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MAXIMAL VALUES OF SYMMETRIC FUNCTIONS IN DISTANCES BETWEEN POINTS

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Abstract. In this note we find the maximal values of several symmetric functions in the variables which are the squares of distances $|z_i - z_j|^2$, $1 \leqslant i < j \leqslant d$, between some d complex points $z_1, \ldots, z_d$ in the unit disc. We compute the maximums of σ_m, for $m = 1,2,3,4$, explicitly and find the conditions on $z_1, \ldots, z_d$ under which those maximal values are attained. This problem is motivated by an inequality of Cassels (1966) and a subsequent conjecture of Alexander.

1. Introduction

Throughout, let

$$\mathbb{U} := \{|z| \leqslant 1,\ z \in \mathbb{C}\}$$

be the unit disc, and let

$$\mathbb{T} := \{|z| = 1,\ z \in \mathbb{C}\}$$

be the unit circle. For any $z_1, \ldots, z_d \in \mathbb{U}$, where $d \geqslant 2$, let

$$Z := \{|z_i - z_j|^2,\ 1 \leqslant i < j \leqslant d\} \tag{1}$$

be the list of squares of distances between the points z_i.

By Hadamard's inequality (see also [14]), the product of all $d(d-1)/2$ elements of Z does not exceed d^d, with equality iff $z_1, \ldots, z_d$ are the vertices of a regular d-gon inscribed in the circle $\mathbb{T}$. For $z_1, \ldots, z_d \in \mathbb{T}$ one can write this well-known inequality in several equivalent forms:

$$\prod_{1 \leqslant i < j \leqslant d} |z_i - z_j|^2 = \prod_{1 \leqslant i < j \leqslant d} |z_i \overline{z_j} - 1|^2 = \prod_{i \neq j} |z_i - z_j| \leqslant d^d.$$

In [5], Cassels considered a very similar product

$$\begin{aligned} P(\rho, Z) &:= \prod_{1 \leqslant i < j \leqslant d} |\rho^2 z_i - z_j|^2 = \prod_{i \neq j} |\rho^2 z_i - z_j| = \prod_{1 \leqslant i < j \leqslant d} |\rho^2 z_i \overline{z_j} - 1|^2 \\ &= \rho^{d(d-1)} \prod_{1 \leqslant i < j \leqslant d} \left((\rho - 1/\rho)^2 + |z_i - z_j|^2\right) \end{aligned}$$

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for $z_1,\dots,z_d \in \mathbb{T}$ and some fixed $\rho \geqslant 1$. The last expression shows that instead of the product of factors $|z_i - z_j|^2$, the product of the shifted factors $a + |z_i - z_j|^2$ is considered. His motivation was an application of such products to the estimates of the Mahler measure of a nonreciprocal algebraic number. (See also the subsequent papers of the author [6] and [7] on the same subject, where such products are quite useful.) Even without applications the evaluation of the maximum of the product $P(\rho,Z)$ itself seems to be a problem of interest.

Assuming that

$$\cos(\pi/d) \leqslant \frac{\rho^2}{\rho^4 - \rho^2 + 1}$$

Cassels showed that the above product $P(\rho,Z)$ also attains its maximum $(1+\rho^2+\ldots+\rho^{2d-2})^d$ iff $z_1,\dots,z_d$ are the vertices of a regular d-gon inscribed in $\mathbb{T}$.

In [1], Alexander observed that the above condition can be slightly improved (to $\cos(\pi/d) \leqslant 2\rho^2/(\rho^4+1)$) and still yields the same conclusion. Note that the range for ρ is very narrow, roughly, $1 \leqslant \rho \leqslant 1 + \pi/(2d)$ for d large, and there is a little chance that using similar methods one can get the same assertion for each $\rho \geqslant 1$. Nevertheless, in [1], Alexander conjectured that

CONJECTURE 1. *For each $\rho \geqslant 1$ we have*

$$P(\rho,Z) \leqslant (1+\rho^2+\ldots+\rho^{2d-2})^d$$

with equality attained iff $z_1,\dots,z_d$ are the vertices of a regular d-gon inscribed in $\mathbb{T}$.

