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NOTE ON STRICHARTZ INEQUALITIES FOR THE WAVE EQUATION WITH POTENTIAL

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Abstract. We obtain Strichartz inequalities for the wave equation with potentials which behave like the inverse square potential $|x|^{-2}$ but might be not a radially symmetric function.

1. Introduction

Consider the following Cauchy problem for the wave equation with a potential $V(x)$:

$$\begin{cases} -\partial_t^2 u + \Delta u - V(x)u = 0, \\ u(x,0) = f(x), \\ \partial_t u(x,0) = g(x), \end{cases} \tag{1}$$

where $u : \mathbb{R}^{n+1} \to \mathbb{C}$, $V : \mathbb{R}^n \to \mathbb{C}$ and Δ is the n dimensional Laplacian.

In this paper we are concerned with the Strichartz inequalities for the wave equation (1). In the free case $V \equiv 0$, the following remarkable estimate was first obtained by Strichartz [13] in connection with Fourier restriction theory in harmonic analysis:

$$\|u\|_{L^{\frac{2(n+1)}{n-1}}(\mathbb{R}^{n+1})} \lesssim \|f\|_{\dot{H}^{1/2}(\mathbb{R}^n)} + \|g\|_{\dot{H}^{-1/2}(\mathbb{R}^n)}, \quad n \geqslant 2, \tag{2}$$

where $\dot{H}^\gamma$ denotes the homogeneous Sobolev space equipped with the norm $\|f\|_{\dot{H}^\gamma} = \|(\sqrt{-\Delta})^\gamma f\|_{L^2}$. Since then, (2) was extended to mixed norm spaces $L_t^q L_x^r$ as follows (see [10, 9] and references therein):

$$\|u\|_{L_t^q(\mathbb{R};L_x^r(\mathbb{R}^n))} \lesssim \|f\|_{\dot{H}^{1/2}(\mathbb{R}^n)} + \|g\|_{\dot{H}^{-1/2}(\mathbb{R}^n)}$$

for $q \geqslant 2$, $2 \leqslant r < \infty$,

$$\frac{2}{q} + \frac{n-1}{r} \leqslant \frac{n-1}{2} \quad \text{and} \quad \frac{1}{q} + \frac{n}{r} = \frac{n-1}{2}. \tag{3}$$

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Here the second equality in (3) is just the gap condition and the case $q = r$ recovers the classical result (2).

Now we turn to the wave equation with a potential. Several works have treated this potential perturbation of the free wave equation. In [1] the potentials satisfy the decay assumption that $V(x)$ decays like $|x|^{-4-\varepsilon}$ at infinity. In [6] this assumption is weakened to $|x|^{-3-\varepsilon}$, which is in turn improved to $|x|^{-2-\varepsilon}$ in [7]. In these papers Strichartz type estimates for the corresponding perturbed wave equations are established. But the main interest in the equation (1) comes from the case where the potential term is homogeneous of degree -2 and therefore scales exactly the same as the Laplacian. For instance, when $V(x) = a|x|^{-2}$ with a real number a, the equation (1) arises in the study of wave propagation on conic manifolds [4]. We also note that the heat flow for the operator $-\Delta + a|x|^{-2}$ has been studied in the theory of combustion [14]. In fact, the decay $|V| \sim |x|^{-2}$ was shown to be critical in [8] which concerns explicitly the Schrödinger case but can be adapted to the wave equation as well.

For the inverse square potentials $V(x) = a|x|^{-2}$ with $a > -(n-2)^2/4$, Planchon, Stalker and Tahvildar-Zadeh [11] first obtained the Strichartz inequalities for the equation (1) with radial Cauchy data f and g. Thereafter, this radially symmetric assumption was removed in [2]. More precisely, the range of the admissible exponents (q, r) for the Strichartz inequalities

$$\|u\|_{L_t^q(\mathbb{R};\dot{H}_r^\sigma(\mathbb{R}^n))} \lesssim \|f\|_{\dot{H}^{1/2}(\mathbb{R}^n)} + \|g\|_{\dot{H}^{-1/2}(\mathbb{R}^n)}$$

