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OPTIMAL L^p HARDY–RELLICH TYPE INEQUALITIES ON THE SPHERE

ABIMBOLA ABOLARINWA* AND KAMILU RAUF

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Abstract. In this paper we study some L^p-Hardy-Rellich type inequalities and the corresponding optimal constant on the geodesic sphere. By the divergence theorem, properties of radial Laplacian and geodesic distance, we obtain an improved version of Hardy-Rellich inequalities holding in dimension $N \geqslant 3$. The result is new for $N = 3,\ 4$. Moreover, we show that the constant obtained is optimally sharp.

1. Introduction

This paper is concerned with the proof of an improved extension of Hardy-Rellich inequalities for L^p-functions on the N-sphere of constant sectional curvature. We apply properties of geodesic distance on the unit sphere, radial Laplacian and the divergence theorem to establish, for $N \geqslant 3$ and $f \in L^p(\mathbb{S}^N, dV)$,

$$\int_{\mathbb{S}^N} \frac{|\Delta_{\mathbb{S}^N} f|^p}{(\sin\theta)^{2-2p}} dV + B \int_{\mathbb{S}^N} \frac{|f|^p}{(\sin\theta)^{2-2p}} dV \geqslant C \int_{\mathbb{S}^N} \frac{|f|^p}{\sin^2\theta} dV, \tag{1}$$

where $B = B(N,p)$ and $C = C(N,p)$ are some constants involving the best constant in the classical Hardy inequality. We further show that the constant $C(N,p)$ is the best possible achieved in the sense that

$$C(N,p) \geqslant \inf_{f \in C^\infty(\mathbb{S}^N)\setminus\{0\}} \frac{\int_{\mathbb{S}^N} \frac{|\Delta_{\mathbb{S}^N} f|^p}{(\sin\theta)^{2-2p}} dV + B \int_{\mathbb{S}^N} \frac{|f|^p}{(\sin\theta)^{2-2p}} dV}{\int_{\mathbb{S}^N} \frac{|f|^p}{d(x,q)^2} dV},$$

where $d(x,q)$ is the geodesic distance between points x and q on $\mathbb{S}^N$. The statement of the above result is given in Theorem 1 and the its proof in Section 3. To the best of our knowledge, this is the first time of having such inequalities extended to dimensions $N = 3$ and 4.

Let $\mathbb{R}^N$, $N \geqslant 3$ be the N-dimensional Euclidean space, the classical Hardy inequality for $f \in C_0^\infty(\mathbb{R}^N)$ and $1 < p < \infty$ is given as follows

$$\int_{\mathbb{R}^N} |\nabla f(x)|^p dx \geqslant \left(\frac{N-p}{p}\right)^p \int_{\mathbb{R}^N} \frac{|f(x)|^p}{|x|^p} dx, \tag{2}$$

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* Corresponding author.

where $((N-p)/p)^p$ is the best constant and never achieved. The Hardy-Rellich [7, 6] states that for $p>1$, $N>\alpha+2$, $\alpha\in\mathbb{R}$, and for any smooth function f on $\Omega\subseteq\mathbb{R}^N$, it holds that

$$\int_\Omega \frac{|\Delta f|^p}{|x|^{\alpha+2-2p}}dx \geqslant \Big(\frac{(N-\alpha-2)[(p-1)(N-2)+\alpha]}{p^2}\Big)^p \int_\Omega \frac{|f|^p}{|x|^{\alpha+2}}dx \tag{3}$$

with sharp constant. Meanwhile, the classical case $p=2$ and $\alpha=2$

$$\int_{\mathbb{R}^N} |\Delta f|^2 dx \geqslant \frac{N^2(N-4)^4}{16}\int_{\mathbb{R}^N} \frac{|f|^2}{|x|^4}dx \tag{4}$$

was first published by F. Rellich in 1955 for $N\geqslant 5$, and the constant $\dfrac{N^2(N-4)^4}{16}$ is optimal but never achieved.

