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ON JAMES TYPE CONSTANTS AND THE NORMAL STRUCTURE IN BANACH SPACES

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Abstract. In this paper, we establish the lower bounds for the weakly convergent sequence coefficient $WCS(X)$ of a Banach space X, in terms of the James type constant $J_{X,t}(\tau)$, the coefficient of weak orthogonality $\mu(X)$ and Domínguez-Benavides coefficient $R(1,X)$. By mean of these bounds, we identify some geometrical properties implying normal structure. Meanwhile, the James type constant $J_{X,t}(\tau)$, the coefficient of weak orthogonality $\mu(X)$ and Domínguez-Benavides coefficient $R(1,X)$ for the Bynum space $l_{2,\infty}$ are computed to show that our estimates are sharp.

1. Introduction

Let X and X^* be a Banach space and its dual space without the Schur property, that is, there is a weakly convergent sequence which is not norm convergent, $S_X = \{x \in X : \|x\| = 1\}$ and $B_X = \{x \in X : \|x\| \leqslant 1\}$ denote the unit sphere and the unit ball of a Banach space X. A mapping $T : C \subseteq X \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leqslant \|x - y\| \quad for \ all \quad x, y \in C.$$

A Banach space X is said to have fixed point property if every nonexpansive mapping $T : C \to C$ has a fixed point, where C is a nonempty bounded closed convex subset of X.

Recall that a Banach space X is called to be uniformly nonsquare, if there exists $\delta > 0$ such that

$$\frac{\|x+y\|}{2} \leqslant 1 - \delta \quad or \quad \frac{\|x-y\|}{2} \leqslant 1 - \delta,$$

whenever $x, y \in S_X$. A Banach space X is said to have (weak) normal structure, if for every (weakly compact)closed bounded convex subset H of X that contains more than one point, there exists a point $x_0 \in H$ such that

$$\sup\{\|x_0 - y\| : y \in H\} < \sup\{\|x - y\| : x, y \in H\}.$$

In reflexive spaces, weak normal structure and normal structure coincide. Normal structure play an important role in metric fixed point theory for nonexpansive mappings.

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It was proved by W. A. Kirk that every reflexive Banach space with normal structure has the fixed point property ([11]). Many geometrical properties in terms of some geometrical constants implying weak normal structure or normal structure have been studied([3-5,7-13,15-16, 19-22]).

2. Preliminaries

Before going to the main results, let us recall some concepts of geometrical constants which will be used in the following sections. The James type constant $J_{X,\,t}(\tau)$ and Schäffer type constant $S_{X,\,t}(\tau)$ were introduced by Takahashi in [14] as follows:

$$J_{X,\,t}(\tau) = \sup\{\mathcal{M}_t(\|x+\tau y\|, \|x-\tau y\|) : x, y \in S_X\},$$

$$S_{X,\,t}(\tau) = \inf\{\mathcal{M}_t(\|x+\tau y\|, \|x-\tau y\|) : x, y \in S_X\},$$

where $\tau \geqslant 0$, $-\infty \leqslant t \leqslant +\infty$ and $\mathcal{M}_t(a,b)$ is the generalized mean defined by

$$\mathcal{M}_t(a,b) := \left(\frac{a^t+b^t}{2}\right)^{\frac{1}{t}} (\,-\infty < t < \infty \; and \; t \neq 0),$$

$$\mathcal{M}_{-\infty}(a,b) := \lim_{t\to-\infty} \mathcal{M}_t(a,b) = \min(a,b),$$

$$\mathcal{M}_{+\infty}(a,b) := \lim_{t\to+\infty} \mathcal{M}_t(a,b) = \max(a,b),$$

$$\mathcal{M}_0(a,b) := \lim_{t\to 0} \mathcal{M}_t(a,b) = \sqrt{ab},$$

where a and b are two positive real numbers. Obviously, $J_{X,\,t}(\tau)$ includes some well known constants or modulus ([1, 7, 8, 12, 15]), such as James constant $J(X) = J_{X,-\infty}(1)$, Alonso's constant $T(X) = J_{X,\,0}(1)$, Baronti's constant $A_2(X) = J_{X,\,1}(1)$, Llorens-Fuster's constant $C_G(\tau, X) = J_{X,0}(\tau)$, Yang's modulus $\gamma(\tau) = J_{X,2}(\tau)^2$ and smooth modulus $\rho_X(\tau) = J_{X,1}(\tau) - 1$. Meanwhile, $S_{X,\,t}(\tau)$ is an extension of Schäffer constant $S(X) = S_{X,\,+\infty}(\tau)$, which also including Gao's constant $f(X) = 2S^2_{X,\,2}(1)$ as a special case [8]. Some geometric properties of Banach spaces X in terms of the constant $J_{X,\,t}(\tau)$ and $S_{X,\,t}(\tau)$ were investigated in [16-19, 21-22].

