1$ and $K = 2/(2-p)$.*

Proof. The regularity 1° is evident. Concerning 2°, we check that

$$U_p(x,x^2,w,v) = x^2w - \frac{pc}{2-p}x^2v^{1-p} \leqslant x^2w - \frac{p}{2-p}x^2w \leqslant 0.$$

Let us show the majorization 3°. If $|x|\vee y<1$, then the estimate is trivial:

$$U_p(x,y,w,v) \geqslant -\frac{pc}{2-p}x^2v^{1-p} \geqslant -Kcx^pv^{1-p}.$$

If $y=1$, then $U_p(x,y,w,v) = w - \frac{pc}{2-p}x^2v^{1-p} \geqslant w - Kcx^pv^{1-p}$. Finally, if $|x|=1$ and $y<1$, then note that

$$\frac{2c}{2-p}v^{1-p} = \frac{pc}{2-p}v^{1-p} + cv^{1-p} \geqslant \frac{pc}{2-p}v^{1-p} + w,$$

which is precisely the desired majorization. It remains to verify the matrix condition 4°. We compute that

$$\mathfrak{D}^2U_p(x,y,w,v) = \begin{bmatrix} 2\left(w - \frac{p}{2-p}cv^{1-p}\right) & 0 & \frac{2p(p-1)c}{2-p}xv^{-p} \\ 0 & 0 & 0 \\ \frac{2p(p-1)c}{2-p}xv^{-p} & 0 & -\frac{p^2(p-1)c}{2-p}x^2v^{-p-1} \end{bmatrix}$$

To check that this matrix is nonpositive definite, observe that the entry in the right-lower corner is negative and the determinant of the 2×2 matrix obtained from $\mathfrak{D}^2U_p$ by removing the middle row and the middle column, equals

$$\begin{aligned} &-2\left(w-\frac{p}{2-p}cv^{1-p}\right)\cdot\frac{p^2(p-1)c}{2-p}x^2v^{-p-1}-\left[\frac{2p(p-1)c}{2-p}xv^{-p}\right]^2\\ &\geqslant -2\left(cv^{1-p}-\frac{p}{2-p}cv^{1-p}\right)\cdot\frac{p^2(p-1)c}{2-p}x^2v^{-p-1}-\left[\frac{2p(p-1)c}{2-p}xv^{-p}\right]^2=0. \end{aligned}$$

This completes the proof.

4. The case $p>2$, a small characteristic

We turn our attention to the case $p>2$. We will consider two cases separately: $c\leqslant p$ and $c>p$. We would like to emphasize here that the proof presented in the next section works for all c, however, for small characteristics the dependence of the constants on p is not optimal. This is why we consider in this section the additional case of 'small c': throughout this part of the paper, we assume that $c\leqslant p$. We will need an additional parameter $\alpha=(4c)^{-1}$.

We start with a technical lemma, which actually holds in both cases $c\leqslant p$, $c>p$.

LEMMA 3. *Let $F(w,v)=2w^{1-\alpha}v^{\alpha(1-p)}-v^{1-p}$. If $1\leqslant wv^{p-1}\leqslant c$, then $w\leqslant F(w,v)\leqslant 2w$ and the Hessian matrix of F satisfies*

$$D^2F(w,v)\leqslant\begin{bmatrix}-(8cw)^{-1} & 0\\ 0 & 0\end{bmatrix}.$$

Proof. Observe that $F(w,v)\leqslant 2w(wv^{p-1})^{-\alpha}\leqslant 2w$; the inequality $F(w,v)\geqslant w$ is equivalent to $2t^{1-\alpha}\geqslant t+1$, where $t=wv^{p-1}\in[1,c]$. The left-hand side of the latter bound is a concave function of t, so it is enough to prove this estimate for $t\in\{1,c\}$. For $t=1$ both sides are equal, for $t=c$ we must show that $2c^{1-1/(4c)}\geqslant c+1$, or

$$\xi(c)=\left(1-\frac{1}{4c}\right)\ln c-\ln\left(\frac{c+1}{2}\right)\geqslant 0.$$

But $\xi(1)=0$, and for $c\geqslant 1$ we have

$$\xi'(c)=\frac{1}{4c^2}\ln c+\frac{1}{c}\left(1-\frac{1}{4c}\right)-\frac{1}{c+1}\geqslant\frac{1}{c}\left(1-\frac{1}{4c}\right)-\frac{1}{c+1}\geqslant 0,$$

the last passage is equivalent to $3c\geqslant 1$.

We turn our attention to the bound for D^2F. First we will show that

$$D^2F(w,v)\leqslant\begin{bmatrix}-\alpha(1-\alpha)w^{-1}t^{-\alpha} & 0\\ 0 & 0\end{bmatrix}. \tag{11}$$

To do this, note that the Hessian matrix of F equals (for brevity, write $t = wv^{p-1}$)

$$\begin{bmatrix} -2\alpha(1-\alpha)w^{-1}t^{-\alpha} & 2(1-\alpha)\alpha(1-p)t^{-\alpha}v^{-1} \\ 2(1-\alpha)\alpha(1-p)t^{-\alpha}v^{-1} & 2\alpha(1-p)(\alpha(1-p)-1)wv^{-2}t^{-\alpha}-p(p-1)v^{-1-p} \end{bmatrix}$$

and hence (11) is equivalent to saying that the matrix

$$\begin{bmatrix} -\alpha(1-\alpha)w^{-1}t^{-\alpha} & 2(1-\alpha)\alpha(1-p)t^{-\alpha}v^{-1} \\ 2(1-\alpha)\alpha(1-p)t^{-\alpha}v^{-1} & 2\alpha(1-p)(\alpha(1-p)-1)wv^{-2}t^{-\alpha}-p(p-1)v^{-1-p} \end{bmatrix}$$