Note that Z defined in (1) is a list of

$$L := \frac{d(d-1)}{2}$$

nonnegative numbers, say, $x_1,\dots,x_L$. For each m in the range $1 \leqslant m \leqslant L$, let

$$\sigma_m = \sigma_m(Z) := \sum_{1\leqslant i_1<\ldots<i_m\leqslant L} x_{i_1}\ldots x_{i_m}$$

be the mth symmetric function in the variables x_i, and let

$$s_m = s_m(Z) := \sum_{j=1}^{L} x_j^m = \sum_{1\leqslant i<j\leqslant d} |z_i - z_j|^{2m}.$$

The relation between σ_m and the power sums $s_m,\dots,s_1$ is given by the following formula (see, e.g., [13]):

$$\sigma_m = \frac{1}{m!}\begin{vmatrix} s_1 & 1 & 0 & \ldots & 0 \\ s_2 & s_1 & 2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{m-1} & s_{m-2} & s_{m-3} & \ldots & m-1 \\ s_m & s_{m-1} & s_{m-2} & \ldots & s_1 \end{vmatrix}. \tag{2}$$

Note that the expression for $P(\rho,Z)\rho^{-2L}$ can be written in the form

$$P(\rho,Z)\rho^{-2L} = \prod_{1\leqslant i<j\leqslant d}(a+|z_i-z_j|^2) = a^L + \sum_{m=1}^{L} a^{L-m}\sigma_m(Z),$$

where $a=(\rho-1/\rho)^2$. So, for any given m in the range $1\leqslant m\leqslant L$, the investigation of the maximum of $\sigma_m(Z)$, where $z_1,\dots,z_d$ all belong to $\mathbb{U}$, seems to be a natural problem.

We remark that the maximum of the sum $\sum_{1\leqslant i<j\leqslant d}|z_i-z_j|$, where $z_1,\dots,z_d\in\mathbb{U}$, has been evaluated by Fejes Tóth in [8] (see also [9]), who showed that it is attained iff $z_1,\dots,z_d$ are the vertices of a regular d-gon inscribed in $\mathbb{T}$. There is a huge literature related to maximization (or minimization) of various functions in $|z_i-z_j|$ when $z_1,\dots,z_d$ lie in a higher dimensional sphere $\mathbb{T}^d$ (energy-minimizing point configurations, so-called Thomson problem, best packing problems, etc.). One can find many references on this in the review paper [4], for instance. See also [2], [3], [11], [12], [15] for some other nice extremal problems when the points $z_1,\dots,z_d$ belong to $\mathbb{T}$ or to a sphere. (On a sphere it is nontrivial already to place 5 points so that that the mutual distance sum between those points is maximal [10].)

In our context it seems likely that the following is true:

CONJECTURE 2. *For any positive integers $d\geqslant 3$ and m, $1\leqslant m\leqslant L-1$, $L=d(d-1)/2$, and any Z as in (1) the maximum of $\sigma_m(Z)$ is attained iff $z_1,\dots,z_d\in\mathbb{T}$ and satisfy*

$$\sum_{j=1}^{d} z_j = \sum_{j=1}^{d} z_j^2 = \dots = \sum_{j=1}^{d} z_j^{\min(m,\lfloor d/2\rfloor)} = 0. \tag{3}$$

The case $m=L$ is excluded, since we already know that

$$\sigma_L(Z) = \prod_{1\leqslant i<j\leqslant d}|z_i-z_j|^2$$

attains its maximum d^d iff $z_1,\dots,z_d\in\mathbb{T}$ and are the vertices of a regular d-gon. The case $d=2$ is also excluded, because it is trivial. As the points $z_j=e^{2\pi i(j-1)/d}$, $j=1,\dots,d$, satisfy the condition (3), one can calculate the maximal value of $\sigma_m(Z)$ by inserting those points into (2) and using Lemma 2 below (where $s_1(Z),\dots,s_m(Z)$ have been evaluated).