obtained in [11, 2] with the gap condition $\sigma = \frac{1}{q} + \frac{n}{r} - \frac{n-1}{2}$ is restricted under $\frac{2}{q} + \frac{n-1}{r} \leqslant \frac{n-1}{2}$ which is the same as that of the wave equation without potential. Here, $\|f\|_{\dot{H}_r^\sigma} = \|(\sqrt{-\Delta})^\sigma f\|_{L^r}$. In [3] these results were further extended to potentials which behave like the inverse square potential but might be not a radially symmetric function. Indeed, the potentials considered in [3] are contained in the weak space, $L^{n/2,\infty}$.

In this paper we consider the Fefferman-Phong class of potentials which is defined for $1 \leqslant p \leqslant n/2$ by

$$V \in \mathscr{F}^p \quad \Leftrightarrow \quad \|V\|_{\mathscr{F}^p} = \sup_{x \in \mathbb{R}^n, r>0} r^{2-n/p} \left(\int_{B_r(x)} |V(y)|^p dy \right)^{\frac{1}{p}} < \infty,$$

where $B_r(x)$ denotes the ball centered at x with radius r. Note that $L^{n/2} = \mathscr{F}^{n/2}$ and $a|x|^{-2} \in L^{n/2,\infty} \subsetneq \mathscr{F}^p$ if $1 \leqslant p < n/2$. Our result is the following theorem.

THEOREM 1. *Let $n \geqslant 3$. Let u be a solution to (1) with Cauchy data $(f, g) \in \dot{H}^{1/2} \times \dot{H}^{-1/2}$ and potential $V \in \mathscr{F}^p$ with small $\|V\|_{\mathscr{F}^p}$ for $p > (n-1)/2$. Then we have*

$$\|u\|_{L_t^q(\mathbb{R};\dot{H}_r^\sigma(\mathbb{R}^n))} \lesssim (1 + \|V\|_{\mathscr{F}^p}) \Big(\|f\|_{\dot{H}^{\frac{1}{2}}} + \|g\|_{\dot{H}^{-\frac{1}{2}}} \Big) \tag{4}$$

for $q > 2$, $2 \leqslant r < \infty$,

$$\frac{2}{q} + \frac{n-1}{r} \leqslant \frac{n-1}{2} \quad \textit{and} \quad \sigma = \frac{1}{q} + \frac{n}{r} - \frac{n-1}{2}. \tag{5}$$

REMARK 1. The class of potentials in the theorem is strictly larger than $L^{n/2,\infty}$. For instance, consider

$$V(x)=f(\frac{x}{|x|})|x|^{-2},\quad f\in L^p(S^{n-1}),\quad (n-1)/2<p<n/2,$$

which is in $\mathscr{F}^p$ but not in $L^{n/2,\infty}$.

Throughout this paper, the letter C stands for a positive constant which may be different at each occurrence. We also denote $A\lesssim B$ to mean $A\leqslant CB$ with unspecified constants $C>0$.

2. Proof of Theorem 1

In this section we prove the Strichartz inequalities (4) by making use of a weighted space-time L^2 estimate for the wave equation.

We first consider the potential term as a source term and then write the solution to (1) as the sum of the solution to the free wave equation plus a Duhamel term, as follows:

$$u(x,t)=\cos(t\sqrt{-\Delta})f+\frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}g+\int_0^t\frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}}\big(V(\cdot)u(\cdot,s)\big)ds. \tag{6}$$

By the classical Strichartz inequalities for the wave equation (see e.g. [9]), we see

$$\|e^{it\sqrt{-\Delta}}f\|_{L_t^q\dot{H}_r^\sigma}=\|(\sqrt{-\Delta})^\sigma e^{it\sqrt{-\Delta}}f\|_{L_t^qL_x^r}\lesssim\|f\|_{\dot{H}^{\frac12}} \tag{7}$$

for (q,r) satisfying $q\geqslant 2$, $2\leqslant r<\infty$ and the condition (5). Applying (7) to (6), we get

$$\|u\|_{L_t^q\dot{H}_r^\sigma}\lesssim\|f\|_{\dot{H}^{\frac12}}+\|g\|_{\dot{H}^{-\frac12}}+\left\|\int_0^t\frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}}\big(V(\cdot)u(\cdot,s)\big)ds\right\|_{L_t^q\dot{H}_r^\sigma}$$

for the same (q,r).