is nonpositive definite. The entry in the left-upper corner is negative, so we will be done if we show that the determinant of the matrix is nonnegative. A little calculation reveals that this determinant is equal to

$$\alpha(1-\alpha)(p-1)t^{-2\alpha}v^{-2}\left[-2\alpha\big(2p-1-\alpha(p-1)\big)+pt^{\alpha-1}\right]$$

and it remains to note that $pt^{\alpha-1} \geqslant pt^{-1} \geqslant p/c$ and $-2\alpha(2p-1-\alpha(p-1)) \geqslant -4\alpha p = -p/c$. This yields (11). Now, we have $1-\alpha = 1-(4c)^{-1} \geqslant 3/4$ and $c^{-1} \geqslant 5^{-c} \geqslant (2/3)^{4c}$, which, in turn, give that $t^{-\alpha} \geqslant c^{-\alpha} \geqslant 2/3$. Thus $(1-\alpha)t^{-\alpha} \geqslant 1/2$ and hence (11) gives the desired claim.

The corresponding Bellman function $U_p : D \to \mathbb{R}$ is given by

$$U_p(x,y,w,v) = 2\left\{y^{p/2+1}F(w,v) - 3px^2y^{p/2}w - pc(144pc)^{p/2}|x|^{p+2}v^{1-p}\right\}.$$

For the sake of convenience, we will also distinguish the function

$$u_1(x,y,w,v) = y^{p/2+1}F(w,v) - 3px^2y^{p/2}w,$$

which can be considered as a main 'building block' of U_p.

LEMMA 4. *The function U_p satisfies the conditions 1°-4° with $\kappa = p/2$ and $K = (9\cdot 144p)^{p/2} = 36^p p^{p/2}$.*

Proof. The regularity condition 1° is obvious, the property 2° is also evident (simply use the fact that $F(w,v) \leqslant 2w$, guaranteed by previous lemma). To show the majorization 3°, recall the estimates $F(w,v) \geqslant w$, $|x| \leqslant 1$ and note that $2pc \leqslant 2p^2 \leqslant 3^p - p/2$ (here is the only place where we use the assumption $c \leqslant p$). Hence

$$U_p(x,y,w,v) \geqslant 2y^{p/2+1}w - 6px^2y^{p/2}w - (3^p - p/2)(144pc)^{p/2}|x|^{p+2}v^{1-p},$$

so it is enough to prove that

$$2y^{p/2+1}w - 6px^2y^{p/2}w + \frac{p}{2}(144pc)^{p/2}|x|^{p+2}v^{1-p} \geqslant 1_{\{|x|\vee y\geqslant 1\}}w. \tag{12}$$

If $6px^2 \leqslant y$, then the left-hand side is not smaller than $y^{p/2+1}w \geqslant 1_{\{|x|\vee y\geqslant 1\}}w$, so the desired estimate holds. On the other hand, if $y < 6px^2 < 6p$, then the left-hand side of

(12) is bigger or equal to

$$\begin{aligned} & y^{p/2+1}w - (6px^2)^{p/2+1}w + \frac{p}{2}(144pc)^{p/2}|x|^{p+2}v^{1-p} \\ & \geqslant 1_{\{|x|\vee y\geqslant 1\}}w - (6px^2)^{p/2+1}\left(w - \frac{1}{12}\cdot(24c)^{p/2}v^{1-p}\right) \geqslant 1_{\{|x|\vee y\geqslant 1\}}w, \end{aligned}$$

where in the last line we have exploited the estimates $p \geqslant 2$ and $cv^{1-p} \geqslant w$. Finally, if $|x| = 1$, then (12) is trivial: the term $\frac{p}{2}(144pc)^{p/2}|x|^{p+2}v^{1-p} \geqslant 144^{p/2}w$ has an overwhelming size.

It remains to verify 4°. Recall the definition of Q''_U given at the very end of Section 2. We first look at the building block u_1. By Lemma 3,

$$\begin{aligned} Q''_{u_1} & \leqslant -y^{p/2+1}(8cw)^{-1}r^2 + (p+2)y^{p/2}F(w,v)d^2 \\ & \quad - 6py^{p/2}wd^2 - 12pxy^{p/2}dr - 3p^2x^2y^{p/2-1}wd^2 \\ & \leqslant -y^{p/2+1}(8cw)^{-1}r^2 - 12pxy^{p/2}dr - 2py^{p/2}wd^2 - 3p^2x^2y^{p/2-1}wd^2. \end{aligned} \tag{13}$$

Now we consider two cases. If $y \geqslant 144pcx^2$, then we skip the last term in the above expression and note that the discriminant of the remaining quadratic function

$$r \mapsto -y^{p/2+1}(8cw)^{-1}r^2 - 12pxy^{p/2}dr - 2py^{p/2}wd^2$$

is equal to $py^pc^{-1}d^2(144pcx^2 - y) \leqslant 0$. Consequently, $Q''_{u_1} \leqslant 0$; since the function $(x,v) \mapsto |x|^{p+2}v^{1-p}$ is convex, we have $Q''_{U_p} \leqslant 2Q''_{u_1} \leqslant 0$.