Note that Conjecture 2 immediately implies Conjecture 1, because the points $z_1,\dots,z_d\in\mathbb{T}$ for which (3) holds for $m\geqslant\lfloor d/2\rfloor$ must be the vertices of a regular d-gon inscribed into $\mathbb{T}$. Indeed, for

$$f(z) := (z-z_1)\dots(z-z_d) = z^d + c_{d-1}z^{d-1} + \dots + c_0$$

with $z_1,\dots,z_d\in\mathbb{T}$ we have

$$c_0z^d + c_1z^{d-1} + \dots + 1 = z^d f(1/z) = c_0\overline{f}(z) = c_0(z^d + \overline{c_{d-1}}z^{d-1} + \dots + \overline{c_0}),$$

so that $c_i = c_0\overline{c_{d-i}}$ for $i=1,\dots,d-1$. Hence, (3) with $m=\lfloor d/2\rfloor$ implies not only $c_1=\dots=c_{\lfloor d/2\rfloor}=0$ but also $c_1=\dots=c_{d-1}=0$.

In this note we shall prove Conjecture 2 for $m=1,2,3$ and 4.

THEOREM 1. *Let $d \geqslant 3$ and let Z be as in (1). Then, for each m in the range $1 \leqslant m \leqslant L = d(d-1)/2$ the maximum of $\sigma_m(Z)$ is attained for $z_1, \dots, z_d \in \mathbb{T}$. Furthermore,*

(i) $\sigma_1(Z) \leqslant d^2$ *with equality iff* $z_1, \dots, z_d \in \mathbb{T}$ *and* $\sum_{j=1}^d z_j = 0$.

(ii) $\sigma_2(Z) \leqslant (d^4 - 3d^2)/2$ *with equality iff* $z_1, \dots, z_d \in \mathbb{T}$ *and*

$$\sum_{j=1}^d z_j = \sum_{j=1}^d z_j^2 = 0.$$

(iii) $\sigma_3(Z) \leqslant (d^6 - 9d^4 + 20d^2)/6$ *(for $d \geqslant 4$) with equality iff* $z_1, \dots, z_d \in \mathbb{T}$ *and*

$$\sum_{j=1}^d z_j = \sum_{j=1}^d z_j^2 = \sum_{j=1}^d z_j^3 = 0.$$

(iv) $\sigma_4(Z) \leqslant (d^8 - 18d^6 + 107d^4 - 210d^2)/24$ *(for $d \geqslant 5$) with equality iff* $z_1, \dots, z_d \in \mathbb{T}$ *and*

$$\sum_{j=1}^d z_j = \sum_{j=1}^d z_j^2 = \sum_{j=1}^d z_j^3 = \sum_{j=1}^d z_j^4 = 0. \tag{4}$$

For $d = 3$ in part (iii) the maximum of $\sigma_3(Z)$ is equal to $d^d = 27$ and is attained at the roots of $z^3 - \theta = 0$, where $\theta \in \mathbb{T}$. For $d = 4$ in part (iv) the condition (4) cannot hold, and the maximum of $\sigma_4(Z)$ is different.

In the next section we shall prove two useful lemmas. Then, in Section 3 we will conclude the proof of Theorem 1.

2. Auxiliary results

LEMMA 1. *Suppose $f_1, \dots, f_\ell$ are holomorphic functions in a bounded domain $D \subset \mathbb{C}$ and continuous up to the boundary of D. Then, the function $|f_1(z)| + \dots + |f_\ell(z)|$ attains its maximum in $\overline{D}$ on the boundary of D.*

Proof. The result is evident if f_i, $i = 1, \dots, \ell$, are all constants. Assume that at least one of the functions f_i is not a constant and that the sum $|f_1(z)| + \dots + |f_\ell(z)|$ attains its maximum at the point $z_0 \in D$. Clearly, for each $j \in \{1, \dots, \ell\}$ there exists $\zeta_j \in \mathbb{T}$ such that $f_j(z_0) = |f_j(z_0)|\zeta_j$. Consider the function

$$g(z) := \sum_{j=1}^{\ell} f_j(z)\overline{\zeta_j}.$$

It is holomorphic in D, continuous up to the boundary of D and not a constant. Furthermore, by our assumption and the definition of ζ_j and z_0, one has

$$\begin{aligned}|g(z)| &= \Big|\sum_{j=1}^{\ell} f_j(z)\overline{\zeta_j}\Big| \leqslant \sum_{j=1}^{\ell} |f_j(z)||\overline{\zeta_j}| = \sum_{j=1}^{\ell} |f_j(z)| \leqslant \sum_{j=1}^{\ell} |f_j(z_0)| \\ &= \sum_{j=1}^{\ell} f_j(z_0)\overline{\zeta_j} = g(z_0) = |g(z_0)|.\end{aligned}$$

This contradicts to the maximum modulus principle for the holomorphic function g, which is not a constant, and hence our initial assumption on z_0 were false.