(i) X is uniformly nonsquare $\Leftrightarrow$ $J_{X,t}(\tau) < 1+\tau$ for some $0 < \tau < +\infty$.

(ii) X is uniformly nonsquare $\Leftrightarrow$ $S_{X,t}(1) > 1$ for some $t > 1$.

(iii) If X is a Banach space with $S_{X,\,t}(\tau) > \frac{1}{g(\tau)}\left(\frac{(1+2\tau-\tau^2)^t+(1+\tau^2)^t}{2}\right)^{\frac{1}{t}}$, where $g(\tau) = \frac{\tau+\sqrt{4+\tau^2}}{2}$, then X has uniform normal structure.

Meanwhile, the Jordan-von Neumann type constant $C_t(X)$ for a Banach space X is also defined in [14] as

$$C_t(X) = \sup\left\{\frac{J^2_{X,t}(\tau)}{1+\tau^2} : 0 \leqslant \tau \leqslant 1\right\}.$$

It is clear that Jordan von-Neumann type constant $C_t(X)$ contain Jordan von-Neumann constant $C_{\mathrm{NJ}}(X) = C_2(X)$ and Zbăganu constant $C_Z(X) = C_0(X)$. In particular, take $\tau = 1$ or $t = -\infty$ in the definition of $C_{\mathrm{t}}(X)$, we can get the following constants:

$$C_{\mathrm{t}}'(X) = \frac{J_{X,t}(1)^2}{2},$$

$$C_{-\infty}(X) = \sup\left\{\frac{J_{X,-\infty}(\tau)^2}{1+\tau^2} : 0 \leqslant \tau \leqslant 1\right\}.$$

Some basic properties of these constants have been studied in some papers ([14, 16, 17, 20]):

(i) Let $-\infty \leqslant t < \infty$, X is uniformly nonsquare $\Leftrightarrow$ $C_{\mathrm{t}}(X) < 2$.

(ii) $\frac{J^2(X)}{2} \leqslant C_{-\infty}(X) \leqslant C_Z(X) \leqslant C_{NJ}(X) \leqslant J(X)$ and $\frac{J^2(X)}{2} \leqslant C_{\mathrm{t}}'(X) \leqslant C_{NJ}(X) \leqslant J(X)$. Moreover, these inequalities are strict in some Banach spaces.

Another coefficient which was used to give sufficient conditions for normal structure is the coefficient of weak orthogonality $\mu(X)$, which is defined as

$$\mu(X) = \inf\{\lambda : \limsup_{n\to\infty} \|x_n + x\| \leqslant \lambda \limsup_{n\to\infty} \|x_n - x\|\},$$

where the infimum is taken over all $x \in X$ and all weakly null sequence $\{x_n\}$. It is proved that $1 \leqslant \mu(X) \leqslant 3$ for all Banach space and $\mu(X) = \mu(X^*)$ in reflexive Banach space([9]).

Let us mention another geometrical constant $R(1,X)$ was considered in the paper, which was defined by Domínguez Benavides [6] as

$$R(1,X) = \sup\{\liminf_{n\to\infty}\{\|x_n + x\|\},$$

where the supremum is taken over all $x \in X$ with $\|x\| \leqslant 1$ and all weakly null sequences $\{x_n\}$ in B_X such that

$$D[(x_n)] := \limsup_{n\to\infty} \limsup_{m\to\infty} \|x_n - x_m\| \leqslant 1.$$

The weakly convergent sequence coefficient $WCS(X)$ was defined in [3] as the supremum of the set of all numbers M with the property that for each weakly convergent sequence $\{x_n\}$, there is some y in the closed convex hull of the sequence such that

$$M \limsup_{n\to\infty} \|x_n - y\| \leqslant \limsup_{n\to\infty}\{\|x_i - x_j\| : i,j \geqslant n\}.$$

