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OPTIMAL BOUNDS OF THE ARITHMETIC MEAN IN TERMS OF NEW SEIFFERT–LIKE MEANS

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Abstract. We provide the optimal bounds for the arithmetic mean in terms of the arithmetic, harmonic, quadratic, inverse quadratic and geometric combination of means generated by sine, tangent, hyperbolic sine and hyperbolic tangent functions.

1. Introduction, definitions and notation

The four means defined for positive x,y by

$$\mathsf{M}_{\mathrm{arcsin}}(x,y)=\begin{cases}\dfrac{x-y}{2\arcsin\frac{x-y}{x+y}} & x\neq y\\ x & x=y\end{cases}, \qquad \text{(first Seiffert mean)}$$

$$\mathsf{M}_{\mathrm{arctan}}(x,y)=\begin{cases}\dfrac{x-y}{2\arctan\frac{x-y}{x+y}} & x\neq y\\ x & x=y\end{cases}, \qquad \text{(second Seiffert mean)}$$

$$\mathsf{M}_{\mathrm{arsinh}}(x,y)=\begin{cases}\dfrac{x-y}{2\,\mathrm{arsinh}\frac{x-y}{x+y}} & x\neq y\\ x & x=y\end{cases}, \qquad \text{(Neuman-Sándor mean)}$$

and

$$\mathsf{M}_{\mathrm{artanh}}(x,y)=\begin{cases}\dfrac{x-y}{2\,\mathrm{artanh}\frac{x-y}{x+y}} & x\neq y\\ x & x=y\end{cases} \qquad \text{(logarithmic mean)}$$

are known in the literature for quite a long time and are subject to intensive research. In [5] one of the coauthors investigated means of the form

$$\mathsf{M}_f(x,y)=\begin{cases}\dfrac{|x-y|}{2f\left(\frac{|x-y|}{x+y}\right)} & x\neq y\\ x & x=y\end{cases}, \tag{1}$$

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and discovered that both sine and tangent functions as well as their hyperbolic companions also create means

$$\mathsf{M}_{\sin}(x,y)=\begin{cases}\dfrac{x-y}{2\sin\frac{x-y}{x+y}} & x\neq y\\ x & x=y\end{cases}, \tag{sine mean}$$

$$\mathsf{M}_{\tan}(x,y)=\begin{cases}\dfrac{x-y}{2\tan\frac{x-y}{x+y}} & x\neq y\\ x & x=y\end{cases}, \tag{tangent mean}$$

$$\mathsf{M}_{\sinh}(x,y)=\begin{cases}\dfrac{x-y}{2\sinh\frac{x-y}{x+y}} & x\neq y\\ x & x=y\end{cases}, \tag{hyperbolic sine mean}$$

and

$$\mathsf{M}_{\tanh}(x,y)=\begin{cases}\dfrac{x-y}{2\tanh\frac{x-y}{x+y}} & x\neq y\\ x & x=y\end{cases}. \tag{hyperbolic tangent mean}$$

The aim of this paper is to determine optimal bounds for the arithmetic mean in terms of these means. In [5] it was shown that $\mathsf{M}_{\tan}<\mathsf{A}<\mathsf{M}_{\sin}$ and $\mathsf{M}_{\sinh}<\mathsf{A}<\mathsf{M}_{\tanh}$. Here we improve the above inequalities by providing the optimal bounds of the arithmetic mean in terms of several weighted means of the pairs of the above-mentioned means.

The main concept we use here is the notion of Seiffert function, i.e. the function f from the formula (1). For every symmetric homogeneous mean defined for positive arguments we have

$$\begin{aligned}M(x,y)&=\frac{x+y}{2}M\left(\frac{x+y-(y-x)}{x+y},\frac{x+y+(y-x)}{x+y}\right)\\&=\frac{|y-x|}{2\dfrac{z}{M(1+z,1-z)}},\end{aligned}\tag{2}$$

where $z=\frac{|x-y|}{x+y}$. Clearly if x,y are positive that z assumes values in $[0,1)$.
From [5] we know that there is a one-to-one correspondence between symmetric and homogeneous means defined on $\mathbb{R}_+$ and the functions $f:[0,1)\to\mathbb{R}$ satisfying $\frac{z}{1+z}\leqslant f(z)\leqslant\frac{z}{1-z}$, and this correspondence is described by the equation:

$$f(z)=\frac{z}{\mathsf{M}_f(1-z,1+z)}.\tag{3}$$

REMARK 1.1. If $M(x,y)$ and $N(x,y)$ are means, then we write $M<N$ to indicate the inequality $M(x,y)<N(x,y)$ holds for all $x\neq y$.