In the remaining case $y < 144pcx^2$, we repeat the above calculation to obtain

$$Q''_{u_1} \leqslant -y^{p/2+1}(8cw)^{-1}r^2 - 12pxy^{p/2}dr.$$

Furthermore, the Hessian of the function $f(x,v) = |x|^{p+2}v^{1-p}$ equals

$$\begin{bmatrix} (p+1)(p+2)|x|^pv^{1-p} & (p+2)(1-p)|x|^{p+1}xv^{-p} \\ (p+2)(1-p)|x|^{p+1}xv^{-p} & p(p-1)|x|^{p+2}v^{-1-p} \end{bmatrix} \geqslant \begin{bmatrix} 2|x|^pv^{1-p} & 0 \\ 0 & 0 \end{bmatrix},$$

as one verifies easily. Therefore,

$$\frac{1}{2}Q''_{U_p} \leqslant -y^{p/2+1}(8cw)^{-1}r^2 - 12pxy^{p/2}dr - pc(144pc)^{p/2}\cdot 2|x|^{2p-2}v^{1-p}d^2. \tag{14}$$

The discriminant of the right-hand side of (14), considered as a quadratic function of r, is equal to

$$144p^2x^2y^{p/2+1}d^2\left(y^{p/2-1} - \left(\frac{c}{wv^{p-1}}\right)\cdot(144pcx^2)^{p/2-1}\right) \leqslant 0.$$

This gives $Q''_{U_p} \leqslant 0$ and completes the proof.

REMARK 2. From the viewpoint of the concavity condition 4°, the function u_1 is the main part of U_p, in the following sense. Namely, this ingredient itself satisfies the inequality $Q''_{u_1} \leqslant 0$ on the set $\{y \geqslant 144pcx^2\}$, and on the compliment of this set we apply the concave 'compensator' $(x,v) \mapsto -|x|^{p+2}v^{1-p}$ (with an appropriate coefficient). In the next section we will encounter a similar phenomenon.

5. A special function, $p > 2$, large characteristic

Here the analysis will be more involved. We will need the parameters $\alpha = (4c)^{-1}$, $\beta = 2/p + (p-2)/p^3$ and $q = p + (p-2)/p$. For the sake of clarity, we split this section into two parts.

5.1. Technical lemmas

We start with the following statement.

LEMMA 5. *Consider the matrix A given by*

$$\begin{bmatrix} 2-25p(p-1)+\frac{p-2}{4p^2} & 2(1-\beta) & (p-1)(25p-2\beta) \\ 2(1-\beta) & -\beta(1-\beta) & \beta(1-\beta)(1-p) \\ (p-1)(25p-2\beta) & \beta(1-\beta)(1-p) & (p-1)(\beta(\beta(p-1)+1)-25p) \end{bmatrix}.$$

Then for any $|\gamma| \leqslant (5p)^{-1}$ we have

$$A \leqslant \begin{bmatrix} 0 & \gamma(1-\beta) & 0 \\ \gamma(1-\beta) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Proof. We must show that for any γ as in the statement, the matrix

$$\begin{bmatrix} 2-25p(p-1)+\frac{p-2}{4p^2} & (2-\gamma)(1-\beta) & (p-1)(25p-2\beta) \\ (2-\gamma)(1-\beta) & -\beta(1-\beta) & \beta(1-\beta)(1-p) \\ (p-1)(25p-2\beta) & \beta(1-\beta)(1-p) & (p-1)(\beta(\beta(p-1)+1)-25p) \end{bmatrix}$$

is nonpositive-definite. We will use Sylvester's criterion and check the signs of the principal minors. Let us perform some operations on the rows and columns to make the calculations simpler. First we add, to the third column, the second column multiplied by $1-p$; then we add, to the third row, the second row multiplied by $1-p$. Finally, we divide the second column by $1-\beta$ and the third column by $p-1$. The obtained matrix is equal to

$$\begin{bmatrix} 2-25p(p-1)+\frac{p-2}{4p^2} & 2-\gamma & 25p-2 \\ (2-\gamma)(1-\beta) & -\beta & 0 \\ (p-1)(25p-2) & 0 & p(\beta-25) \end{bmatrix},$$

and the signs of the principal minors are preserved. Note that it suffices to show that the full determinant is nonpositive. Indeed, using the special form of the above matrix (zeros in appropriate places) and the inequality $0 < \beta < 25$, this will imply that the remaining minors have determinants of alternating signs. A little calculation shows that the determinant is equal to I_1+I_2, where

$$I_1=25\left[(p\beta-2)((p-1)(p\beta-2)-2)+p(1-\beta)\gamma\big(\gamma+2\beta(p-1)-4\big)\right],$$
$$I_2=2\beta(p\beta-2)-\frac{(p-2)\beta^2}{4p}+4\beta(1-\beta)\gamma-\gamma^2\beta(1-\beta)(1+\beta(p-1)).$$

Now, by the formula for β, we get $p\beta-2=(p-2)/p^2$ and $1-\beta=(p-2)(p^2-1)/p^3$. Let us first handle I_1. We have

$$\begin{aligned}I_1&=25\cdot\frac{p-2}{p^2}\left[\frac{(p-1)(p-2)}{p^2}-2+(p^2-1)\gamma\left(\gamma-\frac{2(p^2+3p-2)}{p^3}\right)\right]\\&\leqslant 25\cdot\frac{p-2}{p^2}\left[-1+(p^2-1)|\gamma|\left(|\gamma|+\frac{4}{p}\right)\right]\\&\leqslant 25\cdot\frac{p-2}{p^2}\left[-1+(p^2-1)\cdot\frac{21}{25p^2}\right]\leqslant-\frac{4(p-2)}{p^2}.\end{aligned}$$

Next, discarding the fourth summand in I_2, we see that

$$I_2\leqslant 2\beta(p\beta-2)-\frac{p-2}{p^3}+4\beta(1-\beta)|\gamma|\leqslant 8(p-2)/p^3.$$

Since $p>2$, this implies $I_1+I_2\leqslant 0$ and completes the proof.

Finally, we will need the following fact.

