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ON THE AREAS OF MIDPOINT POLYGONS

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Abstract. For a polygon $V_1...V_n$ in the Euclidean plane, let $V_1^1...V_n^1$ denote its midpoint polygon. By induction, its m-th midpoint polygon $V_1^m...V_n^m$ is defined to be the midpoint polygon of $V_1^{m-1}...V_n^{m-1}$. In this paper, we will give different kinds of formulas of the area of $V_1^m...V_n^m$. We will describe the limit behavior of the area as m goes to infinity, and we will determine the infimum and the supremum of the area among all convex $V_1...V_n$ with a fixed area. Some affine invariants derived from the area will also be discussed.

1. Introduction

Polygon sequences generated by performing iterative processes on an initial polygon have been studied widely, see [6], [4], [8], [3], [5] and the reference therein. A most popular one of such sequences is given by the midpoint polygons. It is also called Kasner polygon sequence, after the work [11] in 1903.

The midpoint polygon sequence and its limit (with suitable normalizations) often provide interesting figures in dynamics, geometry, as well as in topology. For example, in general case, the limit of a midpoint polygon sequence will be an affine regular polygon, which has been proved in the literature many times, see [2], [4], [9] and [7]. It is observed recently that a midpoint polygon sequence and its limit can keep to be knotted for some polygon in the 3-space [13].

An elementary problem about the ratio of the areas of a convex polygon and its midpoint polygon was posted only in later 1990's according to [14] and [1], and the various answers are obtained more recently, see [1], [10] and [12].

In this paper, we will give different kinds of formulas of the area of m-th midpoint polygons for all integers $m > 0$. We will describe the limit behavior of the area as m goes to infinity, and we will determine the infimum and the supremum of the area among all convex polygons with a fixed area. Some affine invariants derived from the area will also be discussed.

We start from some definitions. For an integer $n \geqslant 3$, let $V_1...V_n$ be a polygon in the Euclidean plane $\mathbb{E}^2$, where the vertices of $V_1...V_n$ can be repeated, namely there

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may exist $j \neq k$ such that $V_j = V_k$. The *m-th midpoint polygon* $V_1^m...V_n^m$ of $V_1...V_n$ is defined by induction: Firstly for $m=0$, let $V_j^0 = V_j$ for $1 \leqslant j \leqslant n$. Suppose that $V_1^m...V_n^m$ is defined, then let V_j^{m+1} be the midpoint of the segment $V_j^m V_{j+1}^m$, where we used the convention that $V_j = V_{j+n}$ and $V_j^m = V_{j+n}^m$ for all integers j.

DEFINITION 1.1. For a fixed right-hand coordinate system $O-xy$ of $\mathbb{E}^2$ and any two points $A=(a_1,a_2)$ and $B=(b_1,b_2)$, the operation "$\wedge$" is defined by

$$A \wedge B = \frac{1}{2}(a_1b_2 - a_2b_1).$$

It is linear in both A and B, and $A \wedge B + B \wedge A = 0$.

DEFINITION 1.2. For an integer t, the area function $\mathscr{A}_t$ is defined by

$$\mathscr{A}_t(V_1...V_n) = \sum_{j=1}^{n} V_j \wedge V_{j+t}.$$

This definition of $\mathscr{A}_t$ is independent of the choice of the origin O, and $\mathscr{A}_1(V_1...V_n)$ equals the oriented area of $V_1...V_n$ if $V_1...V_n$ is simple (Lemma 2.1). Here we call a polygon *simple* if it has no repeated vertices and its edges only intersect at vertices. For any polygon $V_1...V_n$ in $\mathbb{E}^2$, we will use $\mathscr{A}_1(V_1...V_n)$ to define its *area*.

A prime fact in the plane geometry is that $\mathscr{A}_1(V_1^1V_2^1V_3^1) = \mathscr{A}_1(V_1V_2V_3)/4$ and $\mathscr{A}_1(V_1^1V_2^1V_3^1V_4^1) = \mathscr{A}_1(V_1V_2V_3V_4)/2$. However, there is no such equality anymore when $n \geqslant 5$. We call a simple polygon *convex* if its interior angles are all smaller than π. For convex polygons we have the following inequalities (see [12]):

$$\frac{1}{2} < \frac{\mathscr{A}_1(V_1^1V_2^1V_3^1V_4^1V_5^1)}{\mathscr{A}_1(V_1V_2V_3V_4V_5)} < \frac{3}{4}, \quad \frac{1}{2} < \frac{\mathscr{A}_1(V_1^1...V_n^1)}{\mathscr{A}_1(V_1...V_n)} < 1, n \geqslant 6.$$

Moreover, the inequalities are sharp, namely any ratio between the lower bound and the upper bound can be realized by a convex polygon.

In this paper, we will describe the limit behavior of $\mathscr{A}_1(V_1^m...V_n^m)$ as m goes to infinity, and for each m we will determine the infimum and supremum of the ratio $\mathscr{A}_1(V_1^m...V_n^m)/\mathscr{A}_1(V_1...V_n)$ among all convex polygons. Our study is based on the following formulas of $\mathscr{A}_1(V_1^m...V_n^m)$.

THEOREM 1.3. *Let r be the largest integer such that $r < n/2$. For an integer k, let C_m^k be the coefficient of x^k in the expansion of $(1+x)^m$. Then $\mathscr{A}_1(V_1^m...V_n^m)$ can be given by any of the three formulas:*

$$(1) \sum_{s=1}^{r} \cos^{2m}\frac{\pi s}{n}\mathscr{A}_s^{\infty}(V_1...V_n), \quad \mathscr{A}_s^{\infty}(V_1...V_n) = n\sin\frac{2\pi s}{n}A_s \wedge B_s,$$

$$(2) \sum_{t=1}^{r} T(m,t)\mathscr{A}_t(V_1...V_n), \quad T(m,t) = \frac{2}{n}\sum_{s=1}^{n-1}\cos^{2m}\frac{\pi s}{n}\sin\frac{2\pi s}{n}\sin\frac{2\pi st}{n},$$

$$(3) \sum_{t=1}^{r} C(m,t)\mathscr{A}_t(V_1...V_n), \quad C(m,t) = \frac{1}{4^m}\sum_{k\equiv t}(C_{2m}^{m-k+1} - C_{2m}^{m-k-1}),$$

where $k \equiv t$ *means that* $k \equiv t \,(\mathrm{mod}\, n)$, *and* A_s *and* B_s *are given by*

$$A_s = \frac{2}{n}\sum_{p=1}^{n}\cos\frac{2\pi sp}{n}V_p, \quad B_s = \frac{2}{n}\sum_{q=1}^{n}\sin\frac{2\pi sq}{n}V_q.$$

Theorem 1.3(1) is an analogy of the following formula given in [13].

$$V_j^m = \frac{1}{n}\sum_{t=1}^{n}V_t + \sum_{s=1}^{r}\cos^m\frac{\pi s}{n}W_{j,s}^m, \quad W_{j,s}^m = \cos\frac{\pi s(m+2j)}{n}A_s + \sin\frac{\pi s(m+2j)}{n}B_s,$$

where $m > 0$. Even though the polygon $W_{1,s}^m \dots W_{n,s}^m$ is dependent on m, there are essentially at most two such polygons, and they have the same area (see [13]). Actually, a direct computation shows that

$$\mathscr{A}_1(W_{1,s}^m \dots W_{n,s}^m) = \mathscr{A}_s^\infty(V_1 \dots V_n), \quad \forall m > 0.$$

Analogous to the result of [13], we have the following theorem.