Now it remains to show that

$$\left\|\int_0^t\frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}}\big(V(\cdot)u(\cdot,s)\big)ds\right\|_{L_t^q\dot{H}_r^\sigma}\lesssim\|V\|_{\mathscr{F}^p}\big(\|f\|_{\dot{H}^{\frac12}}+\|g\|_{\dot{H}^{-\frac12}}\big)$$

for (q,r) satisfying $q>2$, $2\leqslant r<\infty$ and the condition (5). By duality, it is sufficient to show that

$$\begin{aligned}\left\langle(\sqrt{-\Delta})^\sigma\int_0^t\frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}}\big(V(\cdot)u(\cdot,s)\big)ds,G\right\rangle_{x,t}&\\ \lesssim\|V\|_{\mathscr{F}^p}\big(\|f\|_{\dot{H}^{\frac12}}+\|g\|_{\dot{H}^{-\frac12}}\big)\|G\|_{L_t^{q'}L_x^{r'}}.&\end{aligned} \tag{8}$$

The left-hand side of (8) is equivalent to

$$\begin{aligned}\int_{\mathbb{R}}\int_0^t \big\langle (\sqrt{-\Delta})^{\sigma-1}\sin((t-s)\sqrt{-\Delta})\big(V(\cdot)u(\cdot,s)\big), G\big\rangle_x dsdt \\ = \int_{\mathbb{R}}\int_0^t \big\langle Vu, (\sqrt{-\Delta})^{\sigma-1}\sin((t-s)\sqrt{-\Delta})G\big\rangle_x dsdt \\ = \left\langle V^{1/2}u, V^{1/2}(\sqrt{-\Delta})^{\sigma-1}\int_s^\infty \sin((t-s)\sqrt{-\Delta})G\,dt\right\rangle_{x,s}. \end{aligned} \tag{9}$$

Using Hölder's inequality, (9) is bounded by

$$\|u\|_{L^2_{x,s}(|V|)}\left\|(\sqrt{-\Delta})^{\sigma-1}\int_s^\infty \sin((t-s)\sqrt{-\Delta})\,G\,dt\right\|_{L^2_{x,s}(|V|)}.$$

Here, $L^2(|V|)$ denotes a weighted space equipped with the norm

$$\|h\|_{L^2_{x,t}(|V|)} = \left(\int_{\mathbb{R}^{n+1}} |h(x,t)|^2|V(x)|dxdt\right)^{\frac{1}{2}}.$$

We will show that

$$\|u\|_{L^2_{x,t}(|V|)} \lesssim \|V\|^{1/2}_{\mathscr{F}^p}\big(\|f\|_{\dot H^{\frac12}} + \|g\|_{\dot H^{-\frac12}}\big) \tag{10}$$

and

$$\left\|(\sqrt{-\Delta})^{\sigma-1}\int_t^\infty \sin((t-s)\sqrt{-\Delta})\,G\,ds\right\|_{L^2_{x,t}(|V|)} \lesssim \|V\|^{1/2}_{\mathscr{F}^p}\|G\|_{L^{q'}_t L^{r'}_x} \tag{11}$$

for (q,r) satisfying the same conditions in the theorem. Then the desired estimate (8) is proved.

To show (10), we use the following lemma which is a particular case of Proposition 2.3 and 4.2 in [12].