Every inequality between means can be replaced by the inequality between their Seiffert functions, because from (1) we easily deduce that $M_f < M_g$ is equivalent to $g < f$. Obvious inequalities $\sin z < z < \tan z$ and $\tanh z < z < \sinh z$ are the basis for our considerations.

REMARK 1.2. This paper shows striking similarity of the trigonometric functions and their hyperbolic twins.

For the reader's convenience in the sections that follows we place the main results with their proofs, while all lemmas and technical details can be found in the last section of this paper.

2. Arithmetic bounds

Given three means $K < L < M$ one may try to find the best α, β satisfying double inequality $(1-\alpha)K + \alpha M < L < (1-\beta)K + \beta M$ or equivalently $\alpha < \frac{L-K}{M-K} < \beta$. If k, l, m are respective Seiffert functions, then the latter can be written as

$$\alpha < \frac{\frac{1}{l} - \frac{1}{k}}{\frac{1}{m} - \frac{1}{k}} < \beta. \tag{4}$$

Thus the problem reduces to finding the upper and lower bounds for certain function defined on the interval $(0,1)$.

THEOREM 2.1. *The inequalities*

$$(1-\alpha)\mathsf{M}_{\tan} + \alpha \mathsf{M}_{\sin} < \mathsf{A} < (1-\beta)\mathsf{M}_{\tan} + \beta \mathsf{M}_{\sin}$$

hold if and only if $\alpha \leqslant \frac{\cos 1 - \sin 1}{\cos 1 - 1} \approx 0.6551$ *and* $\beta \geqslant \frac{2}{3}$.

Proof. The Seiffert function of the arithmetic mean is the identity function, so by the formula (4) we should investigate the function

$$h(z) = \frac{\frac{1}{z} - \frac{1}{\tan z}}{\frac{1}{\sin z} - \frac{1}{\tan z}} = -\frac{\frac{\sin z}{z} - 1}{\cos z - 1} + 1.$$

To show that h decreases we use Lemma 7.1. We have

$$\frac{\left(\frac{\sin z}{z} - 1\right)'}{(\cos z - 1)'} = \frac{1}{z}\left(\frac{1}{z} - \frac{1}{\tan z}\right) =: \frac{1}{z} p(z). \tag{5}$$

The function p satisfies $\lim_{z \to 0} p(z) = 0$ and $p''(z) = \frac{2}{\sin^3 z}\left(\frac{\sin^3 z}{z^3} - \cos z\right) > 0$ (by Lemma 7.2), so by Property 7.2 the function in (5) increases and this implies that the function h decreases.

We complete the proof by noting that

$$h(z) = -\frac{1-\frac{z^2}{6}+o(z^3)-1}{1-\frac{z^2}{2}+o(z^3)-1}+1 \to \frac{2}{3} \quad \text{as} \quad z \to 0.$$

REMARK 2.1. In the theorems that follow the limit of the function h at $z=0$ can be calculated in a similar way, so we skip the details.

And here comes the hyperbolic version of the previous theorem.

THEOREM 2.2. *The inequalities*

$$(1-\alpha)\mathsf{M}_{\sinh}+\alpha\mathsf{M}_{\tanh} < \mathsf{A} < (1-\beta)\mathsf{M}_{\sinh}+\beta\mathsf{M}_{\tanh}$$

hold if and only if $\alpha \leqslant \frac{\sinh 1-1}{\cosh 1-1} \approx 0.3226$ *and* $\beta \geqslant \frac{1}{3}$.

Proof. The function to be considered here is

$$h(z) = \frac{\frac{1}{z}-\frac{1}{\sinh z}}{\frac{1}{\tanh z}-\frac{1}{\sinh z}} = \frac{\frac{\sinh z}{z}-1}{\cosh z-1}.$$

We follow the same line as in the proof of Theorem 2.1. We have

$$\frac{\left(\frac{\sinh z}{z}-1\right)'}{(\cosh z-1)'} = \frac{1}{z}\left(\frac{1}{\tanh z}-\frac{1}{z}\right) =: \frac{1}{z}p(z). \tag{6}$$

The function p satisfies $\lim_{z\to 0} p(z)=0$ and $p''(z) = \frac{2}{\sinh^3 z}\left(\cosh z-\frac{\sinh^3 z}{z^3}\right) < 0$ (by Lemma 7.3), so by Property 7.2 the function in (6) decreases and so does the function h.