THEOREM 1.4. *Let* $V_1^m \dots V_n^m$ *be the* m*-th midpoint polygon of* $V_1 \dots V_n$, *then*

$$\lim_{m\to\infty}\mathscr{A}_1(V_1^m \dots V_n^m) = 0.$$

If $\mathscr{A}_1(V_1^m \dots V_n^m) \neq 0$ *for some* $m \geqslant 0$, *then there exists a smallest integer* k *such that* $1 \leqslant k < n/2$ *and* $\mathscr{A}_k^\infty(V_1 \dots V_n) \neq 0$. *Then*

$$\lim_{m\to\infty}(\cos^{2m}\frac{\pi k}{n})^{-1}\mathscr{A}_1(V_1^m \dots V_n^m) = \mathscr{A}_k^\infty(V_1 \dots V_n).$$

As a special case, if $V_1 \dots V_n$ *is convex, then* $\mathscr{A}_1^\infty(V_1 \dots V_n) \neq 0$.

Theorem 1.3(2) and Theorem 1.3(3) will be useful in estimation and concrete computation. In fact, the trigonometric coefficient $T(m,t)$ and the combinatoric coefficient $C(m,t)$ are identical (Lemma 2.5). This coefficient is zero when $m+1 < t$ and is positive when $m+1 \geqslant t$ (Lemma 2.6). If $V_1 \dots V_n$ is convex, then the area function $\mathscr{A}_t$ with $1 < t < n/2$ satisfies the sharp inequality (Proposition 4.1)

$$0 < \frac{\mathscr{A}_t(V_1 \dots V_n)}{\mathscr{A}_1(V_1 \dots V_n)} < \min\{t, n-2t\}.$$

With these results we have the following theorem.

THEOREM 1.5. *If* $V_1 \dots V_n$ *is a convex polygon with* $n \geqslant 5$, *then for* $m > 0$,

$$T(m,1) < \frac{\mathscr{A}_1(V_1^m \dots V_n^m)}{\mathscr{A}_1(V_1 \dots V_n)} < \sum_{1\leqslant t<n/2} T(m,t)\min\{t, n-2t\},$$

$$C(m,1) < \frac{\mathscr{A}_1(V_1^m \dots V_n^m)}{\mathscr{A}_1(V_1 \dots V_n)} < \sum_{1\leqslant t<n/2} C(m,t)\min\{t, n-2t\}.$$

Moreover, any ratio satisfying the inequalities can be realized by a convex polygon.

Similarly, the "limit area" $\mathscr{A}_1^\infty(V_1...V_n)$ can be presented as a linear combination of $\mathscr{A}_t(V_1...V_n)$ with positive coefficients (Proposition 3.1), and we have the following estimation of the ratio $\mathscr{A}_1^\infty(V_1...V_n)/\mathscr{A}_1(V_1...V_n)$ for convex polygons.

THEOREM 1.6. *If $V_1...V_n$ is a convex polygon with $n \geqslant 5$, then*

$$\frac{4}{n}\sin^2\frac{2\pi}{n} < \frac{\mathscr{A}_1^\infty(V_1...V_n)}{\mathscr{A}_1(V_1...V_n)} < \frac{4}{n}\sum_{1\leqslant t<n/2}\sin\frac{2\pi}{n}\sin\frac{2\pi t}{n}\min\{t,n-2t\}.$$

Moreover, any ratio satisfying the inequality can be realized by a convex polygon.

By Theorem 1.3(1), if $n=3$ or $n=4$, then $\mathscr{A}_1^\infty(V_1...V_n)=\mathscr{A}_1(V_1...V_n)$, and the ratio in Theorem 1.5 is $1/4^m$ or $1/2^m$ respectively. In Theorem 1.5, the case when $m=1$ gives the results in [12]. Note that all the above ratios of areas are affine invariants. They confine the shape of the polygon and measure the distance between polygons in certain degree. Actually, if we let $\mathscr{A}_1^m(V_1...V_n)=\mathscr{A}_1(V_1^m...V_n^m)$ for $m\geqslant 0$, then we have three kinds of functions defined on polygons, $\mathscr{A}_1^m$, $\mathscr{A}_s^\infty$, $\mathscr{A}_t$, and any ratio of two of them will give us an affine invariant.

THEOREM 1.7. *Let r be the largest integer such that $r<n/2$. Then for any given integers $k,m\geqslant 0$, any $\mathscr{A}_1^m$, $\mathscr{A}_s^\infty$, $\mathscr{A}_t$ can be presented as a linear combination of the functions in any of the following three sets:*

$$\{\mathscr{A}_1^{k+1},\mathscr{A}_1^{k+2},\ldots,\mathscr{A}_1^{k+r}\},\{\mathscr{A}_1^\infty,\mathscr{A}_2^\infty,\ldots,\mathscr{A}_r^\infty\},\{\mathscr{A}_1,\mathscr{A}_2,\ldots,\mathscr{A}_r\}.$$

Moreover, the map from the set of polygons to the r-dimensional vector space $\mathbb{R}^r$ defined by the functions in the same set is surjective.

Let $\mathscr{P}_n$ be the set of polygons $V_1...V_n$, where $\mathscr{A}_1(V_1^m...V_n^m)\neq 0$ for some $m\geqslant 0$. Then by the above theorem, the functions in any of the three sets give us a function from $\mathscr{P}_n$ onto the $(r-1)$-dimensional real projective space $\mathbb{RP}^{r-1}$, which is an affine invariant, and the three invariants differ by projective transformations.

REMARK 1.8. The midpoint polygon is also called the Kasner polygon. In [11], Kasner considered the problem whether a polygon $V_1...V_n$ can be realized as a midpoint polygon of another polygon. He showed that if n is odd, then there exists a unique polygon $U_1...U_n$ such that $U_1^1...U_n^1$ equals $V_1...V_n$; if n is even, then either there exists no such $U_1...U_n$ or there exists infinitely many such $U_1...U_n$; moreover, if n is even and there exists such $U_1...U_n$, then there exists a unique such $U_1...U_n$ such that $U_1...U_n$ can also be realized as a midpoint polygon.

By Theorem 1.7, all the possible $U_1...U_n$ have the same area. Hence any midpoint polygon belongs to a unique two sided infinite sequence of midpoint polygons with the areas determined by the given polygon (Theorem 5.1 and Corollary 5.3).

The structure of the paper is as follows.

In Section 2, we give some basic lemmas, mainly about the definition of the area function and the properties of the binomial coefficients C_m^k.

In Section 3, we give the proofs of Theorem 1.3 and Theorem 1.4.

In Section 4, we will consider the so called "weakly convex" polygons, and prove a generalized version of Theorem 1.5 and Theorem 1.6 for them.

Finally, in Section 5, we will restate Theorem 1.3 in terms of matrices, and use it to prove Theorem 1.7. Then we will give an example about the hexagons, which illustrates our main results.

2. Some preliminary facts

LEMMA 2.1. *The definition of $\mathscr{A}_t$ is independent of the choice of the origin O, and $\mathscr{A}_t(V_1...V_n) = -\mathscr{A}_t(V_n...V_1)$, where $V_n...V_1$ is obtained by reversing the order of the vertices of $V_1...V_n$. If $V_1...V_n$ is simple, then the absolute value of $\mathscr{A}_1(V_1...V_n)$ equals the area of $V_1...V_n$.*

Proof. Let V be an arbitrary point in $\mathbb{E}^2$, then

$$\begin{aligned}
&\sum_{j=1}^{n}(V_j - V)\wedge(V_{j+t}-V)\\
&=\sum_{j=1}^{n} V_j\wedge V_{j+t} - \sum_{j=1}^{n} V_j \wedge V - V\wedge\sum_{j=1}^{n} V_{j+t}\\
&=\sum_{j=1}^{n} V_j\wedge V_{j+t}.
\end{aligned}$$

Namely $\mathscr{A}_t$ is unchanged if we use V as the origin. For $V_n...V_1$, we have

$$\mathscr{A}_t(V_n...V_1) = \sum_{j=1}^{n} V_j\wedge V_{j-t} = -\sum_{j=1}^{n} V_{j-t}\wedge V_j = -\mathscr{A}_t(V_1...V_n).$$

Note that in Definition 1.1 the area of the triangle OAB is $\mathscr{A}_1(OAB)$ if OAB is anticlockwise and is $-\mathscr{A}_1(OAB)$ if OAB is clockwise. If $V_1...V_n$ is a triangle, then we can choose O to be V_1, and the result holds. For general $V_1...V_n$, we will divide it into triangles by adding suitable vertices and edges.

