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REGULAR COSINE FAMILIES OF LINEAR SET–VALUED FUNCTIONS

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Abstract. This paper is concerned with the properties of regular cosine families of continuous linear set-valued functions defined on convex cones of normed spaces. We consider conditions under which a regular cosine family of continuous linear set-valued functions is continuous and then generalize some recent results on commutativity and Hukuhara's derivative of regular cosine families of continuous linear set-valued functions.

1. Introduction

Let X be a vector space. Throughout this paper all vector spaces are supposed to be real. We denote by $n(X)$ the family of all nonempty subsets of X with addition

$$A+B := \{a+b : a \in A, b \in B\}$$

and scalar multiplication

$$\lambda A := \{\lambda a : a \in A\}$$

for every $A,B \in n(X)$ and $\lambda \in \mathbb{R}$.

LEMMA 1. [9] *For subsets $A,B \subseteq X$ and real numbers s,t we have:*

$$s(A+B) = sA+sB, \quad (s+t)A \subseteq sA+tA.$$

Also, if A is convex and $s,t \geqslant 0$ (or $s,t \leqslant 0$), then $(s+t)A = sA+tA$.

A set-valued function $F : [a,b] \to n(X)$ is said to be

- concave if $F(\lambda t+(1-\lambda)s) \subseteq \lambda F(t)+(1-\lambda)F(s)$ for every $s,t \in [a,b]$ and $\lambda \in (0,1)$;
- increasing if $F(s) \subseteq F(t)$ for every $s,t \in [a,b]$ with $s<t$;
- decreasing if $F(t) \subseteq F(s)$ for every $s,t \in [a,b]$ with $s<t$.

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A set-valued function $F:\mathbb{R}\to n(X)$ is said to be even if $F(t)=F(-t)$ for every $t\in\mathbb{R}$.

A subset K of X is said to be a convex cone if $x+y\in K$ and $tx\in K$ for all $x,y\in K$ and $t>0$. For two linear spaces X and Y and a convex cone $K\subseteq X$, the set-valued function $F:K\to n(Y)$ is said to be

- additive if $F(x+y)=F(x)+F(y)$
- linear if $F(x+y)=F(x)+F(y)$ and $F(tx)=tF(x)$

for all $x,y\in K$ and $t>0$.

Assume that X is a normed space, $K\subseteq X$ is a convex cone and $cc(K)$ denotes the family of all nonempty compact convex subsets of K. For $A,B\in cc(K)$, the difference $A-B$ is a set $C\in cc(K)$ satisfying $A=B+C$. Uniqueness of this difference is a conclusion of Lemma 2 in [14].

Let $d(a,B):=\inf_{b\in B}\|a-b\|$ for $a\in A$. Then,

$$\mathfrak{h}(A,B):=\max\{\sup_{a\in A}d(a,B),\sup_{b\in B}d(b,A)\},\qquad (A,B\in cc(X))$$

defines a metric on $cc(X)$, which is called Hausdorff metric.

We understand the continuity of a set-valued function with respect to the Hausdorff metric $\mathfrak{h}$ derived from the norm in X.

DEFINITION 1. [5] Assume that X is a normed space, $K\subseteq X$ is a convex cone and $F:[0,+\infty)\to cc(K)$ is a set-valued function. If all the differences $F(s)-F(t)$ exist for $t,s\in[0,+\infty)$ with $s>t$, then the Hukuhara derivative of F at t is defined by the formula

$$DF(t)=\lim_{s\to t^+}\frac{F(s)-F(t)}{s-t}=\lim_{s\to t^-}\frac{F(t)-F(s)}{t-s}$$

whenever both limits exist with respect to the Hausdorff metric $\mathfrak{h}$ in $cc(K)$ derived from the norm in X. Also,

$$DF(0)=\lim_{s\to 0^+}\frac{F(s)-F(0)}{s}.$$

Consider X,Y and Z are nonempty sets. The superposition $G\circ F$ of set-valued functions $F:X\to n(Y)$ and $G:Y\to n(Z)$ is defined by $(G\circ F)(x)=\cup_{y\in F(x)}G(y)$ for every $x\in X$.

DEFINITION 2. Let X be a normed space and $K\subseteq X$ be a convex cone.

- A family $\{F_t:K\to n(K)\}_{t\geqslant 0}$ is called a cosine family if

  $$F_{t+s}(x)+F_{t-s}(x)=2F_t(F_s(x)),\quad F_0(x)=\{x\}$$

  for every $x\in K$ and $0\leqslant s\leqslant t$. A cosine family $\{F_t:t\geqslant 0\}$ is said to be regular if $\lim_{t\to 0^+}\mathfrak{h}(F_t(x),\{x\})=0$ for every $x\in K$.

- A family $\{F_t : K \to n(K)\}_{t\in\mathbb{R}}$ is called a cosine family if

$$F_{t+s}(x) + F_{t-s}(x) = 2F_t(F_s(x)), \quad F_0(x) = \{x\}$$

for every $x \in K$ and $t,s \in \mathbb{R}$. A cosine family $\{F_t : t \in \mathbb{R}\}$ is said to be regular if $\lim_{t\to 0} \mathfrak{h}(F_t(x),\{x\}) = 0$ for every $x \in K$.

If X is a normed space, K is a convex cone in X and $\{F_t : t \in \mathbb{R}\}$ is a cosine family of set-valued functions $F_t : K \to cc(K)$, then

$$F_s(x) + F_{-s}(x) = 2F_0F_s(x) = 2F_s(x).$$

By Rådström cancelation Lemma, $F_s(x) = F_{-s}(x)$ for all $x \in K$ and $s \in \mathbb{R}$. That is, the set-valued functions $t \mapsto F_t(x)$ are even.

The following Lemma is an immediate consequence of Lemma 1 in [15].

LEMMA 2. *Let X and Y be two topological vector spaces, K be a convex cone in X, $F : K \to cc(Y)$ is an additive set-valued function and $A,B \in cc(K)$. If the difference $A-B$ exists, then $F(A)-F(B)$ exists and $F(A)-F(B) = F(A-B)$.*

By Lemma 4 in [17] (see also Lemma 3 in [19]), we have the following lemma.

LEMMA 3. *Let X and Y be two normed spaces and K be a convex cone in X. If $\{F_i : K \to n(Y)\}_{i\in I}$ is a family of continuous linear set-valued functions, K is of the second category in K and for every $x \in K$, $\cup_{i\in I} F_i(x)$ is bounded in Y, then there exists a positive number M with*

$$||F_i(x)|| := \sup\{||y|| : y \in F_i(x)\} \leqslant M||x||$$

for every $i \in I$ and $x \in K$.

And, by Lemma 2 in [17], we have the following result.

LEMMA 4. *If X, Y and K have the same meaning as in Lemma 3, then the functional*

$$F \mapsto ||F|| := \sup\{\frac{||F(x)||}{||x||} : x \in K, x \neq 0\}$$

is finite for every continuous linear set-valued function $F : K \to cc(Y)$.

LEMMA 5. [17] *Let X and Y be two normed spaces, $\mathfrak{h}$ be the Hausdorff distance derived from the norm in Y and K be a convex cone in X with nonempty interior. Then, there is a positive number M_0 such that for every continuous linear set-valued function $F : K \to cc(Y)$ the inequality $\mathfrak{h}(F(x),F(y)) \leqslant M_0||F||||x-y||$ holds for all $x,y \in K$.*

LEMMA 6. [16] *Consider two metric spaces (X,d_1) and (Y,d_2) and let $\mathfrak{h}_1$ and $\mathfrak{h}_2$ be the corresponding Hausdorff metrics. If $F : X \to n(Y)$ is a set-valued function and M is a positive number satisfying $\mathfrak{h}_2(F(x),F(y)) \leqslant Md_1(x,y)$ for all $x,y \in X$, then $\mathfrak{h}_2(F(A),F(B)) \leqslant M\mathfrak{h}_1(A,B)$ for every $A,B \in n(X)$.*

LEMMA 7. [16] *Let D and Y be a nonempty set and a normed space, respectively. If $F_0, F_n : D \to c(Y)$ are set-valued functions such that the sequence (F_n) uniformly converges to F_0 on D, then*

$$\lim_{n\to\infty} F_n(D) = F_0(D).$$

Since normed spaces and the cones are not supposed to be complete, so our main results generalize some recent results on cosine families of linear set-valued functions.