Owing to several areas of their applications, such as in elliptic operator theory, spectral theory, harmonic analysis, mathematical physics, differential geometry to mention but a few, numerous literatures have been devoted to obtaining improvement and extension of Hardy-Rellich type inequalities. For examples, we find [3, 5, 6, 7, 11]. In paricular, see [8, 9, 13, 15] for the extension to complete manifolds. For more exposition see [1, 2] and the references therein. Recently, Xiao [14] studied L^2-Hardy inequality on the unit sphere and as a consequence derived L^2-Rellich type inequality with sharp constant. Motivated by [14], we obtained some L^p Hardy-Rellich type inequalities in [1] and showed that the constant is sharp in the sense that it cannot be improved. In a similar spirit, the present paper is devoted to obtaining an improved version of the optimal L^p Hardy-Rellich inequalities that can be extended to lower dimensions.

The rest of the paper is planned as follows. In Section 2 we recall some basic facts about the sphere and then present the main results of this paper. Section 3 is devoted to the proof of improved Hardy-Rellich type inequalities and the sharpness of the constant.

2. Preliminaries and main theorem

2.1. Sphere

We deal with the unit N-sphere $\mathbb{S}^N=\{x\in\mathbb{R}^{N+1}:|x|=1\}$ of sectional curvature 1, endowed with canonical Riemannian structure. Let $(\theta_1,\theta_2,\ldots,\theta_N)$ be angular variables on $\mathbb{S}^N$, we set $\theta=\theta_N$, where $x_{N+1}=|x|\cos\theta_N$. The associated weight function is given as $\Theta(\theta,\xi)=(\sin\theta)^{N-1}$, $\xi\in\mathbb{S}^{N-1}$ and by polar coordinate transformation

$$\int_{\mathbb{S}^N} f dV = \int_{\mathbb{S}^{N-1}}\int_0^\pi f(\sin\theta)^{N-1}d\theta d\sigma, \quad f\in L^1(\mathbb{S}^N),$$

where dV and $d\sigma$ denote the standard volume element on $\mathbb{S}^N$ and unit $(N-1)$-sphere respectively. A function $f=f(\theta)$ which depends only on θ is called radial. In this case using the radial part of the Laplace-Beltrami operator $\Delta_{\mathbb{S}^N}$ we have

$$\Delta_{\mathbb{S}^N} f(\theta) = (\sin\theta)^{1-N}\frac{d}{d\theta}\Big((\sin\theta)^{N-1}\frac{d}{d\theta}f\Big)$$

while the gradient of a function f on $\mathbb{S}^N$ is $|\nabla_{\mathbb{S}^N} f(\theta)| = |\frac{d}{d\theta} f(\theta)|$.

The geodesic distance between x and an arbitrary point $q \in \mathbb{S}^N$ is denoted by $d(x,q)$. Note that the points on the sphere are all the same distance from the origin, which is a fixed point and all geodesics of the sphere are closed curves. Consider two points x and y on a sphere of radius $r > 0$ centered at the origin of $\mathbb{R}^N$, the distance between the two points is given by $d(x,y) = r\arccos((x\cdot y)/r)$, while $x\cdot y = r^2\cos\theta$, where θ is the angle between vectors x and y. Hence, the minimal geodesic joining two points on the unit sphere can be taken to be θ. Throughout we denote $\Delta = \Delta_{\mathbb{S}^N}$ and $\nabla = \nabla_{\mathbb{S}^N}$.

LEMMA 1. *Let $\beta \in \mathbb{R}$. Then*

$$\Delta_{\mathbb{S}^N} (\sin\theta)^{-\beta} = \frac{\beta(N-\beta-1)}{(\sin\theta)^{\beta}} - \frac{\beta(N-\beta-2)}{(\sin\theta)^{\beta+2}}. \tag{5}$$

Proof.

$$\begin{aligned}
\Delta_{\mathbb{S}^N} (\sin\theta)^{-\beta} &= (\sin\theta)^{1-N} \frac{d}{d\theta}\Big((\sin\theta)^{N-1} \frac{d}{d\theta}(\sin\theta)^{-\beta}\Big) \\
&= -\beta(\sin\theta)^{1-N} \frac{d}{d\theta}\Big((\sin\theta)^{N-\beta-2}\cos\theta\Big) \\
&= -\beta(N-\beta-2)(\sin\theta)^{-\beta-2}\cos^2\theta + \beta(\sin\theta)^{-\beta} \\
&= \beta(N-\beta-1)(\sin\theta)^{-\beta} - \beta(N-\beta-2)(\sin\theta)^{-(\beta+2)}
\end{aligned}$$

by using the trigonometry identity $\cos^2\theta + \sin^2\theta = 1$, which is formula (5). □