LEMMA 6. *The matrix*

$$\begin{bmatrix}-\beta(1-\beta)w^{-2}/3 & \beta(1-\beta)(p-1)w^{-1}v^{-1}\\ \beta(1-\beta)(p-1)w^{-1}v^{-1} & \big[\beta(p-1)(\beta(p-1)+1)-6p-6\big]v^{-2}\end{bmatrix}$$

is nonpositive-definite.

Proof. Since $\beta(p-1)\leqslant 2$, we have $\beta(p-1)(\beta(p-1)+1)\leqslant 6$ and it is enough to show that

$$\begin{bmatrix}-\beta(1-\beta)w^{-2}/3 & \beta(1-\beta)(p-1)w^{-1}v^{-1}\\ \beta(1-\beta)(p-1)w^{-1}v^{-1} & -6pv^{-2}\end{bmatrix}\leqslant 0.$$

However, this matrix has a negative entry in its left-upper corner and has positive determinant $w^{-2}v^{-2}(1-\beta)\big(2\beta p-(\beta(p-1))^2(1-\beta)\big)$: indeed, we have $2\beta p>4$, $\beta(p-1)\leqslant 2$ and $1-\beta\leqslant 1$.

5.2. The Bellman function

This time the Bellman function $U_p:D\to\mathbb{R}$ it is built from four pieces:

$$U_p=2u_1-2u_4+2\min\{0,u_2,u_3\}=2\min\big\{u_1-u_4,u_1+u_2-u_4,u_1+u_3-u_4\big\},$$

where

$$
\begin{aligned}
u_1(x,y,w,v) &= y^{p/2+1}F(w,v) - 3px^2y^{p/2}w,\\
u_2(x,y,w,v) &= \frac{120p^4}{p-2}\left\{c^\beta x^2 y^{q/2-1}w^{1-\beta}v^{\beta(1-p)} - 25\cdot(144p)^{p/2-1}c^{p/2}|x|^p v^{1-p}\right\},\\
u_3(x,y,w,v) &= \frac{120p^4}{p-2}\left\{\frac{c^\beta y^{q/2}w^{1-\beta}v^{\beta(1-p)}}{8p^2} - 25\cdot(144p)^{p/2-1}c^{p/2}|x|^p v^{1-p}\right\},\\
u_4(x,y,w,v) &= 300^p\left(\frac{p}{p-2}\right)^2 (pc)^{p/2}|x|^{p+2}v^{1-p}.
\end{aligned}
$$

We see some similarities to the function used in the previous section: actually, the function used there was precisely $2u_1 - 2u_4$, modulo the different coefficient appearing in front of u_4. Let us verify that U_p satisfies all the relevant conditions. Clearly, the property 1° is not true, but its relaxed version described in Lemma 1 holds: U_p is obviously a minimum of C^2 functions. Next, we will show the following.

LEMMA 7. *The function U_p satisfies the conditions 2° and 3° with $\kappa = p/2$ and $K = 4\cdot 300^p\left(\frac{p}{p-2}\right)^2 p^{p/2}$.*

Proof. Observe that

$$
\begin{aligned}
U_p(x,x^2,w,v) \leqslant 2u_1(x,x^2,w,v) - 2u_4(x,x^2,w,v) &\leqslant 2u_1(x,x^2,w,v)\\
&= 2|x|^{p+2}\left(F(w,v) - 3pw\right) \leqslant 0,
\end{aligned}
$$

where the last inequality is due to bound $F(w,v) \leqslant 2w$ established in Lemma 3. To prove the condition 3°, observe first that for each $x\in[-1,1]$, the function $y\mapsto U_p(x,y)$ is increasing. Indeed, u_2, u_3 have this monotonicity property, u_4 does not depend on y, and it will be shown below (see Lemma 8) that if $U_p = 2u_1 - 2u_4$, then $y \geqslant 144pcx^2$ (and hence $u_{1y}(x,y,w,v) \geqslant 0$). Therefore, it is enough to show the majorization under the additional assumption $y \in \{0,1\}$. We have

$$
\begin{aligned}
&U_p(x,0,w,v)\\
&= -\frac{6000p^4}{p-2}\cdot(144p)^{p/2-1}c^{p/2}|x|^p v^{1-p} - 2\cdot 300^p\left(\frac{p}{p-2}\right)^2 (pc)^{p/2}|x|^{p+2}v^{1-p}\\
&\geqslant 1_{\{|x|\geqslant 1\}}w - 4\cdot 300^p\left(\frac{p}{p-2}\right)^2 (pc)^{p/2}|x|^p v^{1-p}
\end{aligned}
$$

and

$$U_p(x,1,w,v) \geqslant 2F(w,v) - 6px^2w - \frac{6000p^4}{p-2}\cdot(144p)^{p/2-1}c^{p/2}|x|^p v^{1-p}$$
$$- 2\cdot 300^p\left(\frac{p}{p-2}\right)^2 (pc)^{p/2}|x|^{p+2}v^{1-p}$$
$$\geqslant w - 4\cdot 300^p\left(\frac{p}{p-2}\right)^2 (pc)^{p/2}|x|^p v^{1-p},$$

as one verifies easily (consider the cases $6px^2 \leqslant 1$ and $6px^2 \geqslant 1$ separately).

We turn our attention to the concavity condition 4°; we will exploit Lemma 1. Before we proceed, we make a comment similar to that in Remark 2 above. As previously, 'the main ingredient' of U_p is the function u_1 and the remaining pieces u_2, u_3 and u_4 can be regarded as appropriate compensators guaranteeing the concavity condition. To be more specific, let us first observe that the inequality (13) remains valid. Roughly speaking, all our considerations below aim at making Q''_{u_1} nonpositive. The main problematic term is the summand $-12pxy^{p/2}dr$, which can be positive, and the way of handling it will be different on each of the sets $\{U_p = u_1 - u_4\}$, $\{U_p = u_1 + u_2 - u_4\}$, $\{U_p = u_1 + u_3 - u_4\}$.