2. Main results

For a normed space X, we use the notations X_0, $int_X K$ and $cl_X K$ for the completion X, the interior of K in X and the closure of K in X, respectively. If the symbol $\sim$ denotes Rådström's equivalence relation in $cc(X_0)$ with $(A,B) \sim (D,E) \Leftrightarrow A+E = B+D$ for all $A,B,D,E \in cc(X_0)$ and $[A,B]$ is the equivalence class of (A,B). Then, the vector space Δ of all equivalence classes with operations

$$[A,B]+[D,E] = [A+D, B+E],$$

$$\lambda[A,B] = [\lambda A, \lambda B],\ (\lambda \geqslant 0),$$

$$\lambda[A,B] = [-\lambda B, -\lambda A],\ (\lambda < 0)$$

is a normed space with the norm $\|[A,B]\| := \mathfrak{h}(A,B)$ (see [14]). By Theorems 3.85 and 3.88 in [3], $(cc(X_0), \mathfrak{h})$ is a complete metric space.

2.1. Continuity properties of regular cosine families

From now on, unless explicitly stated otherwise, X and Y are normed spaces and K is a convex cone in X such that $int_X K \neq \emptyset$. Note that $(cc(cl_{X_0}K), \mathfrak{h})$ is a complete metric space. If $F : K \to cc(K)$ is a continuous linear set-valued function, then by Theorem 1 in [2], F has a unique continuous linear extension $\tilde{F} : cl_{X_0}K \to cc(cl_{X_0}K)$ such that $\|\tilde{F}\| = \|F\|$. Identifying $\tilde{F}$ with the unique continuous linear extension of F, we have the following results.

LEMMA 8. *If $\{F_t : t \in \mathbb{R}\}$ is a regular cosine family of continuous linear set-valued functions $F_t : K \to cc(K)$, then the function $t \mapsto \|F_t\|$ is bounded on some neighborhood of zero if and only if the set-valued function $t \mapsto \tilde{F}_t(x)$ is continuous for every $x \in cl_{X_0}K$.*

Proof. Let the function $t \mapsto \|F_t\|$ be bounded on some neighborhood of zero and $x \in K$ be arbitrary. Put $G_t(x) := F_t(x)$ and $H_t(x) := F_{-t}(x)$ for every $t \geqslant 0$. It is easy to see that $\{G_t : t \geqslant 0\}$ and $\{H_t : t \geqslant 0\}$ are regular cosine families. By Theorem 2 in [2], the set-valued function $t \mapsto \tilde{F}_t(x)$ is continuous on $[0,\infty)$ and $(-\infty, 0]$ for every $x \in cl_{X_0}K$. Hence, the set-valued function $t \mapsto \tilde{F}_t(x)$ is continuous for all $x \in cl_{X_0}K$.

Conversely, if the set-valued function $t \mapsto \tilde{F}_t(x)$ is continuous for every $x \in cl_{X_0}K$. Then, putting $E = [-1,1]$, $\cup_{t\in E}\tilde{F}_t(x)$ is compact for every $x \in cl_{X_0}K$. By Lemmas 3

and 4, there is a positive constant M such that $\|\tilde{F}_t\| = \|F_t\| \leqslant M$ for every $t \in E$. Thus, $t \mapsto \|F_t\|$ is bounded on some neighborhood of zero.
It is natural to ask whether the continuity of $t \mapsto F_t(x)$ for every $x \in K$, can be equivalent to the boundedness of $t \mapsto \|F_t\|$ on some neighborhood of zero. In the following, we will list the results of this issue.

THEOREM 1. *If $\{F_t : t \in \mathbb{R}\}$ is a regular cosine family of continuous linear set-valued functions $F_t : K \to cc(K)$, then the following statements are equivalent.*

1. *$t \mapsto F_t(x)$ is continuous for every $x \in K$.*
2. *The function $t \mapsto \|F_t\|$ is bounded on some neighborhood of zero.*
3. *For every $x \in cl_{X_0}K$ the set-valued function $t \mapsto \tilde{F}_t(x)$ is continuous.*

Proof. (1) $\Rightarrow$ (2) Assume by way of contradiction that there exists a sequence (t_n) in $[0,\infty)$ satisfying $\lim_{n\to\infty} t_n = 0$ and $\|F_{t_n}\| = \|\tilde{F}_{t_n}\| \geqslant n$ for all $n \in \mathbb{N}$. By Lemma 3, there exists $x_0 \in cl_{X_0}K$ such that $(\|\tilde{F}_{t_n}(x_0)\|)$ is unbounded. Since $x_0 \in cl_{X_0}K$, so there is (x_n) in K such that $\lim_{n\to\infty} x_n = x_0$. Define real functions $f_n : \mathbb{R} \to \mathbb{R}$ by $f_n(t) = \|[F_t(x_n), \{0\}]\|$ for all $t \in \mathbb{R}$ and $n \in \mathbb{N}$. Since $t \mapsto F_t(x)$ is continuous for every $x \in K$, so $\{f_n : n \in \mathbb{N}\}$ is a family of continuous real functions. On the other hand, $x \mapsto F_t(x)$ is a continuous linear set-valued function for every $t \in \mathbb{R}$, thus $x \mapsto F_t(x)$ is uniformly continuous for every $t \in \mathbb{R}$ and consequently $(F_t(x_n))$ is a Cauchy sequence in $cc(K)$ and therefore bounded for every $t \in \mathbb{R}$. Hence, $(f_n(t))$ is bounded for every $t \in \mathbb{R}$. Since $\mathbb{R}$ is a complete metric space, so by uniform boundedness principle (see [8], pp. 299) there is an open neighborhood U_0 of $\mathbb{R}$ on which the functions f_n are uniformly bounded, that is, there is $L_0 > 0$ such that $|f_n(t)| < L_0$ for all $t \in U_0$ and $n \in \mathbb{N}$. Thus, there are $L_0 > 0$ and $0 \leqslant \delta < \eta$ such that $\|F_t(x_n)\| < L_0$ for every $t \in [\delta, \eta] \subseteq U_0$ and $n \in \mathbb{N}$. As $n \to \infty$, by Theorem 1 in [2] we have:

$$\|\tilde{F}_t(x_0)\| \leqslant L_0$$

for every $t \in [\delta, \eta]$. Now, consider real functions $f_n : [2\delta, 2\eta] \to \mathbb{R}$ by $f_n(t) = \|[F_t(x_n), \{0\}]\|$ for all $t \in [2\delta, 2\eta]$ and $n \in \mathbb{N}$. So as above, there is an open neighborhood V_0 of $[2\delta, 2\eta]$ on which the functions f_n are uniformly bounded, that is, there is $L_0' > 0$ such that $\|\tilde{F}_t(x_0)\| < L_0'$ for every $t \in V_0$ and $n \in \mathbb{N}$.

Put $L = \max\{L_0, L_0', 1\}$. For some $2t_0 \in V_0$, there exists an $n \in \mathbb{N}$ such that $[2t_0, 2t_0 + \frac{t_0}{n}] \subseteq V_0$ and $[t_0, t_0 + \frac{t_0}{2n}] \subseteq [\delta, \eta]$. We claim that $\|\tilde{F}_t(x_0)\|$ is bounded on $[0, \frac{t_0}{2n}]$. Without loss of generality we can assume that $L \geqslant \|F_{t_0}\|$. Since $[t_0, t_0 + \frac{t_0}{2n}] \subseteq [\delta, \eta]$, so for all $t \in [t_0, t_0 + \frac{t_0}{2n}]$ we have:

$$\begin{aligned} \|\tilde{F}_{t-t_0}(x_0)\| &\leqslant \|\tilde{F}_{t+t_0}(x_0)\| + 2\|F_{t_0}\|\|\tilde{F}_t(x_0)\| \\ &\leqslant 3L^2. \end{aligned}$$

Hence, $t \mapsto \|\tilde{F}_t(x_0)\|$ is bounded on some neighborhood $[0, \frac{t_0}{2n}]$ which is a contradiction. Thus, $t \mapsto \|F_t\|$ is bounded on some neighborhood of zero.