It is well known that $1 \leqslant WCS(X) \leqslant 2$, and $WCS(X) > 1$ implies X has the weakly uniformly normal structure. However, the above definition of $WCS(X)$ does not make sense if the space X has the Schur property, therefore we utilize the following equivalent formulation ([2]) in this paper:

$$WCS(X) = \inf\{\lim_{n\neq m} \|x_n - x_m\|\},$$

where the infimum is taken over all weakly null sequences $\{x_n\}$ in X such that $\lim_{n\to\infty} \|x_n\| = 1$ and $\lim_{n\neq m} \|x_n - x_m\|$ exists.

3. Main results

THEOREM 1. *Let X be a Banach space, the following inequality holds.*

$$WCS(X) \geqslant \frac{1+\frac{\tau}{\mu(X)}}{J_{X,t}(\tau)}.$$

Proof. Case 1: If $J_{X,t}(\tau)=1+\tau$, it suffices to note that $WCS(X)\geqslant 1$, $\mu(X)\geqslant 1$,

$$\frac{1+\frac{\tau}{\mu(X)}}{J_{X,t}(\tau)} = \frac{1+\frac{\tau}{\mu(X)}}{1+\tau} \leqslant \frac{1+\tau}{1+\tau} = 1.$$

In this case, our estimate is a trivial one.

Case 2: If $J_{X,t}(\tau)<1+\tau$, then X is uniformly nonsquare and therefore reflexive. Let $\{x_n\}$ be a weakly null sequence in S_X. Assume that $d=\lim_{n\neq m}\|x_n-x_m\|$ exists and consider a normalized functional sequence $\{x_n^*\}$ such that $x_n^*(x_n)=1$. Note that the reflexivity of X guarantees that there exists a $x^*\in X^*$ such that $x_n^* \xrightarrow{w^*} x^*$. Let $0<\varepsilon<1$ and choose N large enough so that $|x^*(x_N)|<\varepsilon$ and

$$d-\varepsilon < \|x_m-x_N\| < d+\varepsilon$$

for all $m>N$. By the definition of $\mu(X)$,

$$\limsup_{n\to\infty}\left\|\frac{x_m+x_N}{d+\varepsilon}\right\| \leqslant \mu(X)\limsup_{n\to\infty}\left\|\frac{x_m-x_N}{d+\varepsilon}\right\| \leqslant \mu(X).$$

Thus, we can choose $M>N$ large enough such that

(i) $|x_N^*(x_M)|<\varepsilon$;

(ii) $|(x_M^*-x^*)(x_N)|<\frac{\varepsilon}{2}$;

(iii) $\|\frac{x_N-x_M}{d+\varepsilon}\|\leqslant 1$;

(iv) $\|\frac{x_N+x_M}{d+\varepsilon}\|\leqslant \mu(X)+\varepsilon$;

then

$$|x_M^*(x_N)| \leqslant |(x_M^*-x^*)(x_N)|+|x^*(x_N)|<\varepsilon.$$

Now, denote $\mu:=\mu(X)$, let us put $x=\frac{x_N-x_M}{d+\varepsilon}$, $y=\frac{x_N+x_M}{(d+\varepsilon)(\mu+\varepsilon)}$. It follows that $x,y\in B_X$ and

$$\begin{aligned}(d+\varepsilon)\|x+\tau y\| &= \left\|(1+\frac{\tau}{(\mu+\varepsilon)})x_N-(1-\frac{\tau}{(\mu+\varepsilon)})x_M\right\| \\ &\geqslant \left(1+\frac{\tau}{(\mu+\varepsilon)}\right)x_N^*(x_N)-\left(1-\frac{\tau}{(\mu+\varepsilon)}\right)x_N^*(x_M) \\ &\geqslant 1+\frac{\tau}{(\mu+\varepsilon)}-\varepsilon,\end{aligned}$$

$$
\begin{aligned}
(d+\varepsilon)\|x-\tau y\| &= \left\|(1+\frac{\tau}{(\mu+\varepsilon)})x_M-(1-\frac{\tau}{(\mu+\varepsilon)})x_N\right\| \\
&\geqslant \left(1+\frac{\tau}{(\mu+\varepsilon)}\right)x_M^*(x_M)-\left(1-\frac{\tau}{(\mu+\varepsilon)}\right)x_M^*(x_N) \\
&\geqslant 1+\frac{\tau}{(\mu+\varepsilon)}-\varepsilon.
\end{aligned}
$$