If $V_1...V_n$ is convex, then we add edges V_1V_j for $2<j<n$, and we have

$$\mathscr{A}_1(V_1...V_n) = \sum_{j=2}^{n-1}\mathscr{A}_1(V_1V_jV_{j+1}).$$

Otherwise, we can assume that the interior angle $\angle V_nV_1V_2 \geqslant \pi$. Then we can add a vertex V and an edge V_1V such that: V is in the interior of some edge V_jV_{j+1} with $1<j<n$, V_1V divides $\angle V_nV_1V_2$ into two angles smaller than π, and the interior of V_1V does not intersect $V_1...V_n$. This edge V_1V divides $V_1...V_n$ into two simple polygons $V_1...V_jV$ and $V_1VV_{j+1}...V_n$, and

$$\mathscr{A}_1(V_1...V_n) = \mathscr{A}_1(V_1...V_jV) + \mathscr{A}_1(V_1V...V_{j+1}V_n).$$

By induction, we can repeat this process until all the polygons are convex. Then we can divide the polygons into triangles. Since in each step the function $\mathscr{A}_1$ is additive and all the triangles will have the same orientation, anticlockwise or clockwise, we get the result by the triangle case.

LEMMA 2.2. *$C_m^k = 0$ if $k < 0$ or $k > m$, and $C_m^k = C_{m-1}^k + C_{m-1}^{k-1}$ when $m > 0$.*

Proof. It can be obtained by comparing the coefficient of x^k in the expansions of the two sides of the equality $(1+x)^m = (1+x)^{m-1} + x(1+x)^{m-1}$.

LEMMA 2.3. *Let $\varepsilon = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$ be the unit n-th root, then*

$$\sum_{s=0}^{n-1} \varepsilon^{sk} = \begin{cases} n & n \mid k \\ 0 & n \nmid k \end{cases}$$

LEMMA 2.4. *For a given integer t,*

$$\sum_{k\equiv t} C_m^k = \frac{1}{n}\sum_{s=0}^{n-1}(1+\varepsilon^s)^m \varepsilon^{-st},$$

where $k \equiv t$ means that $k \equiv t \,(\mathrm{mod}\, n)$.

Proof. By Lemma 2.2, the summation on the left side is for the k such that $k \equiv t$ and $0 \leqslant k \leqslant m$. By the definition of C_m^k, we have

$$\frac{1}{n}\sum_{s=0}^{n-1}(1+\varepsilon^s)^m \varepsilon^{-st} = \frac{1}{n}\sum_{s=0}^{n-1}\sum_{k=0}^{m} C_m^k \varepsilon^{sk}\varepsilon^{-st} = \frac{1}{n}\sum_{k=0}^{m} C_m^k \sum_{s=0}^{n-1}\varepsilon^{s(k-t)} = \sum_{k\equiv t} C_m^k.$$

For the last equality, we have used Lemma 2.3.

LEMMA 2.5. *For $m \geqslant 0$ and any integer t, $T(m,t) = C(m,t)$.*

Proof. By Lemma 2.4,

$$\begin{aligned}
C(m,t) =& \frac{1}{4^m}\sum_{k\equiv t}(C_{2m}^{m-k+1} - C_{2m}^{m-k-1}) \\
=& \frac{1}{4^m n}\sum_{s=0}^{n-1}(1+\varepsilon^s)^{2m}(\varepsilon^{-s(m-t+1)} - \varepsilon^{-s(m-t-1)}) \\
=& \frac{1}{4^m n}\sum_{s=0}^{n-1}(1+\cos\frac{2\pi s}{n} + i\sin\frac{2\pi s}{n})^{2m}\varepsilon^{s(t-m)}(\varepsilon^{-s} - \varepsilon^{s}) \\
=& \frac{1}{4^m n}\sum_{s=1}^{n-1}(2\cos^2\frac{\pi s}{n} + 2i\sin\frac{\pi s}{n}\cos\frac{\pi s}{n})^{2m}\varepsilon^{s(t-m)}(\varepsilon^{-s} - \varepsilon^{s})
\end{aligned}$$

$$=\frac{1}{n}\sum_{s=1}^{n-1}\cos^{2m}\frac{\pi s}{n}\varepsilon^{sm}\varepsilon^{s(t-m)}(\varepsilon^{-s}-\varepsilon^{s})$$
$$=\frac{2}{n}\sum_{s=1}^{n-1}\cos^{2m}\frac{\pi s}{n}\sin\frac{2\pi s}{n}\sin\frac{2\pi st}{n}=T(m,t).$$

For the last equality, we have used the fact that $C(m,t)$ is a real number.

LEMMA 2.6. *For* $m\geqslant 0$ *and* $1\leqslant t<n/2$, $C(m,t)=0$ *when* $m+1<t$, *and* $C(m,t)>0$ *when* $m+1\geqslant t$.

Proof. The term in the summation of $C(m,t)$ is nonzero only if $-m-1\leqslant k\leqslant m+1$. When $m+1<t$, we have $-m-1>-t>t-n$. Then $k\equiv t\,(\mathrm{mod}\,n)$ does not hold when $-m-1\leqslant k\leqslant m+1$. Hence $C(m,t)=0$.

Similarly when $m+1=t$, we have $-m-1=-t>t-n$. Then $k\equiv t\,(\mathrm{mod}\,n)$ holds only if $k=m+1$. Hence $C(m,t)=1/4^m>0$. Since $n\geqslant 3$, by Lemma 2.5, we have $C(m,1)=T(m,1)>0$. By Lemma 2.2, for $m>0$ we have

$$\begin{aligned}C(m,t)=&\frac{1}{4^m}\sum_{k\equiv t}(C_{2m}^{m-k+1}-C_{2m}^{m-k-1})\\=&\frac{1}{4^m}\sum_{k\equiv t}(C_{2m-1}^{m-k+1}+C_{2m-1}^{m-k}-C_{2m-1}^{m-k-1}-C_{2m-1}^{m-k-2})\\=&\frac{1}{4^m}\sum_{k\equiv t}(C_{2m-2}^{m-k+1}+2C_{2m-2}^{m-k}-2C_{2m-2}^{m-k-2}-C_{2m-2}^{m-k-3})\\=&\frac{1}{2}C(m-1,t)+\frac{1}{4^m}\sum_{k\equiv t}(C_{2m-2}^{m-k+1}-C_{2m-2}^{m-k-3})\\=&\frac{1}{2}C(m-1,t)+\frac{1}{4}C(m-1,t-1)+\frac{1}{4}C(m-1,t+1).\end{aligned}$$

Let r be the largest integer such that $r<n/2$. By the above relation and induction, we only need to show that $C(m,r)>0$ when $m+1>r$. By the above relation,

$$C(m,r)=\frac{1}{2}C(m-1,r)+\frac{1}{4}C(m-1,r-1)+\frac{1}{4}C(m-1,r+1).$$

If n is odd, then $n=2r+1$ and $T(m-1,r)+T(m-1,r+1)=0$. If n is even, then $n=2r+2$ and $T(m-1,r+1)=0$. Hence by Lemma 2.5, $C(m,r)$ is a linear combination of $C(m-1,r)$ and $C(m-1,r-1)$ with positive coefficients. Then we have $C(m,r)>0$ by induction.

3. The area formulas

By Lemma 2.5, we know that Theorem 1.3(2) and Theorem 1.3(3) are equivalent. In what follows, we will first show that Theorem 1.3(1) and Theorem 1.3(2) are equivalent. Then we will prove Theorem 1.3(1) and use it to prove Theorem 1.4.