We complete the proof by noting that $\lim_{z\to 0} h(z) = \frac{1}{3}$.

3. Harmonic bounds

Here we look for the optimal bounds for means $K<L<M$ of the form $\frac{1-\alpha}{M}+\frac{\alpha}{K} < \frac{1}{L} < \frac{1-\beta}{M}+\frac{\beta}{K}$ or, in terms of their Seiffert functions,

$$\alpha < \frac{l-k}{m-k} < \beta.$$

THEOREM 3.1. *The inequalities*

$$\frac{1-\alpha}{\mathsf{M}_{\sin}}+\frac{\alpha}{\mathsf{M}_{\tan}} < \frac{1}{\mathsf{A}} < \frac{1-\beta}{\mathsf{M}_{\sin}}+\frac{\beta}{\mathsf{M}_{\tan}}$$

hold if and only if $\alpha \leqslant \frac{1-\sin 1}{\tan 1-\sin 1} \approx 0.2214$ *and* $\beta \geqslant \frac{1}{3}$.

Proof. To prove the theorem we will show that the function

$$h(z) = \frac{z - \sin z}{\tan z - \sin z} = \frac{\frac{z}{\sin z} - 1}{\frac{1}{\cos z} - 1}$$

is decreasing. But

$$\frac{\left(\frac{z}{\sin z} - 1\right)'}{\left(\frac{1}{\cos z} - 1\right)'} = \frac{1}{\tan^2 z} - \frac{z}{\tan^3 z}$$

decreases by Lemma 7.4 so by Lemma 7.1 the function h also decreases. We complete the proof by noting that $\lim_{z\to 0} h(z) = \frac{1}{3}$.

THEOREM 3.2. *The inequalities*

$$\frac{1-\alpha}{\mathsf{M}_{\tanh}} + \frac{\alpha}{\mathsf{M}_{\sinh}} < \frac{1}{\mathsf{A}} < \frac{1-\beta}{\mathsf{M}_{\tanh}} + \frac{\beta}{\mathsf{M}_{\sinh}}$$

hold if and only if $\alpha \leqslant \frac{1-\tanh 1}{\sinh 1 - \tanh 1} \approx 0.5764$ *and* $\beta \geqslant \frac{2}{3}$.

Proof. The function to investigate is

$$h(z) = \frac{z - \tanh z}{\sinh z - \tanh z} = \frac{\frac{z}{\tanh z} - 1}{\cosh z - 1}.$$

We shall show that h decreases. But

$$\frac{\left(\frac{z}{\tanh z} - 1\right)'}{(\cosh z - 1)'} = \frac{\cosh z}{\sinh^2 z} - \frac{z}{\sinh^3 z}$$

decreases by Lemma 7.5 so by Lemma 7.1 the function h also decreases. We complete the proof by noting that $\lim_{z\to 0} h(z) = \frac{2}{3}$.

4. Quadratic bounds

Given three means $K < L < M$ one may try to find the best α, β satisfying double inequality $\sqrt{(1-\alpha)K^2 + \alpha M^2} < L < \sqrt{(1-\beta)K^2 + \beta M^2}$ or equivalently $\alpha < \frac{L^2 - K^2}{M^2 - K^2} < \beta$. If k, l, m are respective Seiffert functions, then the latter can be written as

$$\alpha < \frac{\frac{1}{l^2} - \frac{1}{k^2}}{\frac{1}{m^2} - \frac{1}{k^2}} < \beta. \tag{7}$$

Thus the problem reduces to finding the upper and lower bound for certain function defined on the interval $(0,1)$.

THEOREM 4.1. *The inequalities*

$$\sqrt{(1-\alpha)\mathsf{M}_{\tan}^2 + \alpha \mathsf{M}_{\sin}^2} < \mathsf{A} < \sqrt{(1-\beta)\mathsf{M}_{\tan}^2 + \beta \mathsf{M}_{\sin}^2}$$

hold if and only $\alpha \leqslant 1 - \frac{1}{\tan^2 1} \approx 0.5877$ *and* $\beta \geqslant \frac{2}{3}$.