2.2. Main results

Our main theorem is the following Hardy-Rellich type inequalities

THEOREM 1. *Let $N \geqslant 3$ and $1 < p < \infty$, then there exists a positive constant $A = A(N,\alpha,p)$ such that for all $f \in C^\infty(\mathbb{S}^N)$*

$$\int_{\mathbb{S}^N} \frac{|\Delta_{\mathbb{S}^N} f|^p}{(\sin\theta)^{2-2p}} dV + B(N,p) \int_{\mathbb{S}^N} \frac{|f|^p}{(\sin\theta)^{2-2p}} dV \geqslant C(N,p) \int_{\mathbb{S}^N} \frac{|f|^p}{\sin^2\theta} dV, \tag{6}$$

where

$$B(N,p) = \Big(\frac{N(N-2)(p-1)}{p^2}\Big)^p \quad \text{and} \quad C(N,p) = \Big(\frac{(N-2)^2(p-1)}{p^2}\Big)^p.$$

Moreover, the constant $C(N,p)$ appearing in (6) is sharp.

REMARK 1. The family of inequalities in (6) is new and can be viewed as an extension of [1] and [14] ($p = 2$) since it holds for $N \geqslant 3$. The case $p = 2$ reads

$$\int_{\mathbb{S}^N} \sin^2\theta |\Delta f|^2 dV + \frac{N^2(N-2)^2}{16} \int_{\mathbb{S}^N} \sin^2\theta f^2 dV \geqslant \Big(\frac{N-2}{2}\Big)^2 \int_{\mathbb{S}^N} \frac{|f|^p}{\sin^2\theta} dV.$$

3. Proof of the main Theorem

We start this section with a fundamental lemma that will be applied in the proof of Theorem 1.

LEMMA 2. *([1, Theorem 2.1]) Let* $N \geqslant 3,\ 0 \leqslant \alpha < N-p$, *and* $1 < p < \infty$, *then there exists a constant* $A = A(N,\alpha,p) > 0$ *such that for all* $f \in C^\infty(\mathbb{S}^N)$

$$\int_{\mathbb{S}^N} \frac{|\nabla_{\mathbb{S}^N} f|^p}{(\sin\theta)^\alpha} dV + A\int_{\mathbb{S}^N} \frac{|f|^p}{(\sin\theta)^{\alpha+p-2}} dV \geqslant \Big(\frac{N-p-\alpha}{p}\Big)^p \int_{\mathbb{S}^N} \frac{|f|^p}{(\sin\theta)^{\alpha+p}} dV, \qquad (7)$$

where

$$A(N,\alpha,p) = \min\Big\{1, \frac{p}{2}\Big\}\Big(\frac{N-p-\alpha}{p}\Big)^p + \Big(\frac{N-p-\alpha}{p}\Big)^{p-1}.$$

The idea of the proof is similar to the ones in [1, 8, 12, 14]. It is however included here for completeness sake.

REMARK 2. Consider the extreme case $\alpha = 0$ and $p = 2$ in Lemma 2, we have

$$\int_{\mathbb{S}^N} |\nabla_{\mathbb{S}^N} f|^2 dV + \frac{N(N-2)}{4}\int_{\mathbb{S}^N} |f|^2 dV \geqslant \frac{(N-2)^2}{4}\int_{\mathbb{S}^N} \frac{|f|^2}{\sin^2\theta} dV, \qquad (8)$$

which is exactly the Hardy inequality with sharp constant in [14, Theorem 1].