This together with the definition of $J_{X,t}(\tau)$ give that

$$
(d+\varepsilon)J_{X,t}(\tau)\geqslant 1+\frac{\tau}{(\mu+\varepsilon)}-\varepsilon.
$$

Since the sequence $\{x_n\}$ and ε are arbitrary, we obtain the following inequality from the definition of $WCS(X)$,

$$
WCS(X)\geqslant \frac{1+\frac{\tau}{\mu(X)}}{J_{X,t}(\tau)}. \quad \square
$$

COROLLARY 1. *Let X be a Banach space with*

$$
J_{X,t}(\tau)<1+\frac{\tau}{\mu(X)}
$$

for some $\tau\geqslant 0$ and all $t\in[-\infty,+\infty)$, then X has normal structure.

Proof. Firstly, observe that $J_{X,t}(\tau)<1+\frac{\tau}{\mu(X)}\leqslant 1+\tau$, then X is reflexive, so weak normal structure coincides with normal structure, it is sufficient to prove that $WCS(X)>1$. By the assumption that $J_{X,t}(\tau)<1+\frac{\tau}{\mu(X)}$ and Theorem 1, then

$$
WCS(X)\geqslant \frac{1+\frac{\tau}{\mu(X)}}{J_{X,t}(\tau)}>1.
$$

REMARK 1. In fact, some sufficient conditions which imply normal structure in term of Schäffer type constant $S_{X,\ \tau}(t)$ have been presented in [20]. Let X be a Banach space with

$$
S_{X,\ t}(\tau)>\frac{1}{g(\tau)}\left(\frac{(1+2\tau-\tau^2)^t+(1+\tau^2)^t}{2}\right)^{\frac{1}{t}}
$$

for some $\tau\in(0,1]$, where $g(\tau)=\frac{\tau+\sqrt{4+\tau^2}}{2}$, then X has normal structure. The result can be deduced from the inequality between the Schäffer type constant $S_{X,\ t}(\tau)$ and James type constant $J_{X,t}(\tau)$,

$$
2[S_{X,t}(\tau)]^t[J_{X,t}(\tau)]^t\leqslant(1+2\tau-\tau^2)^t+(1+\tau^2)^t
$$

for all $0\leqslant\tau\leqslant 1$ and $1<t<\infty$. However, the lower bounds for the weakly convergent sequence coefficient $WCS(X)$ were given in terms of the James type constant $J_{X,t}(\tau)$ and some other classical geometrical constant in this paper. By mean of these bounds, we identify some geometrical properties implying normal structure in the following Corollaries.

COROLLARY 2. *Let X be a Banach space fails the Schur property, then X has normal structure if the constants satisfy any one of the following conditions:*

(i) $J(X) < 1 + \frac{1}{\mu(X)}$,

(ii) $\rho_X'(0) < \frac{1}{\mu(X)}$,

(iii) $C_{-\infty}(X) < 1 + \frac{1}{\mu(X)^2}$,

(iv) $C_t'(X) < \frac{(1+\frac{1}{\mu(X)})^2}{2}$.

Proof. (i)The result can be obtained by letting $t = -\infty$ and $\tau = 1$ in Corollary 1.
(ii) From $\rho_X(\tau) = J_{X,1}(\tau) - 1$ and Corollary 1, we can get the result (ii).
(iii) From the definition of Jordan-von Neumann type constant $C_t(X)$, then

$$C_t(X) \geqslant \frac{J_{X,t}^2(\tau)}{1+\tau^2}. \tag{1}$$

Take $\tau = \frac{1}{\mu(X)}$ and $t = -\infty$ in (1), we can get the result (iii) from Corollary 1.
(iv) Let $\tau = 1$ in (1), the assertions are obtained from the Corollary 1.