Proof. By the formula (7) we should investigate the function

$$h(z) = \frac{\dfrac{1}{z^2} - \dfrac{1}{\tan^2 z}}{\dfrac{1}{\sin^2 z} - \dfrac{1}{\tan^2 z}} = \frac{1}{z^2} - \frac{1}{\tan^2 z}.$$

Since $h'(z) = \frac{2}{\sin^3 z}\left(\cos z - \frac{\sin^3 z}{z^3}\right) < 0$ (by Lemma 7.2), the function h decreases. We complete the proof by noting that $\lim_{z\to 0} h(z) = \frac{2}{3}$.

THEOREM 4.2. *The inequalities*

$$\sqrt{(1-\alpha)\mathsf{M}_{\sinh}^2 + \alpha \mathsf{M}_{\tanh}^2} < \mathsf{A} < \sqrt{(1-\beta)\mathsf{M}_{\sinh}^2 + \beta \mathsf{M}_{\tanh}^2}$$

hold if and only if $\alpha \leqslant 1 - \frac{1}{\sinh^2 1} \approx 0.2759$ *and* $\beta \geqslant \frac{1}{3}$.

Proof. The function to be considered here is

$$h(z) = \frac{\dfrac{1}{z^2} - \dfrac{1}{\sinh^2 z}}{\dfrac{1}{\tanh^2 z} - \dfrac{1}{\sinh^2 z}} = \frac{1}{z^2} - \frac{1}{\sinh^2 z}.$$

Its derivative equals $h'(z) = \frac{2}{\sinh^3 z}\left(\cosh z - \frac{\sinh^3 z}{z^3}\right)$. By Lemma 7.3 we have that $h'(z) < 0$, so the function h decreases. We complete the proof by noting that $\lim_{z\to 0} h(z) = \frac{1}{3}$.

5. Bounds by weighted power mean of order -2

In this section we look for the optimal bounds for means $K < L < M$ of the form $\sqrt{\frac{1-\alpha}{M^2} + \frac{\alpha}{K^2}} < \frac{1}{L} < \sqrt{\frac{1-\beta}{M^2} + \frac{\beta}{K^2}}$ or, in terms of their Seiffert functions,

$$\alpha < \frac{l^2 - m^2}{k^2 - m^2} < \beta. \tag{8}$$

THEOREM 5.1. *The inequalities*

$$\sqrt{\frac{1-\alpha}{\mathsf{M}_{\sin}^2} + \frac{\alpha}{\mathsf{M}_{\tan}^2}} < \frac{1}{\mathsf{A}} < \sqrt{\frac{1-\beta}{\mathsf{M}_{\sin}^2} + \frac{\beta}{M_{\tan}^2}}$$

hold if and only if $\alpha \leqslant \cot^4 1 \approx 0.1700$ *and* $\beta \geqslant \frac{1}{3}$.

Proof. Taking into account the formula (8) we should investigate the function

$$h(z) = \frac{z^2 - \sin^2 z}{\tan^2 z - \sin^2 z} = \frac{\cos^2 z}{\sin^4 z}(z^2 - \sin^2 z).$$

We shall show that h decreases. Observe that

$$h'(z) = \frac{\cos z}{\sin^5 z}(z\sin 2z - (z^2+1)\cos 2z - 3z^2 + 1).$$

The function $p(z) = z\sin 2z - (z^2+1)\cos 2z - 3z^2 + 1$ satisfies $p(0) = p'(0) = p''(0) = p'''(0) = 0$ and $p^{(4)}(z) = -16z(3\sin 2z + z\cos 2z)$.
Since $3\sin 2z + z\cos 2z > z\sin 2z + z\cos 2z = \sqrt{2}z\sin(2z+\pi/4) > 0$ we conclude that p is negative and so is h'. Consequently, h decreases. We complete the proof by noting that $\lim_{z\to 0} h(z) = \frac{1}{3}$.

THEOREM 5.2. *The inequalities*

$$\sqrt{\frac{1-\alpha}{\mathsf{M}_{\tanh}^2} + \frac{\alpha}{\mathsf{M}_{\sinh}^2}} < \frac{1}{\mathsf{A}} < \sqrt{\frac{1-\beta}{\mathsf{M}_{\tanh}^2} + \frac{\beta}{M_{\sinh}^2}}$$

hold if and only if $\alpha \leqslant \frac{1}{\sinh^4 1} \approx 0.5243$ *and* $\beta \geqslant \frac{2}{3}$.