PROPOSITION 3.1. *Let r be the largest integer such that $r < n/2$, then*

$$\mathscr{A}_s^\infty(V_1\ldots V_n) = \frac{4}{n}\sum_{t=1}^{r}\sin\frac{2\pi s}{n}\sin\frac{2\pi st}{n}\mathscr{A}_t(V_1\ldots V_n).$$

Proof. We represent $A_s\wedge B_s$ as a linear combination of $\mathscr{A}_t(V_1\ldots V_n)$, $t=1,\ldots,r$, as follows:

$$\begin{aligned}A_s\wedge B_s =&\frac{4}{n^2}\sum_{p=1}^{n}\sum_{q=1}^{n}\cos\frac{2\pi sp}{n}\sin\frac{2\pi sq}{n}V_p\wedge V_q\\ =&\frac{2}{n^2}\sum_{p=1}^{n}\sum_{q=1}^{n}(\cos\frac{2\pi sp}{n}\sin\frac{2\pi sq}{n}-\sin\frac{2\pi sp}{n}\cos\frac{2\pi sq}{n})V_p\wedge V_q\\ =&\frac{2}{n^2}\sum_{p=1}^{n}\sum_{q=1}^{n}\sin\frac{2\pi s(q-p)}{n}V_p\wedge V_q\\ =&\frac{2}{n^2}\sum_{p=1}^{n}\sum_{q=1}^{n}\sin\frac{2\pi sq}{n}V_p\wedge V_{p+q}.\end{aligned}$$

Then we divide the summation about q into two parts.

$$\begin{aligned}A_s\wedge B_s =&\frac{2}{n^2}\sum_{q=1}^{n}\sum_{p=1}^{n}\sin\frac{2\pi sq}{n}V_p\wedge V_{p+q}\\ =&\frac{2}{n^2}\sum_{q=1}^{r}\sum_{p=1}^{n}\sin\frac{2\pi sq}{n}V_p\wedge V_{p+q}+\frac{2}{n^2}\sum_{q=n-r}^{n-1}\sum_{p=1}^{n}\sin\frac{2\pi sq}{n}V_p\wedge V_{p+q}\\ =&\frac{2}{n^2}\sum_{q=1}^{r}\sum_{p=1}^{n}\sin\frac{2\pi sq}{n}V_p\wedge V_{p+q}+\frac{2}{n^2}\sum_{q=1}^{r}\sum_{p=1}^{n}\sin\frac{2\pi sq}{n}V_{p-q}\wedge V_p\\ =&\frac{4}{n^2}\sum_{q=1}^{r}\sin\frac{2\pi sq}{n}\sum_{p=1}^{n}V_p\wedge V_{p+q}.\end{aligned}$$

By the definition of $\mathscr{A}_s^\infty(V_1\ldots V_n)$, we have the result.

Proof of Theorem 1.3. By Proposition 3.1, the right side of the identity

$$n\sum_{s=1}^{r}\cos^{2m}\frac{\pi s}{n}\sin\frac{2\pi s}{n}A_s\wedge B_s = \frac{n}{2}\sum_{s=1}^{n-1}\cos^{2m}\frac{\pi s}{n}\sin\frac{2\pi s}{n}A_s\wedge B_s$$

equals Theorem 1.3(2). The left side is Theorem 1.3(1). Hence, by Lemma 2.5, the three formulas of Theorem 1.3 are identical. We only need to show that the above quantity equals $\mathscr{A}_1(V_1^m\ldots V_n^m)$.

We first show the following identity by induction.

$$V_j^m = \frac{1}{2^m}\sum_{k=0}^{m}C_m^k V_{j+k}.$$

It is clear when $m=0$. Suppose that it is true for m. For $m+1$ we have

$$\begin{aligned}
V_j^{m+1} =&\frac{1}{2}(V_j^m+V_{j+1}^m)\\
=&\frac{1}{2}\Big(\frac{1}{2^m}\sum_{k=0}^{m}C_m^kV_{j+k}+\frac{1}{2^m}\sum_{k=0}^{m}C_m^kV_{j+1+k}\Big)\\
=&\frac{1}{2^{m+1}}\Big(C_m^0V_j+\sum_{k=1}^{m}C_m^kV_{j+k}+\sum_{k=0}^{m-1}C_m^kV_{j+1+k}+C_m^mV_{j+1+m}\Big)\\
=&\frac{1}{2^{m+1}}\Big(C_{m+1}^0V_j+\sum_{k=1}^{m}C_m^kV_{j+k}+\sum_{k=1}^{m}C_m^{k-1}V_{j+k}+C_{m+1}^{m+1}V_{j+1+m}\Big)\\
=&\frac{1}{2^{m+1}}\sum_{k=0}^{m+1}C_{m+1}^kV_{j+k}.
\end{aligned}$$

For the last equality, we have used Lemma 2.2. Then we have

$$V_j^m=\frac{1}{2^m}\sum_{k=0}^{m}C_m^kV_{j+k}=\frac{1}{2^m}\sum_{t=0}^{n-1}\sum_{k\equiv t}C_m^kV_{j+k}=\frac{1}{2^m}\sum_{t=0}^{n-1}\sum_{k\equiv t}C_m^kV_{j+t}.$$

Let E_t^m denote the coefficient of V_{j+t}. By Definition 1.2, we have

$$\begin{aligned}
\mathscr{A}_1(V_1^m\ldots V_n^m)=&\sum_{j=1}^{n}V_j^m\wedge V_{j+1}^m\\
=&\sum_{j=1}^{n}\Big(\sum_{p=0}^{n-1}E_p^mV_{j+p}\Big)\wedge\Big(\sum_{q=0}^{n-1}E_q^mV_{j+1+q}\Big)\\
=&\sum_{j=1}^{n}\Big(\sum_{p=0}^{n-1}E_{p-j}^mV_p\Big)\wedge\Big(\sum_{q=0}^{n-1}E_{q-j-1}^mV_q\Big)\\
=&\sum_{p=0}^{n-1}\sum_{q=0}^{n-1}\Big(\sum_{j=1}^{n}E_{p-j}^mE_{q-j-1}^m\Big)V_p\wedge V_q.
\end{aligned}$$

By Lemma 2.4, the coefficient of $V_p\wedge V_q$ in the last sum is given by

$$\begin{aligned}
\sum_{j=1}^{n}E_{p-j}^mE_{q-j-1}^m =&\frac{1}{4^mn^2}\sum_{j=1}^{n}\sum_{s=0}^{n-1}(1+\varepsilon^s)^m\varepsilon^{-s(p-j)}\sum_{t=0}^{n-1}(1+\varepsilon^t)^m\varepsilon^{-t(q-j-1)}\\
=&\frac{1}{4^mn^2}\sum_{s=0}^{n-1}\sum_{t=0}^{n-1}(1+\varepsilon^s)^m(1+\varepsilon^t)^m\varepsilon^{-sp-tq+t}\sum_{j=1}^{n}\varepsilon^{j(s+t)}\\
=&\frac{1}{4^mn}\sum_{s=0}^{n-1}(1+\varepsilon^s)^m(1+\varepsilon^{-s})^m\varepsilon^{-sp+sq-s}
\end{aligned}$$

$$=\frac{1}{n}\sum_{s=0}^{n-1}\cos^{2m}\frac{\pi s}{n}\varepsilon^{-sp+sq-s}$$
$$=\frac{1}{n}\sum_{s=0}^{n-1}\cos^{2m}\frac{\pi s}{n}\cos\frac{2\pi s(q-p-1)}{n}.$$

For the third equality, we have used Lemma 2.3, and for the last equality, we have used the fact that this coefficient is a real number. Finally we have

$$\begin{aligned}
\mathscr{A}_1(V_1^m...V_n^m) =&\frac{1}{n}\sum_{p=0}^{n-1}\sum_{q=0}^{n-1}\sum_{s=0}^{n-1}\cos^{2m}\frac{\pi s}{n}\cos\frac{2\pi s(q-p-1)}{n}V_p\wedge V_q\\
=&\frac{1}{n}\sum_{s=0}^{n-1}\cos^{2m}\frac{\pi s}{n}\sum_{p=1}^{n}\sum_{q=1}^{n}\cos\frac{2\pi s(q-p-1)}{n}V_p\wedge V_q\\
=&\frac{1}{n}\sum_{s=1}^{n-1}\cos^{2m}\frac{\pi s}{n}\sum_{p=1}^{n}\sum_{q=1}^{n}\sin\frac{2\pi s}{n}\sin\frac{2\pi s(q-p)}{n}V_p\wedge V_q\\
=&\frac{2}{n}\sum_{s=1}^{n-1}\cos^{2m}\frac{\pi s}{n}\sin\frac{2\pi s}{n}\sum_{p=1}^{n}\sum_{q=1}^{n}\cos\frac{2\pi sp}{n}\sin\frac{2\pi sq}{n}V_p\wedge V_q\\
=&\frac{n}{2}\sum_{s=1}^{n-1}\cos^{2m}\frac{\pi s}{n}\sin\frac{2\pi s}{n}A_s\wedge B_s.
\end{aligned}$$

For the third and fourth equalities, we have used $V_p\wedge V_q=-V_q\wedge V_p$.