Proof of Lemma 2

Recall from [10, 13] that for any $u, v \in \mathbb{R}^N$, it holds that $|u+v|^p \geqslant |u|^p + p|u|^{p-2}\langle u,v\rangle$. Now letting $\gamma = -(N-p-\alpha)/p > 0$, $f = \rho^\gamma \phi$, $\rho = \sin\theta$ and $f \in C^\infty(\mathbb{S}^N)$, we have

$$\begin{aligned} |\nabla_{\mathbb{S}^N} f|^p &= |\gamma\rho^{\gamma-1}\nabla\rho\phi + \rho^\gamma\nabla\phi|^p \\ &\geqslant |\gamma|^p\rho^{\gamma p-p}|\nabla\rho|^p|\phi|^p + p\gamma^{p-1}\rho^{\gamma p+1-p}|\phi|^{p-1}\langle|\nabla\rho|^{p-1}, \nabla\phi\rangle. \end{aligned}$$

Multiplying through by $\rho^{-\alpha}$ and applying divergence theorem, we have

$$\begin{aligned} \int_{\mathbb{S}^N} \frac{|\nabla_{\mathbb{S}^N} f|^p}{\rho^\alpha} dV \geqslant |\gamma|^p \int_{\mathbb{S}^N} \rho^{\gamma p-p-\alpha}|\nabla\rho|^p|\phi|^p dV \\ - \frac{|\gamma|^{p-2}\gamma}{\gamma p - p - \alpha + 2}\int_{\mathbb{S}^N} \Delta p^{\gamma p-p-\alpha+2}|\phi|^p dV. \end{aligned} \qquad (9)$$

Note that $|\nabla\rho| = \cos\theta$, $|\gamma| = |-\dfrac{N-p-\alpha}{p}|$, $\gamma p - p - \alpha = -N$ and $\dfrac{|\gamma|^{p-2}\gamma}{\gamma p - p - \alpha + 2} = \dfrac{1}{N-2}\Big(\dfrac{N-p-\alpha}{p}\Big)^{p-1}$. Hence, (9) becomes

$$\begin{aligned}\int_{\mathbb{S}^N} \frac{|\nabla f|^p}{(\sin\theta)^\alpha} dV \geqslant \Big(\frac{N-p-\alpha}{p}\Big)^p \int_{\mathbb{S}^N} \frac{|\varphi|^p}{(\sin\theta)^N} (\cos\theta)^p dV \\ - \frac{1}{(N-2)}\Big(\frac{N-p-\alpha}{p}\Big)^{p-1} \int_{\mathbb{S}^N} \langle \Delta(\sin\theta)^{-(N-2)}, |\varphi|^p\rangle \\ = \Big(\frac{N-p-\alpha}{p}\Big)^p \int_{\mathbb{S}^N} \frac{|\varphi|^p}{(\sin\theta)^N} (\cos\theta)^p dV \\ - \Big(\frac{N-p-\alpha}{p}\Big)^{p-1} \int_{\mathbb{S}^N} \frac{|\varphi|^p}{(\sin\theta)^{N-2}} dV\end{aligned}$$

by applying Lemma 1. Substituting the identity $|\cos\theta|^p \geqslant 1 - \min\{1, \frac{p}{2}\}\sin^2\theta$ into the last inequality yields

$$\begin{aligned}\int_{\mathbb{S}^N} \frac{|\nabla f|^p}{(\sin\theta)^\alpha} dV \geqslant \Big(\frac{N-p-\alpha}{p}\Big)^p \int_{\mathbb{S}^N} \frac{\varphi^p}{(\sin\theta)^N} dV \\ - \left(\min\Big\{1, \frac{p}{2}\Big\}\Big(\frac{N-p-\alpha}{p}\Big)^p + \Big(\frac{N-p-\alpha}{p}\Big)^{p-1}\right) \int_{\mathbb{S}^N} \frac{|\varphi|^p}{(\sin\theta)^{N-2}} dV.\end{aligned}$$

By using the substitution $\phi = \rho^{-\gamma} f = (\sin\theta)^{\frac{N-p-\alpha}{p}} f$, we recover the desired inequality (7). □