Proof. Use the formula (8) we should investigate the function

$$h(z) = \frac{z^2 - \tanh^2 z}{\sinh^2 z - \tanh^2 z} = \frac{\cosh^2 z}{\sinh^4 z}(z^2 - \tanh^2 z).$$

We shall show that h decreases. Its derivative equals

$$h'(z) = \frac{\cosh z}{\sinh^5 z}(z\sinh 2z - (z^2-1)\cosh 2z - 3z^2 - 1).$$

The function $p(z) = z\sinh 2z - (z^2-1)\cosh 2z - 3z^2 - 1$ satisfies $p(0) = p'(0) = p''(0) = p'''(0) = 0$ and

$$p^{(4)}(z) = -16z(3\sinh 2z + z\cosh 2z) < 0.$$

Thus p is negative and so is h'. Consequently, h decreases. We complete the proof by noting that $\lim_{z\to 0} h(z) = \frac{2}{3}$.

6. Geometric bounds

In this section we establish the best bounds for the arithmetic mean by expressions of the form $\mathsf{M}_{\tan}^{1-\alpha}\mathsf{M}_{\sin}^{\alpha}$ and $\mathsf{M}_{\tanh}^{1-\alpha}\mathsf{M}_{\sinh}^{\alpha}$. We begin with the following Theorem

THEOREM 6.1. *The inequalities*

$$\mathsf{M}_{\tan}^{1-\alpha}\mathsf{M}_{\sin}^{\alpha} < \mathsf{A} < \mathsf{M}_{\tan}^{1-\beta}\mathsf{M}_{\sin}^{\beta} \tag{9}$$

hold if and only if $\alpha \leqslant \frac{2}{3}$ *and* $\beta \geqslant \frac{\log\cot 1}{\log\cos 1} \approx 0.7196$.

Proof. Writing the inequalities (9) in terms of Seiffert functions we obtain

$$\frac{1}{\tan^{1-\alpha} z \sin^{\alpha} z} < \frac{1}{z} < \frac{1}{\tan^{1-\beta} z \sin^{\beta} z},$$

or equivalently

$$\cos^{\beta-1} z \sin z - z < 0 < \cos^{\alpha-1} z \sin z - z. \tag{10}$$

By Lemma 7.2 the second inequality in (10) holds for $-1 < \alpha - 1 \leqslant -1/3$ i.e. if $0 < \alpha \leqslant 2/3$, while the first one holds for all $z \in (0,1)$ if and only if $\cos^{\beta-1} 1 \sin 1 - 1 \leqslant 0$.

For the functions $\mathsf{M}_{\sinh}$ and $\mathsf{M}_{\tanh}$ we have similar result:

THEOREM 6.2. *The inequalities*

$$\mathsf{M}_{\sinh}^{1-\alpha} \mathsf{M}_{\tanh}^{\alpha} < \mathsf{A} < \mathsf{M}_{\sinh}^{1-\beta} \mathsf{M}_{\tanh}^{\beta} \tag{11}$$

hold if and only if $\alpha \leqslant \frac{1}{3}$ *and* $\beta \geqslant \frac{\log \sinh 1}{\log \cosh 1} \approx 0.3722$.

Proof. The proof goes along the same line as above. The inequalities (11) can be rewritten as

$$\cosh^{-\beta} z \sinh z - z < 0 < \cosh^{-\alpha} z \sinh z - z. \tag{12}$$

By Lemma 7.3 the second inequality in (12) holds for $-1/3 \leqslant -\alpha < 0$ i.e. if $0 < \alpha \leqslant 1/3$, while the first one holds for all $z \in (0,1)$ if and only if $\cosh^{-\beta} 1 \sinh 1 - 1 \leqslant 0$.

7. Tools and lemmas

In this section we place the all technical details needed to prove our main results.

PROPERTY 7.1. A function $f : (a,b) \to \mathbb{R}$ is convex if and only if for every $a < \theta < b$ its divided difference $\frac{f(x)-f(\theta)}{x-\theta}$ increases for $x \neq \theta$.