LEMMA 2. *Let $k,d \in \mathbb{N}$, $d \geqslant 2$ and $z_1,\dots,z_d \in \mathbb{T}$. Then,*

$$s_k(Z) = \sum_{1\leqslant i<j\leqslant d} |z_i - z_j|^{2k} = \frac{d^2}{2}\binom{2k}{k} + \sum_{s=1}^{k}(-1)^s\binom{2k}{k-s}a_s, \tag{5}$$

where

$$a_s := \Big|\sum_{j=1}^{d} z_j^s\Big|^2 \tag{6}$$

for $s \in \mathbb{Z}$.

Proof. From $z_i, z_j \in \mathbb{T}$ it follows that

$$|z_i - z_j|^2 = (z_i - z_j)(\overline{z_i} - \overline{z_j}) = (z_i - z_j)\left(\frac{1}{z_i} - \frac{1}{z_j}\right) = -\frac{(z_i - z_j)^2}{z_i z_j}.$$

Therefore,

$$|z_i - z_j|^{2k} = (-1)^k \sum_{t=0}^{2k}(-1)^{2k-t}\binom{2k}{t} z_i^{t-k} z_j^{k-t},$$

and consequently

$$\sum_{1\leqslant i<j\leqslant d} |z_i - z_j|^{2k} = \frac{1}{2}\sum_{i,j=1}^{d} |z_i - z_j|^{2k} = \frac{1}{2}\sum_{i,j=1}^{d}\sum_{t=0}^{2k}(-1)^{k+t}\binom{2k}{t} z_i^{t-k} z_j^{k-t}.$$

By changing the summation and taking into account (6), we find that the latter expression is equal to

$$\frac{1}{2}\sum_{t=0}^{2k}(-1)^{k+t}\binom{2k}{t}\sum_{i=1}^{d} z_i^{t-k}\sum_{j=1}^{d} z_j^{k-t} = \frac{1}{2}\sum_{t=0}^{2k}(-1)^{k+t}\binom{2k}{t}a_{t-k}. \tag{7}$$

Since $a_0 = d^2$, the term corresponding to $t = k$ gives the first summand

$$\frac{d^2}{2}\binom{2k}{k}$$

on the right hand side of (5). From (6) and $z_1,\dots,z_d \in \mathbb{T}$ it follows that $a_{-s} = a_s$. Hence,

$$\sum_{t=0}^{k-1}(-1)^{k+t}\binom{2k}{t}a_{t-k} = \sum_{t=k+1}^{2k}(-1)^{k+2k-t}\binom{2k}{2k-t}a_{k-t}$$
$$= \sum_{t=k+1}^{2k}(-1)^{k+t}\binom{2k}{t}a_{t-k},$$

so the sum of all the other terms of the right hand side of (7) (corresponding to $t \neq k$) equals

$$\sum_{t=k+1}^{2k}(-1)^{k+t}\binom{2k}{t}a_{t-k} = \sum_{s=1}^{k}(-1)^s\binom{2k}{k+s}a_s = \sum_{s=1}^{k}(-1)^s\binom{2k}{k-s}a_s,$$

which is the sum on the right hand side of (5). Thus, (7) equals the right hand side of (5), as claimed.

3. Proof of Theorem 1

Fix m in the range $1 \leqslant m \leqslant L$ and i, $1 \leqslant i \leqslant d$. Notice that the mth symmetric function $\sigma_m(Z)$ is of the form $|f_1(z_i)|+\dots+|f_\ell(z_i)|$, where $f_1,\dots,f_\ell$ are polynomials in z_i. Hence, by Lemma 1, the maximum of $\sigma_m(Z)$ is attained when $z_1,\dots,z_d \in \mathbb{T}$. So, from now on, we will assume that $z_1,\dots,z_d \in \mathbb{T}$.