Proof of Theorem 1.4. First note that

$$1>\cos\frac{\pi}{n}>\cos\frac{2\pi}{n}>\cdots>\cos\frac{r\pi}{n}>0,$$

where r is the largest integer such that $r<n/2$. Hence by Theorem 1.3(1), as m goes to infinity, $\mathscr{A}_1(V_1^m...V_n^m)$ converges to 0. If $\mathscr{A}_1(V_1^m...V_n^m)\neq 0$ for some $m\geqslant 0$, then by Theorem 1.3(1) the required integer k exists, and

$$\begin{aligned}
&\lim_{m\to\infty}(\cos^{2m}\frac{\pi k}{n})^{-1}\mathscr{A}_1(V_1^m...V_n^m)\\
=&\lim_{m\to\infty}\sum_{s=k}^{r}(\cos\frac{\pi s}{n}/\cos\frac{\pi k}{n})^{2m}\mathscr{A}_s^\infty(V_1...V_n)\\
=&\mathscr{A}_k^\infty(V_1...V_n).
\end{aligned}$$

If $V_1...V_n$ is convex and $\mathscr{A}_1^\infty(V_1...V_n)=0$, then A_1 and B_1 are linearly dependent. Hence there exists $\theta\in\mathbb{R}^1$ such that

$$\sin\theta\sum_{t=1}^{n}\cos\frac{2\pi t}{n}V_t+\cos\theta\sum_{t=1}^{n}\sin\frac{2\pi t}{n}V_t=0.$$

By Lemma 2.3, we have

$$\sum_{t=1}^{n}\sin(\theta+\frac{2\pi t}{n})=\sin\theta\sum_{t=1}^{n}\cos\frac{2\pi t}{n}+\cos\theta\sum_{t=1}^{n}\sin\frac{2\pi t}{n}=0.$$

Namely the sum of the coefficients of those V_t is zero. Since $n \geqslant 3$, there exists a coefficient which is nonzero. There are integers $1 \leqslant p \leqslant q \leqslant n$ such that $\sin(\theta+\frac{2\pi t}{n})$ is positive (or negative) for $p \leqslant t \leqslant q$ and is non-positive (or non-negative) for $q < t < n+p$. Then

$$\sum_{t=p}^{q}\sin(\theta+\frac{2\pi t}{n})V_t=-\sum_{t=q+1}^{n+p-1}\sin(\theta+\frac{2\pi t}{n})V_t.$$

Multiply the equation by $(\sum_{t=p}^{q}\sin(\theta+\frac{2\pi t}{n}))^{-1}$, then all the coefficients become non-negative and the sum of the coefficients of each side is 1.

The point given by the left side is in the convex hull of $V_p...V_q$, and the point given by the right side is in the convex hull of $V_{q+1}...V_{n+p-1}$. Since $V_1...V_n$ is convex, the two convex hulls do not intersect, and we get a contradiction.

4. The area inequalities

We call a polygon *weakly convex* if it is in the boundary of its convex hull such that its edges only intersect at vertices and it is not a single point. We will first prove the following proposition about weakly convex polygons. Then we will use it to prove Theorem 4.2, which contains Theorem 1.5 and Theorem 1.6 as a part.

PROPOSITION 4.1. *If $V_1...V_n$ is a weakly convex polygon with $n \geqslant 5$, then*

$$0 \leqslant \frac{\mathscr{A}_t(V_1...V_n)}{\mathscr{A}_1(V_1...V_n)} \leqslant \min\{t, n-2t\}, \quad \forall 1 < t < \frac{n}{2}.$$

Moreover, except for the bounds, any ratio satisfying the inequality can be realized by a convex polygon, the bounds can only be realized by weakly convex polygons, and there exist two weakly convex polygons such that one realizes the lower bounds for all $1 < t < n/2$ and the other realizes the upper bounds for all $1 < t < n/2$.

Proof. **We first prove the inequality for a fixed** t. Since $V_1...V_n$ is not a single point and its edges only intersect at vertices, its convex hull has nonzero area. Then since it is in the boundary of its convex hull and its edges only intersect at vertices, it can be obtained by adding vertices to a convex polygon $U_1...U_{n'}$ and as the subscript of V_j increases it turns around $U_1...U_{n'}$ for one lap.

Suppose that among the n vertices exactly k of them coincide at a point U. We call U a point of multiplicity k. Since $U \wedge U = 0$, the quantity $\mathscr{A}_t(V_1...V_n)$ has the form $U \wedge V + W$, where V is a linear combination of the remaining $n-k$ vertices and W is the sum of the terms which do not involve U. Let U move in a line segment

AB. We have $U=(1-s)A+sB$, where $0\leqslant s\leqslant 1$. Then $\mathscr{A}_t(V_1...V_n)$ becomes a linear function of s. Hence it takes extremum when $s=0$ and $s=1$.

In what follows, we will move the vertices of $V_1...V_n$ along line segments such that $\mathscr{A}_t(V_1...V_n)$ does not increase (or does not decrease), $\mathscr{A}_1(V_1...V_n)$ does not change, and the polygon keeps to be weakly convex. When the vertices coincide, we will move them together as one point.

Case 1: If there exist vertices of $V_1...V_n$ in the interior of edges of $U_1...U_{n'}$. Suppose that U is a point of multiplicity k in the interior of U_1U_2. We can move U in two directions along U_1U_2 until it meets other vertices of $V_1...V_n$. We do the movement such that $\mathscr{A}_t(V_1...V_n)$ does not increase (or does not decrease). When U meets other vertices, it becomes a point of multiplicity k' with $k'>k$. We can repeat this process until no vertices of $V_1...V_n$ lie in the interior of edges of $U_1...U_{n'}$.

Case 2: If $n'>3$ and no vertices of $V_1...V_n$ lie in the interior of edges of $U_1...U_{n'}$. Suppose that U_2 is a point of multiplicity k. Let L be the line passing U_2 and parallel to U_1U_3. Let U_1' and U_3' be the intersections of L and the lines containing $U_{n'}U_1$ and U_3U_4 respectively. We can move U_2 in two directions along $U_1'U_3'$ until it meets the intersections. We do the movement such that $\mathscr{A}_t(V_1...V_n)$ does not increase (or does not decrease). When U_2 meets U_1' or U_3', we get a new convex polygon with $n'-1$ edges, and there exist vertices of $V_1...V_n$ in the interior of its edges.