Simple consequence of Property 7.1 is

PROPERTY 7.2. If a function $f : (a,b) \to \mathbb{R}$ is convex and $\lim_{x\to a} f(x) = \Theta$, then the function $\frac{f(x)-\Theta}{x-a}$ increases.

The next lemma can be found in [1, Theorem 1.25] or in [4].

LEMMA 7.1. *Suppose* $f, g : (a,b) \to \mathbb{R}$ *are differentiable with* $g'(x) \neq 0$ *and such that* $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ *or* $\lim_{x\to b} f(x) = \lim_{x\to b} g(x) = 0$. *Then*

1. *if* $\frac{f'}{g'}$ *is increasing on* (a,b), *then* $\frac{f}{g}$ *is increasing on* (a,b),

2. *if* $\frac{f'}{g'}$ *is decreasing on* (a,b), *then* $\frac{f}{g}$ *is decreasing on* (a,b).

LEMMA 7.2. (Mitrinović & Adamović [3]) *Consider the functions* $f_u : [0,\pi/2) \to \mathbb{R}$

$$f_u(x) = \cos^u x \sin x - x, \quad -1 < u < 0.$$

For $-1 < u \leqslant -1/3$ *the functions* f_u *are positive.*
For $-1/3 < u < 0$ *there exists* $0 < x_u < \pi/2$ *such that* f_u *is negative in* $(0,x_u)$ *and positive in* (x_u,∞).

Proof. We have $f_u(0) = f'_u(0) = 0$ and

$$f''_u(x) = u(u-1)\sin x \cos^u x \left[\tan^2 x - \frac{1+3u}{u(u-1)}\right].$$

For $-1 < u \leqslant -1/3$ we have $\frac{3u+1}{u(u-1)} \leqslant 0$, so f_u is convex, thus positive.
For $-1/3 < u < 0$ the equation $\tan^2 x - \frac{1+3u}{u(u-1)} = 0$ has exactly one solution ξ_u, so f_u is concave and negative on $(0,\xi_u)$. Then it becomes convex and tends to infinity, thus assumes zero at exactly one point x_u.

LEMMA 7.3. (Lazarević [2]) *Consider the functions* $g_u : [0,\infty) \to \mathbb{R}$

$$g_u(x) = \cosh^u x \sinh x - x, \quad -1 < u < 0.$$

For $-1/3 \leqslant u < 0$ *the functions* g_u *are positive.*
For $-1 < u < -1/3$ *there exists* $x_u > 0$ *such that* g_u *is negative in* $(0,x_u)$ *and positive in* (x_u,∞).

Proof. We have $g_u(0) = g'_u(0) = 0$ and

$$g''_u(x) = u(u-1)\sinh x \cosh^u x \left[\tanh^2 x + \frac{1+3u}{u(u-1)}\right].$$

If $-1/3 \leqslant u < 0$ we have $\frac{1+3u}{u(u-1)} \geqslant 0$, so g_u is convex thus positive.
For $-1 < u < -1/3$ the equation $\tanh^2 x + \frac{1+3u}{u(u-1)} = 0$ has exactly one solution ξ_u, so g_u is concave and negative on $(0,\xi_u)$. Then it becomes convex and tends to infinity, thus assumes zero at exactly one point x_u.

LEMMA 7.4. *The function* $f(z) = \frac{1}{\tan^2 z} - \frac{z}{\tan^3 z}$ *decreases for* $0 < z < \pi/2$.

Proof. Its derivative equals $\frac{\cos z}{\sin^4 z}(3z\cos z - 3\sin z + \sin^3 z)$. The function $q(z) = 3z\cos z - 3\sin z + \sin^3 z$ satisfies $q(0) = 0$ and $q'(z) = -\frac{3}{2}\sin z(2z - \sin 2z) < 0$, which completes the proof.

LEMMA 7.5. *The function* $f(z) = \frac{\cosh z}{\sinh^2 z} - \frac{z}{\sinh^3 z}$ *decreases for* $z > 0$.

Proof. Its derivative equals $\sin^{-4} z(3z\cosh z - 3\sinh z - \sinh^3 z)$. The function $q(z) = 3z\cosh z - 3\sinh z - \sinh^3 z$ satisfies $q(0) = 0$ and $q'(z) = \frac{3}{2}\sinh z(2z - \sinh 2z) < 0$, which completes the proof.

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