By (5) with $k=1$, we obtain

$$\sigma_1(Z) = s_1(Z) = d^2 - a_1. \tag{8}$$

Here, $a_1 \geqslant 0$ by (6), and hence $\sigma_1(Z) \leqslant d^2$ with equality iff $z_1,\dots,z_d \in \mathbb{T}$ and $a_1 = 0$. This proves (i).

Inserting $k=2$ into (5) we find that

$$s_2(Z) = 3d^2 - 4a_1 + a_2 = 4s_1(Z) + a_2 - d^2. \tag{9}$$

So, by (8), (9) and (2) with $m=2$,

$$2\sigma_2(Z) = s_1(Z)^2 - s_2(Z) = (d^2-a_1)^2 - 3d^2 + 4a_1 - a_2$$
$$= d^4 - 3d^2 - 2d^2a_1 + a_1^2 + 4a_1 - a_2.$$

The inequality $\sigma_2(Z) \leqslant (d^4-3d^2)/2$ is equivalent to

$$-2d^2a_1 + a_1^2 + 4a_1 - a_2 \leqslant 0. \tag{10}$$

Clearly, $0 \leqslant a_1 \leqslant d^2$ by (6). So $a_1 < 2d^2-4$ and

$$-2d^2a_1 + a_1^2 + 4a_1 = a_1(a_1 - 2d^2 + 4) \leqslant 0$$

with equality iff $a_1 = 0$. Trivially, $-a_2 \leqslant 0$ with equality iff $a_2 = 0$. This proves (10). Moreover, equality in (10) is attained iff $a_1 = a_2 = 0$. The proof of part (ii) is completed.

To prove part (iii) note that, by (5),

$$s_3(Z) = 10d^2 - 15a_1 + 6a_2 - a_3 = 15s_1(Z) + 6a_2 - a_3 - 5d^2. \tag{11}$$

Now, using (2) with $m = 3$, we obtain

$$6\sigma_3(Z) = s_1(Z)^3 - 3s_1(Z)s_2(Z) + 2s_3(Z), \tag{12}$$

which in view of (8), (9), (11) equals

$$(d^2 - a_1)^3 - 3(d^2 - a_1)(3d^2 - 4a_1 + a_2) + 20d^2 - 30a_1 + 12a_2 - 2a_3.$$

We need to show that the above expression, denoted by $F(d, a_1, a_2, a_3)$, does not exceed

$$F(d,0,0,0) = d^6 - 9d^4 + 20d^2$$

with equality iff $a_1 = a_2 = a_3 = 0$.

It is evident that $F(d, a_1, a_2, a_3) \leqslant F(d, a_1, a_2, 0)$, with equality iff $a_3 = 0$. Next, the coefficient for a_2 is $12 - 3d^2 + 3a_1$. Consider two cases: $a_1 < d^2 - 4$ and $a_1 \geqslant d^2 - 4$.

In the first case, $a_1 < d^2 - 4$, the coefficient for a_2 is negative. Consequently, $F(d, a_1, a_2, 0) \leqslant F(d, a_1, 0, 0)$ with equality iff $a_2 = 0$. It remains to show that

$$F(d, a_1, 0, 0) = (d^2 - a_1)^3 - 3(d^2 - a_1)(3d^2 - 4a_1) + 20d^2 - 30a_1$$

attains its maximum in $a_1 \in [0, d^2]$ at $a_1 = 0$. Indeed, the difference $F(d, a_1, 0, 0) - F(d, 0, 0, 0)$ is equal to

$$-a_1(a_1^2 - (3d^2 - 12)a_1 + 3d^4 - 21d^2 + 30).$$

Evidently, this is zero if $a_1 = 0$, whereas for $a_1 > 0$ the above expression is negative, because $-a_1 < 0$ and $a_1^2 - (3d^2 - 12) + 3d^4 - 21d^2 + 30 > 0$. The latter inequality is indeed true, since the discriminant of the quadratic polynomial is negative for $d \geqslant 4$:

$$(3d^2 - 12)^2 - 4(3d^4 - 21d^2 + 30) = -3((d^2 - 2)^2 - 12) < 0.$$

We now turn to the alternative case, namely, $a_1 \geqslant d^2 - 4$. Then, $0 \leqslant d^2 - a_1 \leqslant 4$, and hence $s_1(Z)^3 \leqslant (d - a_1)^3 \leqslant 64$. In view of $s_1(Z), s_2(Z), a_3 \geqslant 0$ and $0 \leqslant a_2 \leqslant d^2$, by (11) and (12), we get

$$\begin{aligned} 6\sigma_3(Z) &\leqslant s_1(Z)^3 + 2s_3(Z) \leqslant 64 + 30s_1(Z) + 12a_2 - 10d^2 \\ &\leqslant 61 + 120 + 12d^2 - 10d^2 = 181 + 2d^2. \end{aligned}$$