In each case, $\mathscr{A}_1(V_1...V_n)$ does not change, and $V_1...V_n$ keeps to be weakly convex. We can do the movements in the two cases alternately, and finally we can move $V_1...V_n$ to a triangle ABC such that no vertices of $V_1...V_n$ lie in the interior of its edges. Assume that $V_1...V_n$ is moved to $W_1...W_n$, where $W_1,\dots,W_a$ coincide at A, $W_{a+1},\dots,W_{a+b}$ coincide at B, $W_{a+b+1},...,W_{a+b+c}$ coincide at C, and $a+b+c=n$. Then we only need to prove

$$0\leqslant\frac{\mathscr{A}_t(W_1...W_n)}{\mathscr{A}_1(W_1...W_n)}\leqslant\min\{t,n-2t\}.$$

In the summation formula of $\mathscr{A}_t(W_1...W_n)$ we call $A\wedge B$, $B\wedge C$, $C\wedge A$ the positive terms, and $B\wedge A$, $C\wedge B$, $A\wedge C$ the negative terms. By Lemma 2.1, $\mathscr{A}_1(W_1...W_n)$ and $\mathscr{A}_t(W_1...W_n)$ are independent of the choice of the origin O. If $O=A$, then $\mathscr{A}_1(W_1...W_n)=B\wedge C$, and each $W_j\wedge W_{j+t}$ is either zero, or $B\wedge C$, or $C\wedge B$. Suppose that there are p terms of $B\wedge C$ and q terms of $C\wedge B$. Then

$$\frac{\mathscr{A}_t(W_1...W_n)}{\mathscr{A}_1(W_1...W_n)}=p-q.$$

Proof of the lower bound: If there is a negative term $W_j\wedge W_{j+t}=B\wedge A$, namely $W_j=B$ and $W_{j+t}=A$, then $c<t$. Hence, if there is a negative term for each edge of ABC, then $a,b,c<t$. Then since $t<n/2$, for any negative term $W_j\wedge W_{j+t}$, its "next" term $W_{j+t}\wedge W_{j+2t}$ is a positive term for the same edge. Hence $p\geqslant q$. Otherwise, we can assume that there exists no negative term for BC, and we also have $p\geqslant q$. Hence we always have $p-q\geqslant 0$.

Proof of the upper bound: Since $p\leqslant t$, we have $p-q\leqslant t$. We need to show that $n-2t$ is also an upper bound. If there are two of a, b, c bigger than t, we can assume that $a,b>t$. Then $c<n-2t$, and $p<n-2t$. Hence $p-q<n-2t$. If there

are two of a, b, c having the sum not bigger than t, we can assume that $b+c \leqslant t$. Then $p=q=0$. Below we assume that $b,c \leqslant t$, and the sum of any two of a, b, c is bigger than t. Then $p=b+c-t$. If $a>t$, then $q=0$, and we have

$$p-q=(b+c-t)-0=n-a-t<n-2t.$$

If $a \leqslant t$, then $q=t-a$, and we have

$$p-q=(b+c-t)-(t-a)=n-2t.$$

Then we consider the realization problem. By the above discussion, when $b=c=1$, we have $b+c \leqslant t$ for $1<t<n/2$. Then $p-q=0$ for $1<t<n/2$. On the other hand, let $[x]$ denote the largest integer which is not bigger than x, then when $[n/3] \leqslant a \leqslant b \leqslant c \leqslant [n/3]+1$, we have $p-q=\min\{t,n-2t\}$ for $1<t<n/2$. Actually, if $t \leqslant [n/3]$, then $p=t$ and $q=0$. Hence $p-q=t$. If $t \geqslant [n/3]+1$, then $a,b,c \leqslant t$. Since $n \geqslant 5$, the sum of any two of a, b, c is bigger than $n/2$, which is bigger than t. Hence by the above discussion, $p-q=n-2t$.

Hence there exist two weakly convex polygons such that one realizes the lower bounds for $1<t<n/2$ and the other realizes the upper bounds for $1<t<n/2$. Below we show that the bounds can not be realized by convex polygons, and any other ratios satisfying the inequality can be realized by convex polygons.

If $V_1...V_n$ is convex, then the first movement belongs to Case 2. If $V_{j-t}V_{j+t}$ is not parallel to $V_{j-1}V_{j+1}$, then we can move V_j, and $\mathscr{A}_t(V_1...V_n)$ will change after the first movement. If $V_{j-t}V_{j+t}$ is parallel to $V_{j-1}V_{j+1}$ for all j, then $\mathscr{A}_t(V_1...V_n)$ will change after the second movement, which belongs to Case 1. Hence in each case $\mathscr{A}_t(V_1...V_n)$ will decrease (or increase) when we move $V_1...V_n$ to a triangle, and it can not equal any of the bounds.

To realize the possible ratios, consider a convex polygon $V_1...V_n$ inscribed in a circle. We can move its vertices along the circle such that no vertices coincide. Then the result follows from the facts that $V_1...V_n$ can be moved to converge to any weakly convex polygon $W_1...W_n$ inscribed in the circle, $\mathscr{A}_t$ are continuous functions for $1 \leqslant t<n/2$, and the bounds can be realized by weakly convex polygons.

THEOREM 4.2. *If $V_1...V_n$ is a weakly convex polygon with $n \geqslant 5$, then for $m>0$,*

$$T(m,1) \leqslant \frac{\mathscr{A}_1(V_1^m...V_n^m)}{\mathscr{A}_1(V_1...V_n)} \leqslant \sum_{1 \leqslant t<n/2} T(m,t)\min\{t,n-2t\},$$

$$C(m,1) \leqslant \frac{\mathscr{A}_1(V_1^m...V_n^m)}{\mathscr{A}_1(V_1...V_n)} \leqslant \sum_{1 \leqslant t<n/2} C(m,t)\min\{t,n-2t\},$$

$$\frac{4}{n}\sin^2\frac{2\pi}{n} \leqslant \frac{\mathscr{A}_1^\infty(V_1...V_n)}{\mathscr{A}_1(V_1...V_n)} \leqslant \frac{4}{n}\sum_{1 \leqslant t<n/2} \sin\frac{2\pi}{n}\sin\frac{2\pi t}{n}\min\{t,n-2t\}.$$

Moreover, except for the bounds, any ratio satisfying the inequalities can be realized by a convex polygon, the bounds can only be realized by weakly convex polygons, and there exist two weakly convex polygons such that one realizes all the lower bounds and the other realizes all the upper bounds.

Proof. By Theorem 1.3, Proposition 3.1, Proposition 4.1, Lemma 2.6, Lemma 2.5, we have the inequalities. The two weakly convex polygons realizing the bounds in Proposition 4.1 realize all the lower bounds and the upper bounds. Since $n \geqslant 5$ and $m > 0$, by Lemma 2.5 and Lemma 2.6, the coefficients $T(m,2)$ and $C(m,2)$ are positive. Then since the bounds in Proposition 4.1 can not be realized by convex polygons, the bounds in the theorem can not be realized by convex polygons. Finally, by the last paragraph in the proof of Proposition 4.1, all other ratios satisfying the inequalities can be realized by convex polygons.

COROLLARY 4.3. *If $V_1...V_n$ is weakly convex, then $\mathscr{A}_1^\infty(V_1...V_n) \neq 0$.*

5. The matrix forms of the area formulas

By the recursion formula given in the proof of Lemma 2.6, we can reformulate Theorem 1.3 in terms of matrices in the following Theorem 5.1. It gives a more clear picture of the relations between the functions $\mathscr{A}_1^m$, $\mathscr{A}_s^\infty$, $\mathscr{A}_t$ where $m \geqslant 0$, $1 \leqslant s,t < n/2$. We will use it to prove Theorem 1.7.