LEMMA 8. *We have* $Q''_{u_1-u_4} \leqslant 0$ *on the set* $\{U_p = 2u_1 - 2u_4\}$.

Proof. Let (x,y,w,v) be a point for which U_p and $2u_1 - 2u_4$ coincide. Then we necessarily have $u_2(x,y,w,v) \geqslant 0$ and the formula for this function yields

$$c^\beta x^2 y^{q/2-1} w^{1-\beta} v^{\beta(1-p)} \geqslant 25\cdot(144p)^{p/2-1}c^{p/2}|x|^p v^{1-p}.$$

This in turn, combined with the estimate $y \in [0,1]$, gives

$$y^{p/2-1} \geqslant y^{q/2-1} \geqslant 25\cdot(144pcx^2)^{p/2-1}\cdot\left(\frac{c}{wv^{p-1}}\right)^{1-\beta} \geqslant (144pcx^2)^{p/2-1}.$$

This implies $Q''_{u_1} \leqslant 0$, as we have shown in Lemma 4 above. It remains to note that u_4 is a convex function, so $Q''_{u_1-u_4} \leqslant Q''_{u_1}$.

LEMMA 9. *We have* $Q''_{u_1+u_2-u_4} \leqslant 0$ *on the set* $\{U_p = 2u_1 + 2u_2 - 2u_4\}$.

Proof. If $(x,y,w,v) \in \{U_p = 2u_1 + 2u_2 - 2u_4\}$, then $u_2(x,y,w,v) \leqslant u_3(x,y,w,v)$ and $u_2(x,y,w,v) \leqslant 0$, which is equivalent to saying that

$$y \geqslant 8p^2x^2 \quad \text{and} \quad c^\beta x^2 y^{q/2-1} w^{1-\beta} v^{\beta(1-p)} \leqslant 25\cdot(144p)^{p/2-1}c^{p/2}|x|^p v^{1-p}, \tag{15}$$

respectively. The function u_4 is convex, so (13) gives

$$Q''_{u_1+u_2-u_4} \leqslant Q''_{u_1} + Q''_{u_2} \leqslant 12p|x|y^{p/2}|dr| + Q''_{u_2}. \tag{16}$$

To show that Q''_{u_2} 'overpowers' the term $12p|x|y^{p/2}|dr|$, we compute that

$$\begin{aligned}\frac{p-2}{10p^4}Q''_{u_2} &= c^\beta x^2 y^{q/2-1} w^{1-\beta} v^{\beta(1-p)} \langle A_1(D,R,S),(D,R,S)\rangle \\ &\quad + (q-2)c^\beta x^2 y^{q/2-1} w^{1-\beta} v^{\beta(1-p)} \frac{d^2}{y} \\ &\quad - 25\cdot(144p)^{p/2-1} c^{p/2} x^p v^{1-p} \langle A_2(D,R,S),(D,R,S)\rangle,\end{aligned}$$

where $D=d/x$, $R=r/w$, $S=s/v$,

$$A_1 = \begin{bmatrix} 2 & 2(1-\beta) & 2\beta(1-p) \\ 2(1-\beta) & -\beta(1-\beta) & \beta(1-\beta)(1-p) \\ 2\beta(1-p) & \beta(1-\beta)(1-p) & \beta(p-1)(\beta(p-1)+1) \end{bmatrix}$$

and

$$A_2 = \begin{bmatrix} p(p-1) & 0 & p(1-p) \\ 0 & 0 & 0 \\ p(1-p) & 0 & p(p-1) \end{bmatrix}.$$

It is easy to see that $A_2 \geqslant 0$ and hence the second inequality in (15) gives

$$c^\beta x^2 y^{q/2-1} w^{1-\beta} v^{\beta(1-p)} A_2 \leqslant 25\cdot(144p)^{p/2-1} c^{p/2} x^p v^{1-p} A_2.$$

Moreover, the first inequality in (15) and the bound $q-2 \leqslant 2(p-2)$ imply

$$(q-2)c^\beta x^2 y^{q/2-1} w^{1-\beta} v^{\beta(1-p)} \frac{d^2}{y} \leqslant \frac{p-2}{4p^2} c^\beta x^2 y^{p-2} w^{1-\beta} v^{\beta(1-p)} D^2.$$

Putting the above facts together, we obtain

$$\frac{p-2}{120p^4}Q''_{u_2} \leqslant c^\beta x^2 y^{q/2-1} w^{1-\beta} v^{\beta(1-p)} \langle A(D,R,S),(D,R,S)\rangle,$$

where A is the matrix from Lemma 5. Using this lemma and the estimate $wv^{p-1} \leqslant c$, we get

$$\begin{aligned}\frac{p-2}{120p^4}Q''_{u_2} &\leqslant -c^\beta x^2 y^{q/2-1} w^{1-\beta} v^{\beta(1-p)} \cdot \frac{1-\beta}{5p}|DR| \\ &\leqslant -|x|y^{q/2-1}\cdot\frac{1-\beta}{5p}|dr| \leqslant -\frac{p-2}{10p^2}|x|y^{q/2-1}|dr|,\end{aligned}$$

so $Q''_{u_2} \leqslant -12p^2|x|y^{q/2-1}|dr| \leqslant -12p|x|y^{p/2}|dr|$ and the claim follows by (16).

LEMMA 10. *We have* $Q''_{u_1+u_3-u_4} \leqslant 0$ *on the set* $\{U_p = 2u_1+2u_3-2u_4\}$.