(2) $\Rightarrow$ (3) The proof is an immediate consequence of Lemma 8.

(3) $\Rightarrow$ (1) The proof is clear.

By the proof of Theorem 1, the corresponding result holds for a regular cosine family $\{F_t : t \geqslant 0\}$. Hence, the answer to the considered question in Remark 1 in [2] is yes. That is, the boundedness of the function $t \longrightarrow \|\varphi_t\|$ on some neighborhood of zero in Theorem 2 is essential.

Let $\{F_t : t \in \mathbb{R}\}$ be a regular cosine family of continuous linear set-valued functions $F_t : K \longrightarrow cc(K)$. Since for all $x \in K$ the set-valued functions $t \mapsto F_t(x)$ are even, so

$$2F_t(F_s(x)) = F_{t+s}(x) + F_{t-s}(x) = F_{s+t}(x) + F_{s-t}(x) = 2F_s(F_t(x))$$

for $x \in K$ and $s,t \in \mathbb{R}$. That is, $F_t(F_s(x)) = F_s(F_t(x))$. For $u,v \in \mathbb{R}$ putting $t = \frac{v+u}{2}$ and $s = \frac{v-u}{2}$ in $F_{t+s}(x) + F_{t-s}(x) = 2F_t(F_s(x))$, we have

$$F_v(x) + F_u(x) = 2F_{\frac{u+v}{2}}(F_{\frac{v-u}{2}}(x)).$$

If $x \in F_t(x)$ for all $x \in K$ and $t \in \mathbb{R}$, then

$$F_{\frac{u+v}{2}}(x) \subseteq \frac{F_u(x) + F_v(x)}{2}.$$

By Theorem 4.2 in [9], $t \mapsto F_t(x)$ is continuous and by Theorem 4.1 in [9], this set-valued function is concave. For $0 \leqslant u \leqslant v$, there exists $\lambda \in [0,1]$ such that $u = (1-\lambda)0 + \lambda v$. Thus,

$$\begin{aligned} F_u(x) &\subseteq (1-\lambda)F_0(x) + \lambda F_v(x) \\ &= (1-\lambda)x + \lambda F_v(x) \\ &\subseteq (1-\lambda)F_v(x) + \lambda F_v(x) = F_v(x). \end{aligned}$$

And, for $v \leqslant u \leqslant 0$ we have $F_u(x) \subseteq F_v(x)$. Hence $t \mapsto F_t(x)$ is increasing in $[0,\infty)$ and decreasing in $(-\infty,0]$. Conversely, if $t \mapsto F_t(x)$ is increasing in $[0,\infty)$ or decreasing in $(-\infty,0]$, then $x \in F_t(x)$ for all $x \in K$ and $t \in \mathbb{R}$.

The immediate consequence of the preceding theorem is:

COROLLARY 1. *Let $\{F_t : t \in \mathbb{R}\}$ be a regular cosine family of continuous linear set-valued functions $F_t : K \longrightarrow cc(K)$ such that $\{F_t(x) : t \in \mathbb{R}\}$ is increasing in $[0,\infty)$ for every $x \in K$. Then,*

1. *$t \mapsto F_t(x)$ is continuous for every $x \in K$.*
2. *the function $t \mapsto \|F_t\|$ is bounded on some neighborhood of zero.*
3. *for every $x \in cl_{X_0}K$ the set-valued function $t \mapsto \tilde{F}_t(x)$ is continuous.*

2.2. Commutativity and Hukuhara's derivative of regular cosine families

Recall that if $\{F_t : t \in \mathbb{R}\}$ is a regular cosine family of continuous linear set-valued functions $F_t : K \to cc(K)$ such that $x \in F_t(x)$ for every $x \in K$ and $t \in \mathbb{R}$, then for every $x \in K$ the set-valued function $t \mapsto F_t(x)$ is concave, continuous, even, decreasing in $(-\infty, 0]$ and increasing in $[0, +\infty)$. Also, $F_s \circ F_t = F_t \circ F_s$ for every $s, t \in \mathbb{R}$ (see [18]). For some more properties of sine and cosine equations, see also [4].

THEOREM 2. *If $\{F_t : K \to cc(K)\}_{t \geqslant 0}$ is a regular cosine family of continuous linear set-valued functions such that $t \mapsto \|F_t\|$ is bounded on some neighborhood of zero, then*

$$\lim_{t \to s} \mathfrak{h}(F_t(D), F_s(D)) = 0$$

for every nonempty compact subset D of K.

Proof. Let (t_n) be a sequence in $[0, \infty)$ such that $t_n \to s$. Putting $\phi_n(x) := \tilde{F}_{t_n}(x)$ and $\phi(x) := \tilde{F}_s(x)$ we have $\lim_{n\to\infty} \phi_n(x) = \phi(x)$ for every $x \in cl_{X_0} K$. By Lemma 7 in [16], (ϕ_n) is uniformly convergent to ϕ on each nonempty compact subset D and by Lemma 7, $\lim_{n\to\infty} \phi_n(D) = \phi(D)$. Therefore,

$$\lim_{t \to s} \mathfrak{h}(F_t(D), F_s(D)) = 0$$

for every nonempty compact subset D of K.
The corresponding result (given in Theorem 2) holds for a regular cosine family $\{F_t : K \to cc(K)\}_{t\in\mathbb{R}}$ of continuous linear set-valued functions.

LEMMA 9. *If $F : \mathbb{R} \to cc(X)$ is continuous, then the set-valued function*

$$\phi(t) = \int_a^t F(u)du, \ \ (t \geqslant a)$$

is continuous.

Proof. The proof is identical to the proof of Lemma 10 in [12]. Let $h > 0$ and $t \geqslant a$. By Lemmas 7 and 8 in [1], we have

$$\begin{aligned}
\mathfrak{h}(\phi(t), \phi(t+h)) &= \mathfrak{h}\left(\textstyle\int_a^t F(u)du, \int_a^t F(u)du + \int_t^{t+h} F(u)du\right) \\
&\leqslant \mathfrak{h}\left(\textstyle\int_t^{t+h} F(u)du, \{0\}\right) \\
&\leqslant h \sup_{t \leqslant u \leqslant t+h} \|F(u)\|.
\end{aligned}$$

As $h \to 0$, we have $\mathfrak{h}(\phi(t), \phi(t+h)) \to 0$. That is, ϕ is continuous.

LEMMA 10. *Let $F : \mathbb{R} \to cc(X)$ be continuous, then for every $t \in \mathbb{R}$,*

$$\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} F(u)du = F(t).$$

Proof. Consider $t \in \mathbb{R}$, $\alpha = t-1$ and $\beta = t+1$. Define $H(s) = \int_\alpha^s F(u)du$ for every $s \in [\alpha, \beta]$. Since $F : [\alpha, \beta] \to cc(X)$ is continuous, so by Lemma 9 in [1], H is differentiable and $\lim_{h\to 0} \frac{H(t+h)-H(t)}{h} = F(t)$ or $\lim_{h\to 0} \frac{1}{h}\int_t^{t+h} F(u)du = F(t)$ for all $t \in \mathbb{R}$.