It is easy to see that

$$181+2d^2 < F(d,0,0,0) = d^6-9d^4+20d^2$$

for $d \geqslant 4$, so in the second case the value $F(d,0,0,0)$ of the function $6\sigma_3(Z)$ is not attained. This completes the proof of (iii).

In part (iv), by (2) with $m=4$, we obtain

$$24e_4(Z) = s_1(Z)^4 - 6s_1(Z)^2 s_2(Z) + 3s_2(Z)^2 + 8s_1(Z)s_3(Z) - 6s_4(Z).$$

By (5),

$$\begin{aligned} s_4(Z) &= 35d^2 - 56a_1 + 28a_2 - 8a_3 + a_4 \\ &= 56s_1(Z) + 28a_2 - 8a_3 + a_4 - 21d^2. \end{aligned}$$

Combining this with (9) and (11), we find that $24e_4(Z)$ equals

$$\begin{aligned} \Phi(y,a_2,a_3,a_4) = y^4 &- 6y^2(4y+a_2-d^2) + 3(4y+a_2-d^2)^2 \\ &+ 8y(15y+6a_2-a_3-5d^2) \\ &- 6(56y+28a_2-8a_3+a_4-21d^2), \end{aligned} \tag{13}$$

where

$$y := s_1(Z) = d^2 - a_1$$

by (8).

In all what follows we will show that the maximum of the function $\Phi(y,a_2,a_3,a_4)$ in the range $0 \leqslant y,a_2,a_3,a_4 \leqslant d^2$ (plus some extra restrictions coming from the inequalities $s_2(Z),s_3(Z),s_4(Z) \geqslant 0$ which we will not specify) is attained at the unique point

$$(y,a_2,a_3,a_4) = (d^2,0,0,0).$$

This would complete the proof of (iv), because

$$\Phi(d^2,0,0,0) = d^8 - 18d^6 + 107d^4 - 210d^2$$

by (13).

Clearly, the maximum of $\Phi(y,a_2,a_3,a_4)$ is attained only if $a_4 = 0$, since the only term containing a_4 is $-6a_4$. The coefficient for a_3 is $48-8y = 8(6-y)$. We will show that $y > 6$, and so in the maximum point a_3 must be zero too.

Indeed, in case $y \leqslant 6$ in view of $0 \leqslant a_2,a_3 \leqslant d^2$ and $s_2(Z),s_4(Z) \geqslant 0$ we deduce that

$$\Phi(y,a_2,a_3,0) \leqslant y^4 + 3(4y+a_2-d^2)^2 + 8y(15y+6a_2-a_3-5d^2).$$

From $0 \leqslant s_2(Z) = 4y+a_2-d^2 \leqslant 4y$ and

$$0 \leqslant s_4(Z) = 15y+6a_2-a_3-5d^2 \leqslant 15y+d^2,$$

it follows that

$$\begin{aligned}\Phi(y,a_2,a_3,0) &\leqslant y^4+48y^2+8y(15y+d^2)\\ &\leqslant 6^4+48\cdot 6^2+120\cdot 6^2+48d^2=7344+48d^2,\end{aligned}$$

which is strictly less than $d^8-18d^6+107d^4-210d^2$ for each $d\geqslant 5$. Thus, $y>6$ and so a_3 must be zero.