THEOREM 5.1. *Let r be the largest integer such that $r < n/2$. Then for any integer $k \geqslant 0$, the functions $\mathscr{A}_1^m$, $\mathscr{A}_s^\infty$, $\mathscr{A}_t$ where $m \geqslant 0$, $1 \leqslant s,t < n/2$ satisfy:*

$$(1)\quad \begin{pmatrix} \mathscr{A}_1^k \\ \mathscr{A}_1^{k+1} \\ \vdots \\ \mathscr{A}_1^{k+r-1} \end{pmatrix} = \begin{pmatrix} \cos^{2k}\frac{\pi}{n} & \cos^{2k}\frac{2\pi}{n} & \cdots & \cos^{2k}\frac{r\pi}{n} \\ \cos^{2(k+1)}\frac{\pi}{n} & \cos^{2(k+1)}\frac{2\pi}{n} & \cdots & \cos^{2(k+1)}\frac{r\pi}{n} \\ \vdots & \vdots & & \vdots \\ \cos^{2(k+r-1)}\frac{\pi}{n} & \cos^{2(k+r-1)}\frac{2\pi}{n} & \cdots & \cos^{2(k+r-1)}\frac{r\pi}{n} \end{pmatrix} \begin{pmatrix} \mathscr{A}_1^\infty \\ \mathscr{A}_2^\infty \\ \vdots \\ \mathscr{A}_r^\infty \end{pmatrix}$$

$$(2)\quad \begin{pmatrix} \mathscr{A}_1^k \\ \mathscr{A}_1^{k+1} \\ \vdots \\ \mathscr{A}_1^{k+r-1} \end{pmatrix} = \begin{pmatrix} C(0,1) & C(0,2) & \cdots & C(0,r) \\ C(1,1) & C(1,2) & \cdots & C(1,r) \\ \vdots & \vdots & & \vdots \\ C(r-1,1) & C(r-1,2) & \cdots & C(r-1,r) \end{pmatrix} M_r^k \begin{pmatrix} \mathscr{A}_1 \\ \mathscr{A}_2 \\ \vdots \\ \mathscr{A}_r \end{pmatrix}$$

where when $n \geqslant 5$, M_r is given by the left matrix below if n is odd, and is given by the right matrix below if n is even; when $n=3$, $M_r = 1/4$; when $n=4$, $M_r = 1/2$.

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{4} & & & \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ & & & \frac{1}{4} & \frac{1}{4} \end{pmatrix}_{r\times r} \qquad \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & & & \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ & & & \frac{1}{4} & \frac{1}{2} \end{pmatrix}_{r\times r}$$

REMARK 5.2. We can also reformulate Proposition 3.1 in terms of matrices, which gives the relation between $\mathscr{A}_s^\infty$ and $\mathscr{A}_t$, where $1 \leqslant s,t < n/2$.

Proof of Theorem 1.7. The transformation matrix in Theorem 5.1(1) equals

$$\begin{pmatrix} 1 & \cdots & 1 \\ \cos^2\frac{\pi}{n} & \cdots & \cos^2\frac{r\pi}{n} \\ \vdots & & \vdots \\ \cos^{2(r-1)}\frac{\pi}{n} & \cdots & \cos^{2(r-1)}\frac{r\pi}{n} \end{pmatrix} \begin{pmatrix} \cos^{2k}\frac{\pi}{n} & & & \\ & \cos^{2k}\frac{2\pi}{n} & & \\ & & \ddots & \\ & & & \cos^{2k}\frac{r\pi}{n} \end{pmatrix}$$

which is the product of a Vandermonde matrix and a diagonal matrix. Since

$$1 > \cos\frac{\pi}{n} > \cos\frac{2\pi}{n} > \cdots > \cos\frac{r\pi}{n} > 0,$$

its determinant is nonzero. By Lemma 2.6, the determinant of the transformation matrix in Theorem 5.1(2) is nonzero if the determinant of M_r is nonzero when $n \geqslant 5$. Let D_r denote the determinant of M_r, then we have

$$D_r = \frac{1}{2}D_{r-1} - \frac{1}{16}D_{r-2}, \quad \forall r \geqslant 3.$$

Hence

$$4^r D_r - 4^{r-1}D_{r-1} = 4^{r-1}D_{r-1} - 4^{r-2}D_{r-2} = \cdots = 4^2 D_2 - 4D_1,$$

and we have

$$4^r D_r - 4D_1 = (r-1)(4^2 D_2 - 4D_1).$$

If n is odd, then $D_1 = 1/4$ and $D_2 = 1/16$, hence $D_r = 1/4^r$. If n is even, then $D_1 = 1/2$ and $D_2 = 3/16$, hence $D_r = (r+1)/4^r$. In each case, the determinant of M_r is nonzero. Hence the transformation matrices in Theorem 5.1 are all invertible. Note that $\mathscr{A}_s^\infty = \mathscr{A}_{s+n}^\infty = \mathscr{A}_{-s}^\infty$, $\mathscr{A}_{n/2}^\infty = 0$ when n is even, and $\mathscr{A}_t = \mathscr{A}_{t+n} = -\mathscr{A}_{-t}$, we have that any $\mathscr{A}_1^m$, $\mathscr{A}_s^\infty$, $\mathscr{A}_t$ can be presented as a linear combination of the functions in any of the three sets:

$$\{\mathscr{A}_1^{k+1}, \mathscr{A}_1^{k+2}, \ldots, \mathscr{A}_1^{k+r}\}, \{\mathscr{A}_1^\infty, \mathscr{A}_2^\infty, \ldots, \mathscr{A}_r^\infty\}, \{\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_r\}.$$

To finish the proof, we only need to show that

$$V_1 \ldots V_n \mapsto (\mathscr{A}_1^\infty(V_1 \ldots V_n), \ldots, \mathscr{A}_r^\infty(V_1 \ldots V_n))$$

is a surjective map from the set of polygons to $\mathbb{R}^r$. Consider the polygon given by

$$W_{j,t} = (\cos\frac{2jt\pi}{n}, \sin\frac{2jt\pi}{n}), \quad \forall 1 \leqslant j \leqslant n,$$

where t is an integer. For $1 \leqslant s < n/2$ and $\theta \in \mathbb{R}^1$, by Lemma 2.3, we have

$$\begin{aligned}
&\frac{2}{n}\sum_{p=1}^{n}\cos\frac{2\pi sp}{n}\sin(\theta+\frac{2\pi tp}{n})+i\frac{2}{n}\sum_{q=1}^{n}\sin\frac{2\pi sq}{n}\sin(\theta+\frac{2\pi tq}{n})\\
=&\frac{2}{n}\sum_{p=1}^{n}\varepsilon^{sp}(\sin\theta\cos\frac{2\pi tp}{n}+\cos\theta\sin\frac{2\pi tp}{n})\\
=&\frac{\sin\theta}{n}\sum_{p=1}^{n}\varepsilon^{sp}(\varepsilon^{tp}+\varepsilon^{-tp})+i\frac{\cos\theta}{n}\sum_{p=1}^{n}\varepsilon^{sp}(-\varepsilon^{tp}+\varepsilon^{-tp})\\
=&\begin{cases}\sin\theta+i\cos\theta & s\equiv t\,(\mathrm{mod}\,n)\\ \sin\theta-i\cos\theta & s\equiv -t\,(\mathrm{mod}\,n)\\ 0 & \text{otherwise}\end{cases}
\end{aligned}$$

Hence let $\theta=\pi/2$ and $\theta=0$ respectively, then

$$\begin{aligned}
&\frac{2}{n}\sum_{p=1}^{n}\cos\frac{2\pi sp}{n}W_{p,t}+i\frac{2}{n}\sum_{q=1}^{n}\sin\frac{2\pi sq}{n}W_{q,t}\\
=&\begin{cases}(1,0)+i(0,1) & s\equiv t\,(\mathrm{mod}\,n)\\ (1,0)-i(0,1) & s\equiv -t\,(\mathrm{mod}\,n)\\ 0 & \text{otherwise}\end{cases}
\end{aligned}$$

Now for $x_t, y_t \in \mathbb{R}$, $1 \leqslant t \leqslant r$, consider the polygon $V_1 \ldots V_n$ given by

$$V_j=\sum_{t=1}^{r}x_tW_{j,t}+\sum_{t=1}^{r}y_tW_{j,-t},\quad \forall 1\leqslant j\leqslant n.$$

We have

$$\mathscr{A}_s^{\infty}(V_1\ldots V_n)=(x_s^2-y_s^2)n\sin\frac{2\pi s}{n},\quad \forall 1\leqslant s\leqslant r.$$

Since this can be any real number, this finishes the proof.

As we mentioned in Remark 1.8, for any polygon $V_1 \ldots V_n$ and any integer $m \geqslant 0$, the midpoint polygon $V_1^1 \ldots V_n^1$ can be realized as a $(m+1)$-st midpoint polygon of some polygon $U_1 \ldots U_n$. We can define $\mathscr{A}_1^{-m}(V_1 \ldots V_n)$ to be $\mathscr{A}_1(U_1 \ldots U_n)$. Then by Theorem 1.7, the function $\mathscr{A}_1^{-m}$ is well defined.