Proof. If $(x,y,w,v) \in \{U_p = 2u_1+2u_3-2u_4\}$, then in particular we have $u_3 \leqslant u_2$, so

$$y \leqslant 8p^2x^2. \tag{17}$$

Observe that by (13),

$$Q''_{u_1+u_3-u_4} \leqslant -12pxy^{p/2}dr - 3p^2x^2y^{p/2-1}wd^2 + Q''_{u_3} - Q''_{u_4}. \tag{18}$$

To handle Q''_{u_3}, note that the function $(x,v) \mapsto |x|^p v^{1-p}$ is convex, so $Q''_{u_3} \leqslant \frac{120p^4}{p-2} \cdot (8p^2)^{-1} Q''_{u_5} = \frac{15p^2}{p-2} Q''_{u_5}$, where $u_5(x,y,w,v) = c^\beta y^{q/2} w^{1-\beta} v^{\beta(1-p)}$. Using the estimate $wv^{p-1} \leqslant c$, we obtain

$$2u_{5y}(x,y,w,v) = qc^\beta y^{q/2-1} w^{1-\beta} v^{\beta(1-p)}$$

(u_{5y} is the partial derivative of u_5 with respect to y). Furthermore, we have

$$\begin{bmatrix} u_{5ww}(x,y,w,v) & u_{5wv}(x,y,w,v) \\ u_{5wv}(x,y,w,v) & u_{5vv}(x,y,w,v) \end{bmatrix}$$
$$= c^\beta y^{q/2} w^{1-\beta} v^{\beta(1-p)} \begin{bmatrix} -\beta(1-\beta)w^{-2} & \beta(1-\beta)(p-1)w^{-1}v^{-1} \\ \beta(1-\beta)(p-1)w^{-1}v^{-1} & \beta(p-1)(\beta(p-1)+1)v^{-2} \end{bmatrix}$$

and by Lemma 6, the latter matrix is less than

$$\begin{bmatrix} -2\beta(1-\beta)w^{-2}/3 & 0 \\ 0 & (6p+6)v^{-2} \end{bmatrix}.$$

Consequently, we have shown that Q''_{u_5} does not exceed

$$c^\beta y^{q/2} w^{1-\beta} v^{\beta(1-p)} \left[\frac{qd^2}{y} + \frac{(6p+6)s^2}{v^2} \right] - \frac{2\beta(1-\beta)}{3} c^\beta y^{q/2} w^{-1-\beta} v^{\beta(1-p)} r^2$$

and using (17), we get the final bound for Q''_{u_3}:

$$\begin{aligned} Q''_{u_3} \leqslant \frac{15p^2}{p-2} \Big\{ & c^\beta (8p^2x^2)^{q/2} w^{1-\beta} v^{\beta(1-p)} \left[\frac{qd^2}{8p^2x^2} + \frac{(6p+6)s^2}{v^2} \right] \\ & - \frac{2\beta(1-\beta)}{3} c^\beta y^{q/2} w^{-1-\beta} v^{\beta(1-p)} r^2 \Big\}. \end{aligned} \tag{19}$$

Now, the first term on the right will be overpowered by Q''_{u_4}. To see this, note that the Hessian matrix of the function $u_6(x,v) = |x|^q v^{1-p}$ is equal to

$$\begin{bmatrix} q(q-1)|x|^{q-2}v^{1-p} & q(1-p)|x|^{q-2}xv^{-p} \\ q(1-p)|x|^{q-2}xv^{-p} & p(p-1)|x|^q v^{-1-p} \end{bmatrix},$$

which is easily shown to be bigger than each of the matrices

$$\begin{bmatrix} \frac{q(p-2)}{p^2}|x|^{q-2}v^{1-p} & 0 \\ 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 0 \\ 0 & \frac{p-2}{2p}|x|^q v^{-1-p} \end{bmatrix}.$$

Consequently, we can handle the first term in (19) as follows:

$$\begin{aligned}
&\frac{15p^2}{p-2}\cdot c^{\beta}(8p^2x^2)^{q/2}w^{1-\beta}v^{\beta(1-p)}\left[\frac{qd^2}{8p^2x^2}+\frac{(6p+6)s^2}{v^2}\right]\\
&\leqslant \frac{15p^2}{p-2}\cdot(8p^2x^2)^{q/2}cv^{1-p}\left[\frac{qd^2}{8p^2x^2}+\frac{(6p+6)s^2}{v^2}\right]\\
&=\frac{15p^2}{(p-2)^2}(8p^2)^{q/2}c\left[\frac{1}{8}\cdot\frac{q(p-2)}{p^2}|x|^{q-2}v^{1-p}d^2+2p(6p+6)\cdot\frac{p-2}{2p}|x|^{q}v^{-1-p}s^2\right]\\
&\leqslant -Q''_{u_4},
\end{aligned}$$

since, due to the assumption $c\geqslant p$,

$$15p^2(8p^2)^{q/2}c\cdot\left(\frac{1}{8}+2p(6p+6)\right)\leqslant 15p^2(8p^2)^{(p+1)/2}c\cdot 13p^2\leqslant 300^p p^2(pc)^{p/2}.$$

Combining this with (18) and (19), we get that $Q''_{u_1+u_3-u_4}$ does not exceed

$$\begin{aligned}
&-12pxy^{p/2}dr-3p^2x^2y^{p/2-1}wd^2-\frac{15p^2}{p-2}\cdot\frac{2\beta(1-\beta)}{3}c^{\beta}y^{q/2}w^{-1-\beta}v^{\beta(1-p)}r^2\\
&\leqslant -12pxy^{p/2}dr-3p^2x^2y^{p/2-1}wd^2-\frac{20(p^2-1)}{p^2}y^{p/2+1}w^{-1}r^2,
\end{aligned}$$

where in the last passage we have used the equality $1-\beta=(p-2)(p^2-1)/p^3$ and the estimates $wv^{p-1}\leqslant c$, $y^{q/2}\geqslant y^{p/2+1}$ and $\beta\geqslant 2/p$. It remains to compute that the discriminant of the above expression, considered as a quadratic function of r, is equal to $x^2y^pd^2\left(240-96p^2\right)\leqslant 0$.