LEMMA 11. *If $F : [0,\infty) \to cc(X)$ is continuous, then*

$$\int_0^t \Big(\int_0^s F(u)du\Big)ds = \int_0^t (t-u)F(u)du \ \ (t \geqslant 0). \tag{1}$$

Proof. The proof is identical to that of Lemma 12 in [12]. For sake of convenience we give the proof. Define

$$\phi(t) := \mathfrak{h}\Big(\int_0^t \Big(\int_0^s F(u)du\Big)ds, \int_0^t (t-u)F(u)du\Big) \ \ (t \geqslant 0).$$

By Lemma 9, ϕ is continuous and by Lemma 8 in [1] we have

$$\begin{aligned} \phi(t+h) &= \mathfrak{h}\Big(\textstyle\int_0^{t+h}\big(\int_0^s F(u)du\big)ds, \int_0^{t+h}(t+h-u)F(u)du\Big) \\ &\leqslant \mathfrak{h}\Big(\textstyle\int_0^{t}\big(\int_0^s F(u)du\big)ds, \int_0^{t}(t-u)F(u)du\Big) \\ &+ \mathfrak{h}\Big(\textstyle\int_t^{t+h}\big(\int_0^s F(u)du\big)ds, \int_t^{t+h}(t+h-u)F(u)du + h\int_0^t F(u)du\Big). \end{aligned}$$

Thus,

$$\frac{\phi(t+h)-\phi(t)}{h} \leqslant \mathfrak{h}\Big(\frac{1}{h}\int_t^{t+h}\Big(\int_0^s F(u)du\Big)ds, \frac{1}{h}\int_t^{t+h}(t+h-u)F(u)du + \int_0^t F(u)du\Big)$$

for all $t \geqslant 0$ and $h > 0$. Since F is continuous, so there is $M > 0$ such that $\|F(u)\| \leqslant M$ for $u \in [t, t+1]$. By Lemma 7 in [1],

$$\Big\|\frac{1}{h}\int_t^{t+h}(t+h-u)F(u)du\Big\| \leqslant \frac{1}{h}\int_t^{t+h}(t+h-u)\|F(u)\|du \leqslant \frac{Mh}{2}$$

for every $h \in [0,1]$. Therefore,

$$\lim_{h\to 0^+}\frac{1}{h}\int_t^{t+h}(t+h-u)F(u)du = \{0\}.$$

Consequently, by Lemmas 9 and 10 we have

$$\begin{aligned} \liminf_{h\to 0^+}\frac{\phi(t+h)-\phi(t)}{h} &\leqslant \lim_{h\to 0^+}\mathfrak{h}\Big(\textstyle\frac{1}{h}\int_t^{t+h}\big(\int_0^s F(u)du\big)ds, \int_0^t F(u)du\Big) \\ &+ \lim_{h\to 0^+}\Big\|\textstyle\frac{1}{h}\int_t^{t+h}(t+h-u)F(u)du\Big\| \\ &= \mathfrak{h}\Big(\textstyle\int_0^t F(u)du, \int_0^t F(u)du\Big) + 0 = 0. \end{aligned}$$

Hence, ϕ is nonincreasing. Then, $\phi(t) \leqslant \phi(0)$ for every $t \geqslant 0$. This completes the proof.

LEMMA 12. *Let* $F:[\alpha,\beta]\to cc(X)$ *be continuous and* a,b,A,B *be real numbers satisfying* $a<b$, $Aa+B=\alpha$ *and* $Ab+B=\beta$. *Then,*

$$\int_{\alpha}^{\beta}F(t)dt=A\int_{a}^{b}F(Au+B)du.$$

Proof. Consider $F:[\alpha,\beta]\to cc(X_0)$. By Lemma 3 in [11], $\int_{\alpha}^{\beta}F(t)dt=A\int_a^b F(Au+B)du$. And, $\int_{\alpha}^{\beta}F(t)dt, \int_a^b F(Au+B)du\in cc(X)$, which completes the proof.

LEMMA 13. *Let* $F:\mathbb{R}\to cc(X)$ *be continuous. Then,*

$$\int_a^b F(u)du=\int_{t-b}^{t-a}F(t-u)du$$

for every $t\in\mathbb{R}$.

Proof. Consider $F:\mathbb{R}\to cc(X_0)$. By Lemma 4 in [11], $\int_a^b F(u)du=\int_{t-b}^{t-a}F(t-u)du$ for every $t\in\mathbb{R}$. And, $\int_a^b F(u)du, \int_{t-b}^{t-a}F(t-u)du\in cc(X)$, which completes the proof.

LEMMA 14. *If* $F:K\to cc(X)$ *is continuous linear and* $G:[a,b]\to cc(K)$ *is continuous, then* $\int_a^b F(G(t))dt=F(\int_a^b G(t)dt)$.

Proof. Consider $F:cl_{X_0}K\to cc(X_0)$ and $G:[a,b]\to cc(cl_{X_0}K)$. By Lemma 5 in [11], $\int_a^b F(G(t))dt=F(\int_a^b G(t)dt)$.

We have $\int_a^b G(t)dt\in cc(K)$ and $\int_a^b F(G(t))dt, F(\int_a^b G(t)dt)\in cc(X)$, which complete the proof.

THEOREM 3. *Let* $\{F_t:t\in\mathbb{R}\}$ *be a regular cosine family of continuous linear set-valued functions* $F_t:K\to cc(K)$ *such that* $t\mapsto\|F_t\|$ *is bounded on some neighborhood of zero. For any set* $D\in cc(K)$ *such that* $F_{t+s}(D)+F_{t-s}(D)=2F_tF_s(D)$ *for every* $s,t\in\mathbb{R}$, *the set-valued function* $\phi:\mathbb{R}\to cc(K)$ *satisfying*

$$\phi(s)=\int_0^s(s-v)F_v(D)dv,\ \ (s\geqslant 0)$$

$$\phi(s)=\phi(-s)\ \ (s\leqslant 0)$$

is a continuous even solution of

$$\phi(t+s)+\phi(t-s)=2F_t(\phi(s))+2\phi(t) \tag{2}$$

with $\phi(0)=\{0\}, D\phi(0)=\{0\}$.

Proof. By Theorem 2 (which also holds for a regular cosine family on all reals), set-valued functions $t \mapsto F_t(D)$ are continuous. Define

$$\phi(s) = \int_0^s (s-v)F_v(D)dv, \quad s \geqslant 0,$$

$$\phi(s) = \phi(-s), \quad s \leqslant 0.$$

By Lemma 9, ϕ is continuous. By Lemmas 10 and 11, $D\phi(t) = \lim_{h\to 0} \frac{\phi(t+h)-\phi(t)}{h} = \int_0^t F_v(D)dv$ for every $t \geqslant 0$. It is easy to see that ϕ is even, $\phi(0) = \{0\}$ and $D\phi(0) = \lim_{h\to 0} \frac{\phi(h)-\phi(0)}{h} = \{0\}$. If $s \in [0,t]$, then by Lemma 12,

$$\int_0^s (s-v)F_{t+v}(D)dv = \int_t^{t+s} (t+s-v)F_v(D)dv. \tag{3}$$

And, by Lemma 13,

$$\int_0^s (s-v)F_{t-v}(D)dv = \int_{t-s}^{t} (s-t+v)F_v(D)dv. \tag{4}$$

By Lemma 1, Lemma 8 in [1] and (3) we have

$$\begin{aligned}
\phi(t+s)+\phi(t-s) &= \textstyle\int_0^{t+s}(t+s-v)F_v(D)dv + \int_0^{t-s}(t-s-v)F_v(D)dv \\
&= \textstyle\int_0^{t-s}(t+s-v)F_v(D)dv + \int_{t-s}^{t}(t+s-v)F_v(D)dv \\
&+ \textstyle\int_t^{t+s}(t+s-v)F_v(D)dv + \int_0^{t-s}(t-s-v)F_v(D)dv \\
&= \textstyle 2\int_0^{t-s}(t-v)F_v(D)dv + \int_0^{s}(s-v)F_{t+v}(D)dv \\
&+ \textstyle\int_{t-s}^{t}(t+s-v)F_v(D)dv.
\end{aligned}$$

By the equality

$$\int_{t-s}^{t}(t+s-v)F_v(D)dv = \int_{t-s}^{t}(s-t+v)F_v(D)dv + 2\int_{t-s}^{t}(t-v)F_v(D)dv,$$