Proof of Theorem 1

Let $f \in C^\infty(\mathbb{S}^N)$. For $\varepsilon > 0$, define $f_\varepsilon := (|f|^2 + \varepsilon^2)^{p/2} - \varepsilon^p \in C^\infty(\mathbb{S}^N)$ with the same support as f. We have

$$\begin{aligned}\Delta f_\varepsilon &= p(|f|^2+\varepsilon^2)^{\frac{p}{2}-1}|\nabla f|^2 + p(p-2)(|f|^2+\varepsilon^2)^{\frac{p}{2}-2} f^2 |\nabla f|^2 \\ &\quad + p(|f|^2+\varepsilon^2)^{\frac{p}{2}-1} f\Delta f \\ &\geqslant p(p-1)(|f|^2+\varepsilon^2)^{\frac{p}{2}-2} f^2|\nabla f|^2 + p(|f|^2+\varepsilon^2)^{\frac{p}{2}-1} f\Delta f \\ &\geqslant \frac{4(p-1)}{p}|\nabla h_\varepsilon| + p(|f|^2+\varepsilon^2)^{\frac{p}{2}-1} f\Delta f,\end{aligned}$$

where $h_\varepsilon := (|f|^2+\varepsilon^2)^{\frac{p}{4}} - \varepsilon^{\frac{p}{2}} \in C^\infty(\mathbb{S}^N)$. Therefore

$$-p(|f|^2+\varepsilon^2)^{\frac{p}{2}-1} f\Delta f \geqslant \frac{4(p-1)}{p}|\nabla h_\varepsilon|^2 - \Delta f_\varepsilon.$$

Integrating the last inequality over $\mathbb{S}^N$ and using compactness of the sphere yields

$$-p\int_{\mathbb{S}^N} (|f|^2+\varepsilon^2)^{\frac{p}{2}-1} f\Delta f dV \geqslant \frac{4(p-1)}{p} \int_{\mathbb{S}^N} |\nabla h_\varepsilon|^2 dV. \qquad (10)$$

Applying Lemma 2 (i.e. (7) with $p=2$ and $\alpha=0$) on the right hand side of (10), we obtain

$$-\int_{\mathbb{S}^N}(|f|^2+\varepsilon^2)^{\frac{p}{2}-1}f\Delta f dV \geqslant \frac{(N-2)^2(p-1)}{p^2}\int_{\mathbb{S}^N}\frac{|h_\varepsilon|^2}{\sin^2\theta}dV - \frac{N(N-2)(p-1)}{p^2}\int_{\mathbb{S}^N}|h_\varepsilon|^2 dV.$$

Sending $\varepsilon\to 0$, we have by Lebesgue dominated convergence theorem

$$\frac{(N-2)^2(p-1)}{p^2}\int_{\mathbb{S}^N}\frac{|f|^p}{\sin^2\theta}dV \leqslant \int_{\mathbb{S}^N}|f|^{p-1}\Delta f dV + \frac{N(N-2)(p-1)}{p^2}\int_{\mathbb{S}^N}|f|^p dV.$$

By Hölder's inequality

$$\int_{\mathbb{S}^N}|f|^{p-1}\Delta f dV \leqslant \Big(\int_{\mathbb{S}^N}\frac{|f|^p}{\sin^2\theta}dV\Big)^{\frac{p-1}{p}}\Big(\int_{\mathbb{S}^N}\frac{|\Delta f|^p}{(\sin\theta)^{2-2p}}dV\Big)^{\frac{1}{p}}.$$

Therefore

$$\left(\int_{\mathbb{S}^N}\frac{|\Delta f|^p}{(\sin\theta)^{2-2p}}dV\right)^{\frac{1}{p}} \geqslant \frac{(N-2)^2(p-1)}{p^2}\left(\int_{\mathbb{S}^N}\frac{|f|^p}{\sin^2\theta}dV\right)^{\frac{1}{p}} - \frac{N(N-2)(p-1)}{p^2}\left(\int_{\mathbb{S}^N}\frac{|f|^p}{(\sin\theta)^{2-2p}}dV\right)^{\frac{1}{p}}.$$

Denoting by $B(N,p)=\Big(\frac{(N-2)^2(p-1)}{p^2}\Big)^p$ and $C(N,p)=\Big(\frac{N(N-2)(p-1)}{p^2}\Big)^p$, we arrived at (6) which is the required inequality.