6. Sharpness of the exponent in (5)

Now we will address the optimality of the exponent $\max\{1/p,1/2\}$ in (5). We consider two natural cases.

6.1. The case $1<p<2$

We will show that in this range, the best exponent κ_p in the estimate

$$||\langle X\rangle_\infty^{1/2}||_{L^{p,\infty}(W)}\leqslant K_p[W]_{A_p}^{\kappa_p}||X||_{L^p(W)},$$

cannot be smaller than $1/p$. We construct an appropriate example. For a given $c>1$, consider the region $D_{p,c}$ and let $PQ\subset D_{p,c}$ be the line segment with endpoints P, Q lying on the lower boundary of $D_{p,c}$, tangent to the upper boundary of this set at $(1,c^{1/(p-1)})$. This line segment is contained in a line of the form $w=av+b$ for some $a,b\in\mathbb{R}$. Consider the process $(B,aB+b)$, where B be a Brownian motion starting from 1. This two-dimensional process starts from $(1,c^{1/(p-1)})$ and, for sufficiently small times, takes values in the segment PQ. Let us stop this process when it reaches P or Q and denote the obtained pair by (W,X). We have $(W_\infty,X_\infty)\in\{P,Q\}$, so $W_\infty X_\infty^{p-1}=1$. Note that W is an A_p weight with $[W]_{A_p}=c$: indeed, for any stopping time τ we have

$$\mathbb{E}(W|\mathscr{F}_\tau)\mathbb{E}(W^{1/(1-p)}|\mathscr{F}_\tau)^{p-1}=\mathbb{E}(W_\infty|\mathscr{F}_\tau)\mathbb{E}(X_\infty|\mathscr{F}_\tau)^{p-1}=W_\tau X_\tau^{p-1}\in[1,c],$$

since the range of (X,W) is contained in PQ (and hence, in particular, in the region $D_{p,c}$). This implies $[W]_{A_p}\leqslant c$, and to see that we actually have equality here, put $\tau=0$: then we have

$$\mathbb{E}(W|\mathscr{F}_\tau)\mathbb{E}(W^{1/(1-p)}|\mathscr{F}_\tau)^{p-1}=W_0X_0^{p-1}=c.$$

Now, take $\lambda=\mathbb{E}X=c^{1/(p-1)}$. We have $\langle X\rangle\geqslant X_0^2=\lambda^2$ almost surely, so

$$\frac{\|\langle X\rangle^{1/2}\|_{L^{p,\infty}(W)}}{\|X\|_{L^p(W)}}\geqslant\frac{\lambda(\mathbb{E}W)^{1/p}}{(\mathbb{E}W^{1/(1-p)})^{1/p}}=\frac{\lambda}{\lambda^{1/p}}=c^{1/p}=[W]_{A_p}^{1/p}.$$

Since the parameter $c>1$ was arbitrary, the optimality of the exponent $1/p$ is established.

6.2. The case $p>2$

Now we will prove that for these p, the smallest κ_p permitted in the estimate

$$||\langle X\rangle_\infty^{1/2}||_{L^{p,\infty}(W)}\leqslant K_p[W]_{A_1}^{\kappa_p}||X||_{L^p(W)} \tag{20}$$

must be at least $1/2$ (note that the A_1 characteristic of W is used). Since $[W]_{A_1}\geqslant[W]_{A_p}$, this will give the desired sharpness. Note that if (20) holds, then for any mean-one A_1 weight W and any martingale X bounded by 2 we have

$$\begin{aligned}
\mathbb{E}\langle X\rangle W&=2\int_0^\infty\lambda W(\langle X\rangle^{1/2}\geqslant\lambda)\mathrm{d}\lambda\\
&=2\int_0^{[W]_{A_1}^{\kappa_p}}\lambda W(\langle X\rangle^{1/2}\geqslant\lambda)\mathrm{d}\lambda+2\int_{[W]_{A_1}^{\kappa_p}}^\infty\lambda W(\langle X\rangle^{1/2}\geqslant\lambda)\mathrm{d}\lambda\\
&\leqslant[W]_{A_1}^{2\kappa_p}\mathbb{E}W+2\int_{[W]_{A_1}^{\kappa_p}}^\infty\lambda^{1-p}\cdot K_p^p[W]_{A_1}^{p\kappa_p}\|X\|_{L^p(W)}^p\mathrm{d}\lambda\\
&=[W]_{A_1}^{2\kappa_p}\left\{\mathbb{E}W+\frac{2^{p+1}K_p^p}{p-2}\mathbb{E}W\right\}=[W]_{A_1}^{2\kappa_p}\left\{1+\frac{2^{p+1}K_p^p}{p-2}\right\}.
\end{aligned} \tag{21}$$