Lemma 14 and (4) we have

$$\begin{aligned}
\phi(t+s)+\phi(t-s) &= \textstyle\int_0^{s}(s-v)F_{t+v}(D)dv + \int_0^{s}(s-v)F_{t-v}(D)dv \\
&+ \textstyle 2\int_0^{t}(t-v)F_v(D)dv \\
&= \textstyle 2F_t(\int_0^{s}(s-v)F_v(D)dv) + 2\int_0^{t}(t-v)F_v(D)dv \\
&= 2F_t(\phi(s)) + 2\phi(t).
\end{aligned}$$

That is, ϕ is a solution of equation (2) for $0 \leqslant s \leqslant t$. Now we prove that $F_t(\phi(s)) + \phi(t) = F_s(\phi(t)) + \phi(s)$ for all $s,t \in \mathbb{R}$. If $0 \leqslant s \leqslant t$, then by Lemmas 12, 13 and Lemma 8 in [1],

$$\int_0^t (t-v)F_{s+v}(D)dv = \int_s^{t+s}(t+s-v)F_v(D)dv$$

and

$$\begin{aligned} \int_0^t (t-v)F_{s-v}(D)dv &= \int_0^s (t-v)F_{s-v}(D)dv + \int_s^t (t-v)F_{v-s}(D)dv \\ &= \int_0^s (t-s+v)F_v(D)dv + \int_0^{t-s}(t-s-v)F_v(D)dv. \end{aligned}$$

Consequently, by Lemma 14 we have

$$\begin{aligned} 2F_s(\phi(t)) + 2\phi(s) &= 2F_s(\int_0^t (t-v)F_v(D)dv) + 2\int_0^s (s-v)F_v(D)dv \\ &= \int_0^t (t-v)F_{s+v}(D)dv + \int_0^t (t-v)F_{s-v}(D)dv \\ &+ 2\int_0^s (s-v)F_v(D)dv \\ &= \int_s^{t+s}(t+s-v)F_v(D)dv + \int_0^s (t-s+v)F_v(D)dv \\ &+ \int_0^{t-s}(t-s-v)F_v(D)dv + 2\int_0^s (s-v)F_v(D)dv. \end{aligned}$$

Also, by Lemma 1,

$$\int_0^s (t-s+v)F_v(D)dv + 2\int_0^s (s-v)F_v(D)dv = \int_0^s (t+s-v)F_v(D)dv$$

and by Lemma 8 in [1],

$$\int_s^{t+s}(t+s-v)F_v(D)dv = \int_s^t (t+s-v)F_v(D)dv + \int_t^{t+s}(t+s-v)F_v(D)dv.$$

Therefore,

$$\begin{aligned} 2F_s(\phi(t)) &+ 2\phi(s) \\ &= \int_t^{t+s}(t+s-v)F_v(D)dv + \int_0^t (t+s-v)F_v(D)dv \\ &+ \int_0^{t-s}(t-s-v)F_v(D)dv. \end{aligned}$$

From

$$\begin{aligned} \int_0^t (t+s-v)F_v(D)dv &+ \int_0^{t-s}(t-s-v)F_v(D)dv \\ &= \int_{t-s}^t (t+s-v)F_v(D)dv + \int_0^{t-s}(2t-2v)F_v(D)dv \\ &= \int_{t-s}^t (s-t+v)F_v(D)dv + 2\int_{t-s}^t (t-v)F_v(D)dv \\ &+ 2\int_0^{t-s}(t-v)F_v(D)dv \\ &= \int_{t-s}^t (s-t+v)F_v(D)dv + 2\int_0^t (t-v)F_v(D)dv, \end{aligned}$$

we have:

$$\begin{aligned} 2F_s(\phi(t)) &+ 2\phi(s) \\ &= \int_t^{t+s}(t+s-v)F_v(D)dv + \int_{t-s}^t (s-t+v)F_v(D)dv \\ &+ 2\int_0^t (t-v)F_v(D)dv. \end{aligned}$$

According to (3), (4) and Lemma 14,

$$
\begin{aligned}
2F_s(\phi(t)) \quad &+ \quad 2\phi(s)\\
&= \quad \textstyle\int_0^s (s-v)F_{t+v}(D)dv + \int_0^s (s-v)F_{t-v}(D)dv\\
&+ \quad 2\textstyle\int_0^t (t-v)F_v(D)dv\\
&= \quad 2F_t(\textstyle\int_0^s (s-v)F_v(D)dv) + 2\phi(t)\\
&= \quad 2F_t(\phi(s)) + 2\phi(t).
\end{aligned}
$$

Hence, $F_t(\phi(s)) + \phi(t) = F_s(\phi(t)) + \phi(s)$ for every $s,t \in \mathbb{R}$. Since for all $x \in K$, $t \mapsto F_t(x)$ and ϕ are even, so ϕ satisfies (2) for all $s,t \in \mathbb{R}$.

EXAMPLE 1. Let $\{F_t : t \in \mathbb{R}\}$ be a regular cosine family of continuous linear set-valued functions $F_t : K \to cc(K)$ and $x \in F_t(x)$ for every $x \in K$ and $t \in \mathbb{R}$. Then, for every set $D \in cc(K)$ satisfying $0 \in D$ and $F_{t+s}(D) + F_{t-s}(D) = 2F_t(F_s(D))$ for every $s,t \in \mathbb{R}$, the set-valued function $\phi : \mathbb{R} \to cc(K)$ via

$$\phi(s) = \int_0^s (s-u)F_u(D)du, \ (s \geqslant 0)$$

and

$$\phi(s) = \phi(-s), \ (s \leqslant 0)$$

is a continuous even solution of (2) with $\phi(0) = \{0\}$, $D\phi(0) = \{0\}$ and $0 \in \phi(s)$ for all $s \in \mathbb{R}$.

LEMMA 15. *Let (A_n) and (B_n) be two sequences in $cc(X)$ such that $A_n \to A$ and $B_n \to B$. If there exist the Hukuhara differences $A_n - B_n$ in $cc(X)$ for every $n \in \mathbb{N}$, then there exists the Hukuhara difference $A - B$ and $A_n - B_n \to A - B$.*

Proof. There is no loss of generality in supposing that (A_n) and (B_n) are two sequences in $cc(X_0)$ such that $A_n \to A$ and $B_n \to B$ in $cc(X_0)$. By Lemma 1 in [13], there exists Hukuhara difference $A - B$ in $cc(X_0)$ and $A_n - B_n \to A - B$. Now, put $C := A - B$ and $C_n := A_n - B_n$ for $n \in \mathbb{N}$, by definition of the Hukuhara difference $A = B + C$ and $A_n = B_n + C_n$ for $n \in \mathbb{N}$. Since for all $n \in \mathbb{N}$, A_n, B_n, A, B are compact subsets in $cc(X)$, so $B_n + C_n, B + C \in cc(X)$ for $n \in \mathbb{N}$ and consequently $C_n, C \in cc(X)$ for all $n \in \mathbb{N}$.

The next Lemma is the normed space version of Lemma 11 in [11] which can be easily obtained via a similar argument if we just replace Lemma 1 in [13] with Lemma 15.