Inserting $a_3=a_4=0$ into Φ defined in (13) we obtain

$$\begin{aligned}\Phi(y,a_2,0,0) &= y^4-6y^2(4y+a_2-d^2)+3(4y+a_2-d^2)^2\\ &\quad +8y(15y+6a_2-5d^2)-6(56y+28a_2-21d^2).\end{aligned}$$

Here, the terms involving a_2 are

$$-6y^2a_2+3a_2^2+6a_2(4y-d^2)+48ya_2-168a_2=3a_2(a_2-2y^2+24y-2d^2-56).$$

The factor $a_2-2y^2+24y-2d^2-56$ is negative, because $d^2-a_2\geqslant 0$ and

$$\begin{aligned}2y^2-24y+2d^2+56-a_2 &\geqslant 2y^2-24y+d^2+56\\ &=2(y-6)^2+d^2-16>0\end{aligned}$$

by $d\geqslant 5$. This shows that $\Phi(y,a_2,0,0)$ attains the maximum only if $a_2=0$.

Inserting $a_2=0$ into Φ we find that

$$\begin{aligned}G(y):=\Phi(y,0,0,0) &= y^4-6y^2(4y-d^2)+3(4y-d^2)^2\\ &\quad +8y(15y-5d^2)-6(56y-21d^2)\\ &= y^4-24y^3+(6d^2+168)y^2-(64d^2+336)y+3d^4+126d^2.\end{aligned}$$

Note that

$$G'(y)=4y^3-72y^2+(12d^2+336)y-(64d^2+336)$$

and

$$G''(y)=12y^2-144y+12d^2+336=12((y-6)^2+d^2-8)>0.$$

Hence, $G'(y)$ is increasing in $\mathbb{R}$. As $G'(0)<0$, the only root y_d of $G'(y)=0$ is positive. This means that $G(y)$ is decreasing in $[0,y_d]$ and increasing in $[y_d,+\infty)$. Observe that

$$\begin{aligned}G'(d^2) &= 4d^6-72d^4+12d^4+336d^2-64d^2-336\\ &=4d^6-60d^4+272d^2-336>0\end{aligned}$$

for $d\geqslant 5$. As $G'(y)\leqslant 0$ for $0\leqslant y\leqslant y_d$, this yields $y_d<d^2$. Hence, the maximum of $G(y)$ in the interval $[0,d^2]$ is attained at one of its endpoints $y=0$ or $y=d^2$. The inequality

$$3d^4+126d^2=G(0)<G(d^2)=d^8-18d^6+107d^4-210d^2$$

is equivalent to

$$d^6 - 18d^4 + 104d^2 - 336 > 0,$$

which clearly holds for $d \geqslant 5$. Therefore, the maximum of the function G in $[0, d^2]$ is attained at the right endpoint of the interval only, that is,

$$\max_{0 \leqslant y \leqslant d^2} G(y) = G(d^2) = \Phi(d^2, 0, 0, 0).$$

This concludes the proof of (iv). The proof of Theorem 1 is completed.

Note that by the same method one can obtain similar results for linear forms in $\sigma_m(Z)$. For instance, by (8) and (9), we obtain the following identity:

$$\begin{aligned} 2(\sigma_2(Z) - k\sigma_1(Z)) &= s_1(Z)^2 - s_2(Z) - 2ks_1(Z) \\ &= (d^2 - a_1)^2 - (3d^2 - 4a_1 + a_2) - 2k(d^2 - a_1) \\ &= d^4 - (2k+3)d^2 - a_1(2d^2 - 2k - 4 - a_1) - a_2. \end{aligned}$$

From $0 \leqslant a_1, a_2 \leqslant d^2$, it is easy to see that its right hand side is less than or equal to $d^4 - (2k+3)d^2$ when $k \leqslant d^2/2 - 2$ with equality attained iff $a_1 = a_2 = 0$. (The case $(a_1, a_2) = (d^2, 0)$ for $k = d^2/2 - 2$ is impossible, since $a_1 = d^2$ yields $a_2 = d^2$ by (6).) Thus, the following statement (more general than Theorem 1 (ii)) is true:

PROPOSITION 1. *Let $d \geqslant 3$, $k \in (-\infty, d^2/2 - 2]$ and let Z be as in (1) with $z_1, \dots, z_d \in \mathbb{U}$. Then,*

$$\sigma_2(Z) - k\sigma_1(Z) \leqslant \frac{d^4 - (2k+3)d^2}{2}$$

with equality iff $z_1, \dots, z_d \in \mathbb{T}$ and $\sum_{j=1}^d z_j = \sum_{j=1}^d z_j^2 = 0$.

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