Now we will construct an example related to the above inequality. Pick a large positive integer N. Consider a one-dimensional Brownian motion started at $1/2$ and introduce the nondecreasing family $\tau_0,\tau_1,\tau_2,\ldots,\tau_{2N}$ of stopping times given recursively by

$$\begin{aligned}
\tau_{2n+1}&=\inf\{t\geqslant\tau_{2n}:B_t\geqslant 3/2\text{ or }B_t\leqslant-1/2\},\\
\tau_{2n+2}&=\inf\{t\geqslant\tau_{2n+1}:B_t\leqslant-3/2\text{ or }B_t\geqslant 1/2\}
\end{aligned}$$

for $n = 0, 1, 2, \dots, N-1$. The stopped process $X = B^{\tau_{2N}} = (B_{\tau_{2N}\wedge t})_{t\geqslant 0}$ enjoys the following behavior. It starts from $1/2$; then, on the interval $[\tau_0, \tau_1]$, it evolves until it reaches $3/2$ or $-1/2$. If the first possibility occurs, the evolution is over; in the second case, the process continues its move, on the time interval $[\tau_1, \tau_2]$, until it gets to $-3/2$ or to $1/2$. Again, if the first scenario occurs, then the process terminates; otherwise, the pattern is repeated. Note that $X_{\tau_{2N}} = 1/2$ with probability 4^{-N} (indeed: $X_{\tau_{2N}} = 1/2$ means that $X_{\tau_n} = (-1/2)^n$ for all n). Furthermore, X is bounded in absolute value by 2 (actually, it is even bounded by $3/2$).

Next, fix $c > 1$ and consider the weight $W = (W_t)_{t\geqslant 0}$ given as follows:

$$W_t = \begin{cases} (2-c^{-1})^{2n}\left[1-(1-c^{-1})(B_t - 1/2)\right] & \text{for } t \in [\tau_{2n}, \tau_{2n+1}], \\ (2-c^{-1})^{2n+1}\left[1+(1-c^{-1})(B_t + 1/2)\right] & \text{for } t \in [\tau_{2n+1}, \tau_{2n+2}] \end{cases}$$

for $n = 0, 1, 2, \dots, N-1$, and $W_t = W_{\tau_{2N}}$ for $t \geqslant \tau_{2N}$. It is easy to see that W is a continuous-path martingale. Furthermore, we know from the above construction that if $t \in (\tau_{2n}, \tau_{2n+1}]$, then $B_t \in [-1/2, 3/2]$; for $t \in (\tau_{2n+1}, \tau_{2n+2}]$ we have $B_t \in [-3/2, 1/2]$. This implies that if $t \in (\tau_n, \tau_{n+1}]$, then we have the inclusion $W_t \in [(2-c^{-1})^n c^{-1}, (2-c^{-1})^{n+1}]$. In particular, this implies that for $t \in (\tau_n, \tau_{n+1}]$ we have

$$W_t \geqslant (2-c^{-1})^n c^{-1} = (2c-1)^{-1} \cdot (2-c^{-1})^{n+1} \geqslant (2c-1)^{-1} \sup_{s\leqslant t} W_s,$$

so W is an A_1 weight with $[W]_{A_1} \leqslant 2c-1$. Observe that on the set $X_{\tau_{2N}} = 1/2$ we have $W_{\tau_{2N}} = (2-c^{-1})^{2N}$.

Let us look at both sides of (21). The above considerations imply that the right-hand side of (21) is not bigger than

$$(2c-1)^{2\kappa_p}\left\{1 + \frac{2^{p+1}K_p^p}{p-2}\right\}.$$

On the other hand, we may write

$$\begin{aligned} \mathbb{E}\langle X\rangle W &\geqslant \mathbb{E}\langle X\rangle W 1_{\{X_{\tau_{2N}}=1/2\}} \\ &= \mathbb{E}\langle X\rangle (2-c^{-1})^{2N} 1_{\{X_{\tau_{2N}}=1/2\}} \\ &= \sum_{n=0}^{2N-1} \mathbb{E}\left(\langle X\rangle_{\tau_{n+1}} - \langle X\rangle_{\tau_n}\right)(2-c^{-1})^{2N} 1_{\{X_{\tau_{2N}}=1/2\}}. \end{aligned} \tag{22}$$

Recall that $\{X_{\tau_{2N}} = 1/2\}$ is equivalent to saying that $B_{\tau_{n+1}} - B_{\tau_n} = (-1)^{n+1}$ for all n. It follows from the independence and the symmetry of increments of Brownian motion that

$$\mathbb{E}\left(\langle X\rangle_{\tau_{n+1}} - \langle X\rangle_{\tau_n} \middle| X_{\tau_{2N}} = 1/2\right) = \mathbb{E}\left(\langle X\rangle_{\tau_1} - \langle X\rangle_{\tau_0}\right) = \mathbb{E}X_{\tau_1}^2 = 1,$$

which implies

$$\begin{aligned} \mathbb{E}\left(\langle X\rangle_{\tau_{n+1}} - \langle X\rangle_{\tau_n}\right)(2-c^{-1})^{2N} 1_{\{X_{\tau_{2N}}=1/2\}} &= (2-c^{-1})^{2N} \cdot \mathbb{P}(X_{\tau_{2N}} = 1/2) \\ &= \left(1-(2c)^{-1}\right)^{2N}. \end{aligned}$$

Coming back to (22), we see that

$$\mathbb{E}\langle X\rangle W \geqslant 2N\left(1-(2c)^{-1}\right)^{2N}.$$

Plugging all the above observations into (21), we obtain

$$2N\left(1-(2c)^{-1}\right)^{2N} \leqslant (2c-1)^{2\kappa_p}\left\{1+\frac{2^{p+1}K_p^p}{p-2}\right\}.$$

Now put $N=\lfloor c\rfloor$ and let $c\to\infty$. Then the left-hand side increases linearly with c and hence $2\kappa_p \geqslant 1$. This shows that $\kappa_p \geqslant 1/2$ and completes the proof.

Acknowledgement. The author would like to thank an anonymous referee for the careful reading of the first version of the paper and helpful suggestions. The research was supported by Narodowe Centrum Nauki (Poland), grant no. DEC-2014/14/E/ST1/00532.

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(Received January 21, 2019)

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