LEMMA 16. *If a continuous set-valued function $\phi : \mathbb{R} \to cc(K)$ fulfills (2) and $\phi(0) = \{0\}$, then for all $0 \leqslant s \leqslant t$ Hukuhara differences $\phi(t) - \phi(s)$ exist.*

THEOREM 4. *Let $\{F_t : t \in \mathbb{R}\}$ be a regular cosine family of continuous linear set-valued functions $F_t : K \to cc(K)$ such that $t \mapsto \|F_t\|$ is bounded on some neighborhood* *of zero. If a Hukuhara differentiable set-valued function $\phi : \mathbb{R} \to cc(K)$ is an even solution of (2) such that $D\phi$ is continuous, $\phi(0) = \{0\}$, $D\phi(0) = \{0\}$ and $\lim_{t\to 0^+} \frac{D\phi(t)}{t}$ exists, then there is a set $D \in cc(K)$ satisfying*

$$F_{t+s}(D) + F_{t-s}(D) = 2F_t(F_s(D)), \quad (s,t \in \mathbb{R})$$

$$\phi(s) = \int_0^s (s-u)F_v(D)dv, \quad (s \geqslant 0)$$

$$\phi(s) = \phi(-s), \quad (s \leqslant 0).$$

Proof. Since by assumption ϕ is even, so

$$\phi(t+s) + \phi(t-s) = 2F_s(\phi(t)) + 2\phi(s), \ (s,t \in \mathbb{R}). \tag{5}$$

Consider $0 \leqslant s \leqslant t$, and replace t by $t+v$ in (5). Then,

$$\phi(t+s+v) + \phi(t-s+v) = 2F_s(\phi(t+v)) + 2\phi(s) \tag{6}$$

where $v > 0$. By (5), (6) and Lemma 16 we obtain

$$\frac{\phi(t+s+v) - \phi(t+s)}{v} + \frac{\phi(t-s+v) - \phi(t-s)}{v} = 2F_s\left(\frac{\phi(t+v) - \phi(t)}{v}\right).$$

As $v \to 0^+$, we get

$$D\phi(t+s) + D\phi(t-s) = 2F_s(D\phi(t)) \tag{7}$$

for $0 \leqslant s \leqslant t$. By Lemma 16, the Hukuhara differences $\phi(t) - \phi(s)$ exist for $0 \leqslant s \leqslant t$. Consider $0 \leqslant s < t$ and $0 < v \leqslant t-s$ and replace s by $s+v$ in (2). Then,

$$\phi(t+s+v) + \phi(t-s-v) = 2F_t(\phi(s+v)) + 2\phi(t). \tag{8}$$

Adding both sides of (2) and (8) yields

$$\begin{aligned} \phi(t+s+v) &+ \phi(t-s-v) + 2F_t(\phi(s)) + 2\phi(t) \\ &= \phi(t+s) + \phi(t-s) + 2F_t(\phi(s+v)) + 2\phi(t). \end{aligned}$$

Hence,

$$\phi(t+s+v) - \phi(t+s) = 2F_t(\phi(s+v) - \phi(s)) + \phi(t-s) - \phi(t-s-v).$$

Dividing by v and letting $v \to 0^+$ we have

$$D\phi(t+s) = 2F_t(D\phi(s)) + D\phi(t-s) \tag{9}$$

for $0 \leqslant s < t$. From (9) we have

$$F_v(D\phi(t+s)) = 2F_v(F_t(D\phi(s))) + F_v(D\phi(t-s))$$

and replacing in (7) t by $t+s$ and s by v and next t by $t-s$ and s by v, we have

$$\frac{1}{2}D\phi(t+s+v)+\frac{1}{2}D\phi(t+s-v)=2F_v(F_t(D\phi(s)))+\frac{1}{2}D\phi(t-s+v)+\frac{1}{2}D\phi(t-s-v).$$

By (9), we get

$$F_{t+v}(D\phi(s))+F_{t-v}(D\phi(s))=2F_v(F_t(D\phi(s)))$$

for $0\leqslant v\leqslant t-s$. Dividing by s and letting $s\to 0^+$, we have

$$F_{t+v}(D)+F_{t-v}(D)=2F_v(F_t(D)), \tag{10}$$

where $D:=\lim_{t\to 0^+}\frac{D\phi(t)}{t}$. Define

$$\psi(t)=\int_0^t(t-v)F_v(D)dv,\ \ (t\geqslant 0)$$

and

$$\psi(t)=\psi(-t),\ \ (t\leqslant 0).$$

By Theorem 3, ψ is continuous, holds in (2) and $D\psi(t)=\int_0^t F_v(D)dv$. Moreover, by Lemma 10 we have

$$\lim_{t\to 0^+}\frac{D\psi(t)}{t}=\lim_{t\to 0^+}\frac{1}{t}\int_0^t F_v(D)dv=F_0(D)=D.$$

To end the proof it suffices to show that $\phi=\psi$. Define $h(t)=\mathfrak{h}(D\phi(t),D\psi(t))$ for every $t\geqslant 0$. Then,

$$\begin{aligned}
h(t+s) \quad - \quad & h(t)\\
= \quad & \mathfrak{h}(D\phi(t+\tfrac{s}{2}+\tfrac{s}{2}),D\psi(t+\tfrac{s}{2}+\tfrac{s}{2}))-\mathfrak{h}(D\phi(t),D\psi(t))\\
= \quad & \mathfrak{h}(2F_{t+\frac{s}{2}}(D\phi(\tfrac{s}{2}))+D\phi(t),2F_{t+\frac{s}{2}}(D\psi(\tfrac{s}{2}))+D\psi(t))-\mathfrak{h}(D\phi(t),D\psi(t))\\
\leqslant \quad & 2\mathfrak{h}(F_{t+\frac{s}{2}}(D\phi(\tfrac{s}{2})),F_{t+\frac{s}{2}}(D\psi(\tfrac{s}{2}))).
\end{aligned}$$

By Lemmas 5 and 6, there is $M_0\geqslant 0$ with

$$\frac{h(t+s)-h(t)}{s}\leqslant\mathfrak{h}(F_{t+\frac{s}{2}}(\frac{D\phi(\frac{s}{2})}{\frac{s}{2}}),F_{t+\frac{s}{2}}(\frac{D\psi(\frac{s}{2})}{\frac{s}{2}}))\leqslant M_0\|F_{t+\frac{s}{2}}\|\mathfrak{h}(\frac{D\phi(\frac{s}{2})}{\frac{s}{2}},\frac{D\psi(\frac{s}{2})}{\frac{s}{2}}).$$

By Theorem 1, $t\mapsto\tilde{F}_t(x)$ is continuous for every $x\in cl_{X_0}K$, consequently $\cup_{s\in[0,1]}\tilde{F}_{t+\frac{s}{2}}(x)$ is bounded for every $x\in cl_{X_0}K$. By Lemma 3, there exists $M>0$ such that $\|F_{t+\frac{s}{2}}\|\leqslant M$ for $s\in[0,1]$. Thus,

$$\frac{h(t+s)-h(t)}{s}\leqslant MM_0\mathfrak{h}(\frac{D\phi(\frac{s}{2})}{\frac{s}{2}},\frac{D\psi(\frac{s}{2})}{\frac{s}{2}}).$$

Hence, $\liminf_{s\to 0^+}\frac{h(t+s)-h(t)}{s}\leqslant 0$. By Zygmund's Lemma (see [6], p. 174) h is non-increasing. So, $h(t)\leqslant h(0)$ for all $t\geqslant 0$. That is, $D\phi=D\psi$. Since $D\phi=D\psi$, $\phi(0)=\psi(0)$ and ϕ,ψ are even, so $\phi=\psi$.

EXAMPLE 2. Let $\{F_t : t \in \mathbb{R}\}$ is a regular cosine family of continuous linear set-valued functions $F_t : K \to cc(K)$ and $x \in F_t(x)$ for every $x \in K$ and $t \in \mathbb{R}$. If a set-valued function $\phi : \mathbb{R} \to cc(k)$ is a concave continuous even solution of (2) with $\phi(0) = \{0\}$, $D\phi(0) = \{0\}$ and $0 \in \phi(t)$ for all $t \in \mathbb{R}$, then by Corollary 1, $t \mapsto \|F_t\|$ is bounded on some neighborhood of zero. By Lemma 16, the differences $\phi(t) - \phi(s)$ exist for all $0 \leqslant s \leqslant t$. And, by Theorem 3.2 in [10], there exists

$$\lim_{h \to 0^+} \frac{\phi(t+h) - \phi(t)}{h} := D^+\phi(t), \ (t > 0)$$

and

$$\lim_{h \to 0^+} \frac{\phi(t) - \phi(t-h)}{h} := D^-\phi(t), \ (t > 0).$$

By (2), we have

$$\phi(t+s) - \phi(t) = 2F_t(\phi(s)) + \phi(t) - \phi(t-s)$$

for all $0 < s \leqslant t$. Divide by s and let $s \to 0^+$, then $D^+\phi(t) = D^-\phi(t) =: D\phi(t)$ for all $t > 0$. That is, ϕ is Hukuhara differentiable at every $t > 0$. Since by assumption ϕ is even, so

$$\phi(t+s) + \phi(t-s) = 2F_s(\phi(t)) + 2\phi(s), \ (s,t \in \mathbb{R}). \tag{11}$$