The next is to prove that the constant $\Big(\frac{(N-2)^2(p-1)}{p^2}\Big)^p$ is sharp. It then suffices to show that

$$\Big(\frac{(N-2)^2(p-1)}{p^2}\Big)^p \geqslant \inf_{f\in C^\infty(\mathbb{S}^N)\setminus\{0\}}\frac{\int_{\mathbb{S}^N}\frac{|\Delta_{\mathbb{S}^N}f|^p}{(\sin\theta)^{2-2p}}dV + B\int_{\mathbb{S}^N}\frac{|f|^p}{(\sin\theta)^{2-2p}}dV}{\int_{\mathbb{S}^N}\frac{|f|^p}{\theta^2}dV}.$$

The proof is similar to [1] and we follow it closely, see also [14, 15].

Let $\varphi(t)\in[0,1]$ be the cut-off function such that $\varphi(t)=1$ for $|t|\leqslant 1$ and $\varphi(t)\equiv 0$ for $|t|>2$. Set $H(t)=1-\varphi(t)$. For sufficiently small ε, define $f_\varepsilon(\theta)=H\Big(\frac{\theta}{\varepsilon}\Big)\theta^{\frac{2-N}{p}}$ for $0<\theta\leqslant\pi$ and $f_\varepsilon(\theta)=0$ for $\theta=0$. Without loss of generality, we assume $0<\varepsilon<1/2$ and $f_\varepsilon(\theta)$ is a smooth radial function on $\mathbb{S}^N$. Let $Vol(\mathbb{S}^{N-1})$ denote the volume of the unit $(N-1)$-sphere, then we have

$$\int_{\mathbb{S}^N}\frac{f_\varepsilon^p}{\theta^2}dV \geqslant Vol\Big(\mathbb{S}^{N-1}\Big)\int_{2\varepsilon}^{\pi}\theta^{-N}(\sin\theta)^{N-1}d\theta$$

and

$$\int_{\mathbb{S}^N} \frac{f_\varepsilon^p}{(\sin\theta)^{2-2p}} dV \leqslant Vol\left(\mathbb{S}^{N-1}\right) \int_\varepsilon^\pi \theta^{2p-1} d\theta.$$

Since $f_\varepsilon(\theta)$ is radial we compute

$$\begin{aligned}
\Delta_{\mathbb{S}^N} f_\varepsilon(\theta) &= \left(\frac{d^2}{d\theta^2} + (N-1)\cot\theta \frac{d}{d\theta}\right)\left(H\left(\frac{\theta}{\varepsilon}\right)\theta^{\frac{2-N}{p}}\right) \\
&= H\left(\frac{\theta}{\varepsilon}\right)\left(\left(\frac{2-N}{p}\right)\left(\frac{2-N-p}{p}\right)\theta^{\frac{2-N-2p}{p}} + (N-1)\left(\frac{2-N}{p}\right)\theta^{\frac{2-N-p}{p}}\cot\theta\right) \\
&\quad + \frac{1}{\varepsilon}H'\left(\frac{\theta}{\varepsilon}\right)\left(\frac{2(2-N)}{p}\theta^{\frac{2-N-p}{p}} + (N-1)\theta^{\frac{2-N}{p}}\cot\theta\right) + \frac{1}{\varepsilon^2}H''\left(\frac{\theta}{\varepsilon}\right)\theta^{\frac{2-N}{p}}.
\end{aligned}$$

Hence

$$\int_{\mathbb{S}^N} \frac{|\Delta f_\varepsilon|^p}{(\sin\theta)^{2-2p}} dV \leqslant \mathrm{I} \quad + \quad \mathrm{II} \quad + \quad \mathrm{III},$$

where

$$\begin{aligned}
\mathrm{I} := Vol(\mathbb{S}^{N-1}) \int_\varepsilon^\pi H^p\left(\frac{\theta}{\varepsilon}\right)\Big|\left(\frac{2-N}{p}\right)\left(\frac{2-N-p}{p}\right)\theta^{\frac{2-N-2p}{p}} \\
+ (N-1)\left(\frac{2-N}{p}\right)\theta^{\frac{2-N-p}{p}}\cot\theta\Big|^p (\sin\theta)^{N-3+2p} d\theta \\
\leqslant Vol(\mathbb{S}^{N-1}) \int_\varepsilon^\pi \Big|\left(\frac{2-N}{p}\right)\left(\frac{2-N-p}{p}\right) \\
+ (N-1)\left(\frac{2-N}{p}\right)\cos\theta\Big|^p \theta^{2-N-p}(\sin\theta)^{N-3+2p} d\theta.
\end{aligned}$$