COROLLARY 5.3. *The formulas in Theorem 5.1 hold for any integer k.*

Proof. Let M and D be the following two matrices, respectively.

$$\begin{pmatrix}1 & \cdots & 1\\ \cos^2\frac{\pi}{n} & \cdots & \cos^2\frac{r\pi}{n}\\ \vdots & & \vdots\\ \cos^{2(r-1)}\frac{\pi}{n} & \cdots & \cos^{2(r-1)}\frac{r\pi}{n}\end{pmatrix}\qquad\begin{pmatrix}\cos^2\frac{\pi}{n} & & & \\ & \cos^2\frac{2\pi}{n} & & \\ & & \ddots & \\ & & & \cos^2\frac{r\pi}{n}\end{pmatrix}$$

For any polygon $V_1...V_n$ and any integer $k \geqslant 0$, there exists a polygon $U_1...U_n$ such that $U_1^{k+1}...U_n^{k+1}$ equals $V_1^1...V_n^1$. Then by Theorem 5.1(1),

$$\begin{pmatrix} \mathscr{A}_1^{-k}(V_1...V_n) \\ \mathscr{A}_1^{-k+1}(V_1...V_n) \\ \vdots \\ \mathscr{A}_1^{-k+r-1}(V_1...V_n) \end{pmatrix} = \begin{pmatrix} \mathscr{A}_1^{0}(U_1...U_n) \\ \mathscr{A}_1^{1}(U_1...U_n) \\ \vdots \\ \mathscr{A}_1^{r-1}(U_1...U_n) \end{pmatrix} = M \begin{pmatrix} \mathscr{A}_1^{\infty}(U_1...U_n) \\ \mathscr{A}_2^{\infty}(U_1...U_n) \\ \vdots \\ \mathscr{A}_r^{\infty}(U_1...U_n) \end{pmatrix}$$
$$= MD^{-k-1}M^{-1} \begin{pmatrix} \mathscr{A}_1^{0}(V_1^1...V_n^1) \\ \mathscr{A}_1^{1}(V_1^1...V_n^1) \\ \vdots \\ \mathscr{A}_1^{r-1}(V_1^1...V_n^1) \end{pmatrix} = MD^{-k} \begin{pmatrix} \mathscr{A}_1^{\infty}(V_1...V_n) \\ \mathscr{A}_2^{\infty}(V_1...V_n) \\ \vdots \\ \mathscr{A}_r^{\infty}(V_1...V_n) \end{pmatrix}$$

Hence Theorem 5.1(1) also holds for negative integers. Similarly Theorem 5.1(2) holds for all integers.

COROLLARY 5.4. *For any polygon $V_1...V_n$ and any integer $k \geqslant 0$,*

$$(1)\ \mathscr{A}_s^{\infty}(V_1^k...V_n^k) = \cos^{2k}\frac{\pi s}{n}\mathscr{A}_s^{\infty}(V_1...V_n), \quad \forall 1 \leqslant s < \frac{n}{2}.$$
$$(2) \begin{pmatrix} \mathscr{A}_1(V_1^k...V_n^k) \\ \mathscr{A}_2(V_1^k...V_n^k) \\ \vdots \\ \mathscr{A}_r(V_1^k...V_n^k) \end{pmatrix} = M_r^k \begin{pmatrix} \mathscr{A}_1(V_1...V_n) \\ \mathscr{A}_2(V_1...V_n) \\ \vdots \\ \mathscr{A}_r(V_1...V_n) \end{pmatrix}$$

Proof. In the proof of Corollary 5.3, let $V_1...V_n = U_1^k...U_n^k$, then we can get (1) for the polygon $U_1...U_n$. The proof of (2) is similar.

As the end of the paper, we give an example about the hexagons, which illustrates our main results. Since $n = 6$, by Theorem 1.3 and Proposition 3.1, we have

$$(1)\ \mathscr{A}_1(V_1^m...V_6^m) = \frac{3^m}{4^m}\mathscr{A}_1^{\infty}(V_1...V_6) + \frac{1}{4^m}\mathscr{A}_2^{\infty}(V_1...V_6),$$
$$(2)\ \mathscr{A}_1(V_1^m...V_6^m) = \frac{3^m+1}{2\times 4^m}\mathscr{A}_1(V_1...V_6) + \frac{3^m-1}{2\times 4^m}\mathscr{A}_2(V_1...V_6),$$
$$(3)\ \mathscr{A}_1^{\infty}(V_1...V_6) = \frac{1}{2}\mathscr{A}_1(V_1...V_6) + \frac{1}{2}\mathscr{A}_2(V_1...V_6).$$

If $\mathscr{A}_1^m(V_1...V_6) \neq 0$ for some $m \geqslant 0$, then by Theorem 1.4,

$$\lim_{m\to\infty} \frac{\mathscr{A}_1(V_1^{m+1}...V_n^{m+1})}{\mathscr{A}_1(V_1^m...V_n^m)}$$

exists. It is $3/4$ if $\mathscr{A}_1^{\infty}(V_1...V_6) \neq 0$, and is $1/4$ if $\mathscr{A}_1^{\infty}(V_1...V_6) = 0$.

For convex hexagons, by Theorem 1.5 and Theorem 1.6, for $m > 0$ we have

$$\frac{3^m+1}{2\times 4^m} < \frac{\mathscr{A}_1(V_1^m\ldots V_6^m)}{\mathscr{A}_1(V_1\ldots V_6)} < \frac{3^{m+1}-1}{2\times 4^m}, \quad \frac{1}{2} < \frac{\mathscr{A}_1^\infty(V_1\ldots V_6)}{\mathscr{A}_1(V_1\ldots V_6)} < \frac{3}{2}.$$

Moreover, any ratio satisfying the inequalities can be realized by a convex hexagon. The inequalities are derived from (2), (3) and Proposition 4.1, which says that

$$0 < \frac{\mathscr{A}_2(V_1\ldots V_6)}{\mathscr{A}_1(V_1\ldots V_6)} < 2.$$

By Theorem 1.7, each of the function pairs $\{\mathscr{A}_1^1,\mathscr{A}_1^2\}$, $\{\mathscr{A}_1^\infty,\mathscr{A}_2^\infty\}$ and $\{\mathscr{A}_1,\mathscr{A}_2\}$ defines a surjective map from the set of hexagons to $\mathbb{R}^2$, and they differ by linear transformations of $\mathbb{R}^2$. Especially, for any pair of real numbers $(x,y)\in\mathbb{R}^2$, there exists a hexagon with area x such that its midpoint hexagon has area y.

Since $n = 6$ is even, by [11], $V_1\ldots V_6$ is a midpoint hexagon if and only if

$$V_1+V_3+V_5 = V_2+V_4+V_6.$$

In this case, from any point U_1 in $\mathbb{E}^2$ we can construct a hexagon $U_1\ldots U_6$ such that $U_1^1\ldots U_6^1$ equals $V_1\ldots V_6$. By Theorem 1.7, they have the same area. Moreover, the hexagon $V_1\ldots V_6$ corresponds to a unique two sided infinite sequence of midpoint hexagons H_m, such that $H_0 = V_1\ldots V_6$ and H_{m+1} is the midpoint hexagon of H_m. By Theorem 5.1 and Corollary 5.3, the formulas (1) and (2) hold for any integer m. Then combined with Corollary 5.4, among convex H_0, we can estimate any ratio of two of $\mathscr{A}_1^k(H_m)$, $\mathscr{A}_1^\infty(H_m)$, $\mathscr{A}_2^\infty(H_m)$ and $\mathscr{A}_2(H_m)$, where k and m can be any integers. Actually, all the ratios are fractional linear functions of $\mathscr{A}_2(H_0)/\mathscr{A}_1(H_0)$.

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