Consider $0 \leqslant s \leqslant t$ and replace t by $t+v$ in (11), then

$$\phi(t+s+v) + \phi(t-s+v) = 2F_s(\phi(t+v)) + 2\phi(s) \tag{12}$$

where $v > 0$. By (11) and (12), we get

$$\frac{\phi(t+s+v) - \phi(t+s)}{v} + \frac{\phi(t-s+v) - \phi(t-s)}{v} = 2F_s(\frac{\phi(t+v) - \phi(t)}{v}).$$

As $v \to 0^+$, we get $D\phi(t+s) + D\phi(t-s) = 2F_s(D\phi(t))$ for all $0 \leqslant s \leqslant t$. Putting $t = \frac{u+v}{2}$ and $s = \frac{v-u}{2}$ we have

$$D\phi(v) + D\phi(u) = 2F_{\frac{v-u}{2}}(D\phi(\frac{v+u}{2}))$$

where $0 \leqslant u \leqslant v$. By assumption $x \in F_t(x)$, we have $D\phi(\frac{u+v}{2}) \subseteq \frac{D\phi(v)+D\phi(u)}{2}$. Let $[a,b] \subseteq [0,\infty)$ and fix it. By Theorem 3.2 in [10], $D\phi$ is increasing and for $t \in [a,b]$, $D\phi(t) \subseteq D\phi(b)$. Thus, $D\phi$ is bounded on $[a,b]$ and by Theorem 4.4 in [9], $D\phi$ is continuous on $(0,\infty)$ and by Theorem 4.1 in [9], is concave. Therefore, there exists

$$\lim_{t \to 0^+} \frac{D\phi(t)}{t} \in cc(K).$$

Since $D\phi(0) = \{0\}$, $D\phi$ is increasing and $0 \in D\phi(t)$ for $t \geqslant 0$. Hence by Theorem 4, there is a set $D \in cc(K)$ with $0 \in D$ and

$$F_{t+s}(D) + F_{t-s}(D) = 2F_t(F_s(D)), \quad (s,t \in \mathbb{R})$$

$$\phi(s) = \int_0^s (s-u)F_u(D)du, \quad (s \geqslant 0)$$

$$\phi(s) = \phi(-s), \quad (s \leqslant 0).$$

LEMMA 17. *If set-valued functions $F,G,H : K \to cc(K)$ are continuous and linear, then there exists at most one continuous linear set-valued function $\varphi : [0,\infty)\times K \to cc(K)$ which is twice differentiable with respect to the first variable and it satisfies the following differentiable problem*

$$D_t^2\varphi(t,x) = \varphi(t,H(x)), \varphi(0,x) = F(x), D_t\varphi(t,x)|_{t=0} = G(x). \tag{13}$$

Proof. Let $\phi,\psi : [0,\infty)\times K \to cc(K)$ be two solutions of problem (13). By Lemmas 9 and 10, we have

$$D\phi(t,x) = G(x) + \int_0^t \phi(u,H(x))du$$

and

$$\phi(t,x) = F(x) + tG(x) + \int_0^t (\int_0^s \phi(u,H(x))du)ds.$$

Also,

$$D\psi(t,x) = G(x) + \int_0^t \psi(u,H(x))du$$

and

$$\psi(t,x) = F(x) + tG(x) + \int_0^t (\int_0^s \psi(u,H(x))du)ds.$$

By Theorem 1 in [2], F,G,H and ϕ,ψ have continuous linear extensions $\tilde F,\tilde G,\tilde H$: $cl_{X_0}K \to cc(cl_{X_0}K)$ and $\tilde\phi,\tilde\psi : [0,\infty)\times cl_{X_0}K \to cc(cl_{X_0}K)$, respectively. By Lemma 7 in [1], we obtain $\tilde\phi(t,x) = \tilde F(x)+t\tilde G(x)+\int_0^t(\int_0^s \tilde\phi(u,\tilde H(x))du)ds$ and $\tilde\psi(t,x) = \tilde F(x)+t\tilde G(x)+\int_0^t(\int_0^s \tilde\psi(u,\tilde H(x))du)ds$. Thus, $\tilde\phi(t,x)$ and $\tilde\psi(t,x)$ are two solutions of problem (13). By Theorem 2 in [7], $\tilde\phi(t,x) = \tilde\psi(t,x)$ and consequently $\phi(t,x) = \psi(t,x)$ for every $(t,x) \in [0,\infty)\times cc(K)$.

From now, we use the abbreviation $G_t(x)$ for $\lim_{h\to 0}\frac{F_{t+h}(x)-F_t(x)}{h}$.

THEOREM 5. *Let $\{F_t : t \geqslant 0\}$ be a regular cosine family of continuous linear set-valued functions $F_t : K \to cc(K)$ such that $t \mapsto \|F_t\|$ is bounded on some neighborhood of zero. If $\lim_{h\to 0^+}\frac{F_{t+h}(x)-F_t(x)}{h}$ exists, then $\{F_t : t \geqslant 0\}$ is differentiable. Moreover, if $\lim_{h\to 0^+}\frac{G_h(x)}{h} := H(x)$ exists, then*

$$F_t(F_s(x)) = F_s(F_t(x))$$

for $x \in K$ and $s,t \geqslant 0$.

Proof. Since $F_{t+s}(x)+F_{t-s}(x) = 2F_t(F_s(x))$ for all $x \in K$ and $0 \leqslant s \leqslant t$, so $\frac{F_{2t}(x)-x}{2t} = F_t(\frac{F_t(x)-x}{t}) + \frac{F_t(x)-x}{t}$. Let $t \to 0^+$, then

$$\lim_{t\to 0^+} F_t(\frac{F_t(x)-x}{t}) = \{0\}. \tag{14}$$

By Lemmas 5 and 6, there exists $M_0 > 0$ such that

$$\begin{aligned}\mathfrak{h}(F_t(C_t(x)),C(x)) &\leqslant \mathfrak{h}(F_t(C_t(x)),F_t(C(x)))+\mathfrak{h}(F_t(C(x)),C(x))\\ &\leqslant M_0\|F_t\|\mathfrak{h}(C_t(x),C(x))+\mathfrak{h}(F_t(C(x)),C(x)),\end{aligned}$$

where $C_t(x) := \frac{F_t(x)-x}{t}$ and $C(x) := \lim_{t\to 0^+}\frac{F_t(x)-x}{t}$. Since $t\mapsto \|F_t\|$ is bounded on some neighborhood of zero, so there exist positive constants δ and M such that $\|F_t\|\leqslant M$ for $t\in[0,\delta]$. Moreover, by Theorem 3 in [1],

$$\lim_{t\to 0^+}\mathfrak{h}(F_t(D),D)=0$$

for every nonempty compact subset D of K. Therefore,

$$\lim_{t\to 0^+}\mathfrak{h}(F_t(C_t(x)),C(x))=0. \tag{15}$$

From (14) and (15) we have $C(x)=\lim_{t\to 0^+}\frac{F_t(x)-x}{t}=\{0\}$ for every $x\in K$. By Lemma 2, we obtain $F_{t+h}(x)-F_t(x)=2F_t(F_h(x)-x)+F_t(x)-F_{t-h}(x)$ for $0<h\leqslant t$. Dividing this equality by h we get

$$\frac{F_{t+h}(x)-F_t(x)}{h}=2F_t(\frac{F_h(x)-x}{h})+\frac{F_t(x)-F_{t-h}(x)}{h}.$$

Letting $h\to 0^+$, we have

$$\lim_{t\to 0^+}\frac{F_{t+h}(x)-F_t(x)}{h}=\lim_{t\to 0^+}\frac{F_t(x)-F_{t-h}(x)}{h}=G_t(x)\quad (t>0).$$

This implies that the family $\{F_t : t\geqslant 0\}$ is differentiable.