$$\begin{aligned}
\mathrm{II} := +Vol(\mathbb{S}^{N-1})\frac{1}{\varepsilon^p}\int_\varepsilon^{2\varepsilon} \left|H'\left(\frac{\theta}{\varepsilon}\right)\right|^p \Big|\frac{2(2-N)}{p}\theta^{\frac{2-N-2}{p}} \\
+ (N-1)\theta^{\frac{2-N}{p}}\cot\theta\Big|^p (\sin\theta)^{N-3+2p} d\theta \\
\leqslant Vol(\mathbb{S}^{N-1})\frac{1}{\varepsilon^p}\left(\max_{t\in[0,2]} H'(t)\right)^p \int_\varepsilon^{2\varepsilon} \Big|\frac{2(2-N)}{p}\sin\theta \\
+ (N-1)\theta\cos\theta\Big|^p \theta^{2-N-p}(\sin\theta)^{N-3+2p} d\theta \\
\leqslant Vol(\mathbb{S}^{N-1})\frac{1}{\varepsilon^p}\left(\max_{t\in[0,2]} H'(t)\right)^p \int_\varepsilon^{2\varepsilon} \Big|\frac{2(2-N)}{p}\theta + (N-1)\theta\Big|^p \theta^{2-N-p}\theta^{N-3+2p} d\theta \\
= Vol(\mathbb{S}^{N-1})\frac{1}{\varepsilon^p}\left(\max_{t\in[0,2]} H'(t)\right)^p \left(\frac{2(2-N)+(N-1)p}{p}\right) \int_\varepsilon^{2\varepsilon} \theta^{2p-1} d\theta.
\end{aligned}$$

$$\begin{aligned}
\text{III} &:= Vol(\mathbb{S}^{N-1})\frac{1}{\varepsilon^{2p}}\int_{\varepsilon}^{2\varepsilon}\left|H''\Big(\frac{\theta}{\varepsilon}\Big)\right|^p \theta^{2-N}(\sin\theta)^{N-3+2p}d\theta\\
&\leqslant Vol(\mathbb{S}^{N-1})\frac{1}{\varepsilon^{2p}}\Big(\max_{t\in[0,2]} H''(t)\Big)^p\int_{\varepsilon}^{2\varepsilon}\theta^{2-N}\theta^{N-3+2p}d\theta\\
&= Vol(\mathbb{S}^{N-1})\frac{1}{\varepsilon^{2p}}\Big(\max_{t\in[0,2]} H''(t)\Big)^p\int_{\varepsilon}^{2\varepsilon}\theta^{2p-1}d\theta.
\end{aligned}$$

Thus

$$\inf_{f\in C^\infty(\mathbb{S}^N)}\frac{\int_{\mathbb{S}^N}\frac{|\Delta f|^p}{(\sin\theta)^{2-2p}}dV + B\int_{\mathbb{S}^N}\frac{|f|^p}{(\sin\theta)^{2-2p}}dV}{\int_{\mathbb{S}^N}\frac{|f|^p}{\sin^2\theta}dV} \leqslant \frac{\int_{\mathbb{S}^N}\frac{|\Delta f_\varepsilon|^p}{(\sin\theta)^{2-2p}}dV + B\int_{\mathbb{S}^N}\frac{|f_\varepsilon|^p}{(\sin\theta)^{2-2p}}dV}{\int_{\mathbb{S}^N}\frac{|f_\varepsilon|^p}{\theta^2}dV}$$

$$\leqslant \frac{\text{I} \;+\; \text{II} \;+\; \text{III}}{Vol\Big(\mathbb{S}^{N-1}\Big)\int_{2\varepsilon}^{\pi}\theta^{2-N-p}(\sin\theta)^{N-1}d\theta} + \frac{B\int_{\varepsilon}^{2\varepsilon}\theta^{2p-1}d\theta}{\int_{2\varepsilon}^{\pi}\theta^{2-N-p}(\sin\theta)^{N-1}d\theta}$$