LEMMA 1. *Let $n \geqslant 3$. Assume that $V \in \mathscr{F}^p$ for $p > (n-1)/2$. Then we have*

$$\|\cos(t\sqrt{-\Delta})f\|_{L^2_{x,t}(|V|)} \lesssim \|V\|^{1/2}_{\mathscr{F}^p}\|(\sqrt{-\Delta})^{1/2}f\|_{L^2}, \tag{12}$$

$$\left\|\frac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}}g\right\|_{L^2_{x,t}(|V|)} \lesssim \|V\|^{1/2}_{\mathscr{F}^p}\|(\sqrt{-\Delta})^{-1/2}g\|_{L^2}, \tag{13}$$

and

$$\left\|\int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}}F(\cdot,s)\,ds\right\|_{L^2_{x,t}(|V|)} \lesssim \|V\|_{\mathscr{F}^p}\|F\|_{L^2_{x,t}(|V|^{-1})}.$$

Indeed, applying Lemma 1 to (6), we see

$$\|u\|_{L^2_{x,t}(|V|)} \lesssim \|V\|^{1/2}_{\mathscr{F}^p}\big(\|f\|_{\dot H^{\frac12}} + \|g\|_{\dot H^{-\frac12}}\big) + \|V\|_{\mathscr{F}^p}\|u\|_{L^2_{x,t}(|V|)}. \tag{14}$$

Since we are assuming that $\|V\|_{\mathscr{F}^p}$ is small enough, the last term on the right-hand side of (14) can be absorbed into the left-hand side. Hence, we get the first estimate (10). To obtain the second estimate (11), we first note that the first two estimates (12) and (13) in Lemma 1 directly imply

$$\big\|(\sqrt{-\Delta})^{\gamma}e^{it\sqrt{-\Delta}}f\big\|_{L^2_{x,t}(|V|)}\lesssim\|V\|^{1/2}_{\mathscr{F}^p}\big\|(\sqrt{-\Delta})^{\gamma+1/2}f\big\|_{L^2} \tag{15}$$

for $\gamma\in\mathbb{R}$. Using (15) and the dual estimate of (7), we then have

$$\begin{aligned}\Big\|(\sqrt{-\Delta})^{\sigma-1}\int_{\mathbb{R}}\sin((t-s)\sqrt{-\Delta})\,G\,ds\Big\|_{L^2_{x,t}(|V|)}&\\ &\lesssim\Big\|(\sqrt{-\Delta})^{\sigma-1}\int_{\mathbb{R}}e^{i(t-s)\sqrt{-\Delta}}G\,ds\Big\|_{L^2_{x,t}(|V|)}\\ &\lesssim\|V\|^{1/2}_{\mathscr{F}^p}\Big\|(\sqrt{-\Delta})^{\sigma-1/2}\int_{\mathbb{R}}e^{-is\sqrt{-\Delta}}G\,ds\Big\|_{L^2}\\ &\lesssim\|V\|^{1/2}_{\mathscr{F}^p}\|G\|_{L^{q'}_tL^{r'}_x}\end{aligned}$$

for (q,r) satisfying $q\geqslant2$, $2\leqslant r<\infty$ and the condition (5). Here we are going to use the following Christ-Kiselev lemma ([5]) to conclude that

$$\Big\|(\sqrt{-\Delta})^{\sigma-1}\int_{-\infty}^{t}\sin((t-s)\sqrt{-\Delta})\,G\,ds\Big\|_{L^2_{x,t}(|V|)}\lesssim||V||^{1/2}_{\mathscr{F}^p}||G||_{L^{q'}_tL^{r'}_x} \tag{16}$$

if $2>q'$.

LEMMA 2. *Let X and Y be two Banach spaces and let T be a bounded linear operator from $L^{\alpha}(\mathbb{R};X)$ to $L^{\beta}(\mathbb{R};Y)$ such that*

$$Tf(t)=\int_{\mathbb{R}}K(t,s)f(s)ds.$$

Then the operator

$$\widetilde{T}f(t)=\int_{-\infty}^{t}K(t,s)f(s)ds$$

has the same boundedness when $\beta>\alpha$, and $\|\widetilde{T}\|\lesssim\|T\|$.

The desired estimate (11) follows directly from (16) by changing some variables. This completes the proof.

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