Let $s\geqslant 0$, define $\varphi(t,x):=F_s(F_{t+s}(x))$ and $\psi(t,x):=F_{t+s}(F_s(x))$ for all $x\in K$ and $t\geqslant 0$. We have

$$\varphi(0,x)=F_s(F_s(x))=\psi(0,x),\ (x\in K).$$

By Lemma 2, we have

$$\begin{aligned}D_t^+\varphi(t,x) &= \lim_{h\to 0^+}\tfrac{F_s(F_{t+h+s})(x)-F_s(F_{t+s})(x)}{h}\\ &= F_s(\lim_{h\to 0^+}\tfrac{F_{t+s+h}(x)-F_{t+s}(x)}{h})\\ &= F_s(G_{t+s}(x))\end{aligned}$$

for $x\in K$ and $t\geqslant 0$. And, similarly $D_t^-\varphi(t,x)=F_s(G_{t+s}(x))$ for $t>0$ and $x\in K$. Moreover, we obtain

$$\begin{aligned}D_t^+\psi(t,x) &= \lim_{h\to 0^+}\tfrac{F_{t+s+h}(F_s(x))-F_{t+s}(F_s(x))}{h})\\ &= \tfrac{1}{2}\lim_{h\to 0^+}[\tfrac{F_{t+2s+h}(x)-F_{t+2s}(x)}{h}+\tfrac{F_{t+h}(x)-F_t(x)}{h}]\\ &= \tfrac{1}{2}[G_{t+2s}(x)+G_t(x)]=D_t^-\psi(t,x).\end{aligned}$$

Also, for $0<s<t$ we obtain

$$
\begin{aligned}
2F_t(G_s(x)) &= 2F_t(\lim_{h\to 0^+} \tfrac{F_{s+h}(x)-F_s(x)}{h}) \\
&= \lim_{h\to 0^+} \tfrac{2F_t(F_{s+h}(x))-2F_t(F_s(x))}{h} \\
&= \lim_{h\to 0^+} \tfrac{F_{t+s+h}(x)+F_{t-s-h}(x)-(F_{t+s}(x)+F_{t-s}(x))}{h} \\
&= \lim_{h\to 0^+} [\tfrac{F_{t+s+h}(x)-F_{t+s}(x)}{h} - \tfrac{F_{t-s}(x)-F_{t-s-h}(x)}{h}] \\
&= G_{t+s}(x) - G_{t-s}(x).
\end{aligned}
$$

And,

$$
\begin{aligned}
\mathfrak{h}(2F_s(G_s(x)), G_{2s}(x)) &\leqslant \mathfrak{h}(\tfrac{F_{2s}(x)-F_{2s-h}(x)}{h}), G_{2s}(x)) \\
&+ \mathfrak{h}(\tfrac{(F_{2s}(x)+x)-(F_{2s-h}(x)+F_h(x))}{h} + \tfrac{F_h(x)-x}{h}, 2F_s(G_s(x)) \\
&\leqslant \mathfrak{h}(\tfrac{F_{2s}(x)-F_{2s-h}(x)}{h}, G_{2s}(x)) \\
&+ \mathfrak{h}(\tfrac{2F_s(F_s(x))-2F_s(F_{s-h}(x))}{h}, 2F_s(G_s(x))) \\
&+ \mathfrak{h}(\tfrac{F_h(x)-x}{h}, \{0\}) \\
&\leqslant \mathfrak{h}(G_{2s}(x), \tfrac{F_{2s}(x)-F_{2s-h}(x)}{h}) \\
&+ 2M_0\|F_s\|\mathfrak{h}(\tfrac{F_s(x)-F_{s-h}(x)}{h}, G_s(x)) + \mathfrak{h}(\tfrac{F_h(x)-x}{h}, \{0\}).
\end{aligned}
$$

Therefore,

$$G_{t+s}(x) = G_{t-s}(x) + 2F_t(G_s(x)), \ \ (x \in K, 0 \leqslant s \leqslant t). \tag{16}$$

By equation (16), $D_t\varphi(t,x) = F_s(G_{t+s}(x))$ and $D_t\psi(t,x) = \frac{1}{2}(G_{t+2s}(x) + G_t(x))$ we have $D_t\varphi(t,x)|_{t=0} = F_s(G_s(x)) = D_t\psi(t,x)|_{t=0}$ for $x \in K$. Putting

$$H_t(x) := \lim_{h\to 0^+} \frac{G_{t+h}(x) - G_t(x)}{h},$$

we have

$$\lim_{s\to 0^+} \frac{G_{t+2s}(x) - G_t(x)}{2s} = \lim_{s\to 0^+} F_{t+s}(\frac{G_s(x)}{s}) = F_t(H(x))$$

for $x \in K, t \geqslant 0$ and

$$\lim_{s\to 0^+} \frac{G_t(x) - G_{t-2s}(x)}{2s} = \lim_{s\to 0^+} F_{t-s}(\frac{G_s(x)}{s}) = F_t(H(x))$$

for $x \in K, t > 0$.

$$
\begin{aligned}
D_t^+ D_t\varphi(t,x) &= D_t^+ F_s(G_{t+s}(x)) \\
&= \lim_{h\to 0^+} F_s(\tfrac{G_{t+s+h}(x)-G_{t+s}(x)}{h}) \\
&= F_s(H_{t+s}(x)) = \varphi(t, H(x)) = D_t^- D_t\varphi(t,x),
\end{aligned}
$$

and

$$\begin{aligned} D_t^+ D_t \psi(t,x) &= \tfrac{1}{2} D_t^+ (G_{t+2s}(x) + G_t(x)) \\ &= \tfrac{1}{2} \lim_{h\to 0^+} \left(\frac{G_{t+2s+h}(x) - G_{t+2s}(x)}{h} + \frac{G_{t+h}(x) - G_t(x)}{h} \right) \\ &= \tfrac{1}{2} (H_{t+2s}(x) + H_t(x)) = D_t^- D_t \psi(t,x) \end{aligned}$$

where $H_t(x) = F_t(H(x))$.

Hence, we have

$$D_t^2 \psi(t,x) = F_{t+s}(H_s(x)) = F_{t+s}(F_s(H(x))) = \psi(t, H(x)).$$

Therefore, the set-valued functions φ and ψ are solutions of problem

$$D_t^2 \varphi(t,x) = \varphi(t, H(x)), \ \varphi(0,x) = F(x), \ D_t \varphi(t,x)|_{t=0} = G(x)$$

with $F(x) := F_s(F_s(x)), G(x) := F_s(G_s(x))$ and $H(x) := D_t^2 F_t(x)|_{t=0}$. By Lemma 17, $\varphi(t,x) = \psi(t,x)$. Thus, $F_s(F_{t+s}(x)) = F_{t+s}(F_t(x))$ for $s,t \geqslant 0, x \in K$. This completes the proof.
Theorem 5 shows that a regular cosine family $\{F_t : t \geqslant 0\}$ of continuous linear set-valued functions can be extended to a regular cosine family $\{F_t : t \in \mathbb{R}\}$.

EXAMPLE 3. Let $\{F_t : t \geqslant 0\}$ be a regular cosine family of continuous linear set-valued functions $F_t : K \to cc(K)$ such that $x \in F_t(x)$ for all $x \in K$ and $t \geqslant 0$. By Corollary 1, $t \mapsto \|F_t\|$ is bounded on some neighborhood of zero and by Theorem 2 in [2], $\{\tilde{F}_t : t \geqslant 0\}$ is a regular cosine family of continuous linear set-valued functions $\tilde{F}_t : cl_{X_0} K \to cc(cl_{X_0} K)$ such that $x \in \tilde{F}_t(x)$ for all $x \in cl_{X_0} K$ and $t \geqslant 0$. By Theorem 4.2 in [18], $\tilde{F}_t(\tilde{F}_s(x)) = \tilde{F}_s(\tilde{F}_t(x))$ and consequently $F_t(F_s(x)) = F_s(F_t(x))$ for all $x \in K$ and $t \geqslant 0$.

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