$$\leqslant \frac{\int_{\varepsilon}^{\pi}\left|\Big(\frac{2-N}{p}\Big)\Big(\frac{2-N-p}{p}\Big)+(N-1)\Big(\frac{2-N}{p}\Big)\cos\theta\right|^p\theta^{2-N-p}(\sin\theta)^{N-3+2p}d\theta}{\int_{2\varepsilon}^{\pi}\theta^{2-N-p}(\sin\theta)^{N-1}d\theta}$$

$$+\frac{\frac{1}{\varepsilon^p}\Big(\max_{t\in[0,2]}H'(t)\Big)^p\Big(\frac{2(2-N)+(N-1)p}{p}\Big)\int_{\varepsilon}^{2\varepsilon}\theta^{2p-1}d\theta}{\int_{2\varepsilon}^{\pi}\theta^{2-N-p}(\sin\theta)^{N-1}d\theta}$$
$$+\frac{\frac{1}{\varepsilon^{2p}}\Big(\max_{t\in[0,2]}H''(t)\Big)^p\int_{\varepsilon}^{2\varepsilon}\theta^{2p-1}d\theta}{\int_{2\varepsilon}^{\pi}\theta^{2-N-p}(\sin\theta)^{N-1}d\theta} + \frac{B\int_{\varepsilon}^{2\varepsilon}\theta^{2p-1}d\theta}{\int_{2\varepsilon}^{\pi}\theta^{2-N-p}(\sin\theta)^{N-1}d\theta}.$$

Passing to the limit as $\varepsilon\to 0^+$ yields

$$\begin{aligned}
&\inf_{f\in C^\infty(\mathbb{S}^N)}\frac{\int_{\mathbb{S}^N}\frac{|\Delta f|^p}{(\sin\theta)^{2-2p}}dV + B\int_{\mathbb{S}^N}\frac{|f|^p}{(\sin\theta)^{2-2p}}dV}{\int_{\mathbb{S}^N}\frac{|f|^p}{\sin^2\theta}dV}\\
&\leqslant \lim_{\varepsilon\to 0^+}\frac{\int_{\varepsilon}^{\pi}\left|\Big(\frac{2-N}{p}\Big)\Big(\frac{2-N-p}{p}\Big)+(N-1)\Big(\frac{2-N}{p}\Big)\cos\theta\right|^p\theta^{2-N-p}(\sin\theta)^{N-3+2p}d\theta}{\int_{2\varepsilon}^{\pi}\theta^{2-N-p}(\sin\theta)^{N-1}d\theta}\\
&= \lim_{\varepsilon\to 0^+}\frac{-\left|\Big(\frac{2-N}{p}\Big)\Big(\frac{2-N-p}{p}\Big)+(N-1)\Big(\frac{2-N}{p}\Big)\cos\varepsilon\right|^p(\varepsilon)^{2-N-p}(\sin\varepsilon)^{N-3+2p}}{-(2\varepsilon)^{2-N-p}(\sin 2\varepsilon)^{N-1}}\\
&= \Big(\frac{(2-N)^2(p-1)}{p^2}\Big)^p,
\end{aligned}$$

since $\lim_{\varepsilon\to 0^+}\int_{2\varepsilon}^{\pi}\theta^{2-N-p}(\sin\theta)^{N-1}d\theta\to+\infty$. Application of L'Hopital rule gives the transition from last inequality sign to the next equality sign. The proof is complete. □

4. Conclusion

In this paper, we have considered optimal L^p-Hardy-Rellich type inequalities on the N-Sphere of constant sectional curvature. We obtained improved inequalities involving the best constants in the classical Hardy inequalities for dimension $N \geqslant 3$. Our computation makes use of properties of geodesic distance, radial Laplacian and divergence theorem. We show that the constant obtained is the best possible.

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Abimbola Abolarinwa
Department of Physical Sciences
Landmark University
P. M. B. 1001, Omu-Aran, Kwara State, Nigeria
e-mail: a.abolarinwa1@gmail.com

Kamilu Rauf
Department of Mathematics
University of Ilorin
P. M. B. 1515, Ilorin, Kwara State, Nigeria
e-mail: krauf@unilorin.edu.ng