THEOREM 2. *Let $\tau \geqslant 0$ and $t \in [-\infty, +\infty)$, then for any Banach space X,*

$$J_{X,t}(\tau) \geqslant \frac{1}{WCS(X)} + \frac{\tau}{R(1,X) + (1 - \frac{1}{WCS(X)})}.$$

Proof. Case 1: If $J_{X,t}(\tau) = 1 + \tau$ and it suffices to note that $WCS(X) \geqslant 1$ and the following inequality

$$R(1,X) \geqslant WCS(X) - 1 + \frac{1}{WCS(X)} \geqslant \frac{1}{WCS(X)},$$

then

$$J_{X,t}(\tau) = 1 + \tau \geqslant \frac{1}{WCS(X)} + \frac{\tau}{R(1,X) + (1 - \frac{1}{WCS(X)})}.$$

In this case, the estimate is proved.

Case 2: If $J_{X,t}(\tau) < 1 + \tau$, then X is reflexive. Let $\{x_n\}$ be a weakly null sequence in S_X such that $d = \lim_{n \neq m} \|x_n - x_m\|$ exists and x_N, x_M^*, x_N^* be chosen as in Theorem 1. Note that

$$\lim_{n\neq m} \left\| \frac{x_m - x_n}{d+\varepsilon} \right\| \leqslant 1, \quad \left\| \frac{x_N}{d+\varepsilon} \right\| \leqslant 1.$$

By the definition of $R(1,X)$, we can choose $M > N$ large enough such that

(i) $x_N^*(x_M) < \varepsilon$;

(ii) $\|(x_M^* - x^*)(x_N)\| < \frac{\varepsilon}{2}$;

(iii) $\|\frac{x_N}{d+\varepsilon} + x_M\| \leqslant R(1,X) + \varepsilon$;

then

$$\|x_N + x_M\| \leqslant \|\frac{x_N}{d+\varepsilon} + x_M\| + (1 - \frac{1}{d+\varepsilon})\|x_N\| \leqslant R(1,X) + \varepsilon + (1 - \frac{1}{d+\varepsilon}),$$

$$|x_M^*(x_N)| \leqslant |(x_M^* - x^*)(x_N)| + |x^*(x_N)| < \varepsilon.$$

Now, denote $R := R(1,X)$, let us put $x = \frac{x_N - x_M}{d+\varepsilon}$, $y = \frac{x_N + x_M}{R+\varepsilon+(1-\frac{1}{d+\varepsilon})}$, it is easy to check that $x, y \in B_X$, for all $\tau \geqslant 0$

$$\begin{aligned}
\|x+\tau y\| &= \left\|\left(\frac{1}{d+\varepsilon} + \frac{\tau}{R+\varepsilon+(1-\frac{1}{d+\varepsilon+\varepsilon})}\right)x_N - \left(\frac{1}{d+\varepsilon} - \frac{\tau}{R+\varepsilon+(1-\frac{1}{d+\varepsilon})}\right)x_M\right\| \\
&\geqslant \left(\frac{1}{d+\varepsilon} + \frac{\tau}{R+\varepsilon+(1-\frac{1}{d+\varepsilon})}\right)x_N^*(x_N) - \left(\frac{1}{d+\varepsilon} - \frac{\tau}{R+\varepsilon+(1-\frac{1}{d+\varepsilon})}\right)x_N^*(x_M) \\
&\geqslant \frac{1}{d+\varepsilon} + \frac{\tau}{R+\varepsilon+(1-\frac{1}{d+\varepsilon})},
\end{aligned}$$

$$\begin{aligned}
\|x-\tau y\| &= \left\|\left(\frac{1}{d+\varepsilon} + \frac{\tau}{R+\varepsilon+(1-\frac{1}{d+\varepsilon})}\right)x_M - \left(\frac{1}{d+\varepsilon} - \frac{\tau}{R+\varepsilon+(1-\frac{1}{d+\varepsilon})}\right)x_N\right\| \\
&\geqslant \left(\frac{1}{d+\varepsilon} + \frac{\tau}{R+\varepsilon+(1-\frac{1}{d+\varepsilon})}\right)x_M^*(x_M) - \left(\frac{1}{d+\varepsilon} - \frac{\tau}{R+\varepsilon+(1-\frac{1}{d+\varepsilon})}\right)x_M^*(x_N) \\
&\geqslant \frac{1}{d+\varepsilon} + \frac{\tau}{R+\varepsilon+(1-\frac{1}{d+\varepsilon})}.
\end{aligned}$$

This together with the definition of $J_{X,t}(\tau)$ give that

$$J_{X,t}(\tau) \geqslant \frac{1}{d+\varepsilon} + \frac{\tau}{R+\varepsilon+(1-\frac{1}{d+\varepsilon})}.$$

Since the sequence $\{x_n\}$ and ε are arbitrary, we can get the following estimates from the definition of $WCS(X)$,

$$J_{X,t}(\tau) \geqslant \frac{1}{WCS(X)} + \frac{\tau}{R+(1-\frac{1}{WCS(X)})}.$$

Recently, Zuo and Tang [22] have proved the following Theorem and Corollary.

THEOREM 3. *Let $\tau \geqslant 0$ and $t \in [-\infty, +\infty)$, then for any Banach space X,*

$$J_{X,t}(\tau) \geqslant \frac{1}{WCS(X)}\left(1 + \frac{\tau}{R(1,X)}\right).$$

COROLLARY 3. *Let X be a Banach space with*

$$J_{X,t}(\tau) < 1 + \frac{\tau}{R(1,X)}$$

for some $\tau \geqslant 0$ and all $t \in [-\infty,+\infty)$, then X has normal structure.

REMARK 2. (1)Take $t = -\infty$ and $\tau = 1$ in Theorem 2, then we can get the following inequality

$$J(X) \geqslant \frac{1}{WCS(X)} + \frac{1}{R(1,X) + (1 - \frac{1}{WCS(X)})}.$$

It was proved in [13] that $J(X) \geqslant R(1,X)$, therefore

$$\begin{aligned} J(X) &\geqslant \frac{1}{WCS(X)} + \frac{1}{R(1,X) + (1 - \frac{1}{WCS(X)})}, \\ &\geqslant \frac{1}{WCS(X)} + \frac{1}{J(X) + (1 - \frac{1}{WCS(X)})}, \end{aligned}$$

which is equivalently the following inequality

$$WCS(X) \geqslant \frac{2}{2J(X) + 1 - \sqrt{5}}.$$

The result improves the Theorem 3.2 in [5], it is clear that

$$WCS(X) \geqslant \frac{2}{2J(X) + 1 - \sqrt{5}} > \frac{J(X) + 1}{[J(X)]^2},$$

provided that $J(X) < \frac{1+\sqrt{5}}{2}$.
(2)It is easy to check that

$$\frac{1}{WCS(X)} + \frac{\tau}{R(1,X) + (1 - \frac{1}{WCS(X)})} \geqslant \frac{1}{WCS(X)}\left(1 + \frac{\tau}{R(1,X)}\right).$$

The inequality is strict for the case $WCS(X) > 1$, therefore Theorem 2 improve the Theorem 3 and Corollary 3. Meanwhile, we can also get the following Corollary in [22].

COROLLARY 4. *Let X be a Banach space with*

$$C_t(X) < 1 + \frac{1}{R(1,X)^2},$$

for some $t \in [-\infty,+\infty)$, then X has normal structure.

Proof. From the definition of Jordan-von Neumann type constant $C_t(X)$, then

$$C_t(X) \geqslant \frac{J_{X,t}^2(\tau)}{1+\tau^2}. \tag{2}$$

Let $\tau = \frac{1}{R(1,X)}$ in (2), then

$$J_{X,t}(\tau) < 1 + \frac{\tau}{R(1,X)}.$$

The assertions are obtained from the Corollary 3.

REMARK 3. The Bynum space $l_{2,\infty}$, which is the space l_2 renormed according to $\|x\|_{2,\infty} = \max\{\|x^+\|, \|x^-\|\}$, where x^+ and x^- are the positive and the negative part of x, respectively, defined as $x^+(i) = \max\{x(i), 0\}$ and $x^- = x^+ - x$. In the sequence, we use the computation to conclude that the Bynum space $l_{2,\infty}$ is a limiting space for both Theorem 1, Corollary 1, Theorem 2, Theorem 3 and Corollary 3. Using the same method in [9], it is not hard to see that $J_{X,t}(\tau) = 1 + \frac{\tau}{\sqrt{2}}$, $\mu(l_{2,\infty}) = \sqrt{2}$ ([9]), $R(1,X) = \sqrt{2}$ ([6]) and $WCS(X) = 1$, then the estimates in Theorem 1, Corollary 1, Theorem 2, Theorem 3 and Corollary 3 become equality. However, the Bynum space $l_{2,\infty}$ lacks normal structure, therefore the results obtained in the paper are sharp.

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