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WEIGHTED ESTIMATES FOR ROUGH SINGULAR INTEGRALS WITH APPLICATIONS TO ANGULAR INTEGRABILITY, II

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Abstract. This paper is devoted to studying certain singular integral operators with rough radial kernel h and sphere kernel Ω as well as the corresponding maximal operators along polynomial curves. The authors establish several weighted estimates for such operators by assuming that the kernels $h \equiv 1$ and $\Omega \in \mathscr{F}_\beta(\mathrm{S}^{n-1})$, or $h \in \Delta_\gamma(\mathbb{R}_+)$ and $\Omega \in W\mathscr{F}_\beta(\mathrm{S}^{n-1})$. Here $\mathscr{F}_\beta(\mathrm{S}^{n-1})$ denotes the Grafakos-Stefanov kernel and $W\mathscr{F}_\beta(\mathrm{S}^{n-1})$ denotes the variant of Grafakos-Stefanov kernel. As applications, the boundedness of such operators on the mixed radial-angular spaces $L^p_{|x|}L^q_\theta(\mathbb{R}^n)$ are obtained. Meanwhile, the corresponding vector-valued versions are also given. Moreover, the bounds are independent of the coefficients of the polynomials in the definition of operators.

1. Introduction

In this paper we continue with the program started in [21], which proved two results related to the boundedness of singular integral operators and the corresponding truncated maximal operators on the mixed radial-angular spaces. In what follows, let $\mathbb{R}^n$, $n \geqslant 2$, be the Euclidean space of dimension n and S^{n-1} denote the unit sphere in $\mathbb{R}^n$ equipped with the normalized Lebesgue measure $d\sigma$. We now recall the definition of mixed radial-angular spaces.

DEFINITION 1. (*Mixed radial-angular space*). For $1 \leqslant p < \infty$ and $1 \leqslant q < \infty$, the mixed radial-angular spaces $L^p_{|x|}L^q_\theta(\mathbb{R}^n)$ are defined as the collection of all measurable functions u defined in $\mathbb{R}^n$ for which $\|u\|_{L^p_{|x|}L^q_\theta(\mathbb{R}^n)} < \infty$, where

$$\|u\|_{L^p_{|x|}L^q_\theta(\mathbb{R}^n)} := \Big(\int_0^\infty \|u(\rho\cdot)\|^p_{L^q(\mathrm{S}^{n-1})}\rho^{n-1}d\rho\Big)^{1/p}.$$

The mixed radial-angular spaces $L^p_{|x|}L^q_\theta(\mathbb{R}^n)$ with $p = \infty$ or $q = \infty$ can be defined by applying the usual modifications.

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It is easy to check that the spaces $L^p_{|x|}L^q_\theta(\mathbb{R}^n)$ have the following basic properties:
(a) If $1 \leqslant p \leqslant \infty$ and $q = p$, then

$$\|u\|_{L^p_{|x|}L^q_\theta(\mathbb{R}^n)} = \|u\|_{L^p(\mathbb{R}^n)}. \tag{1}$$

(b) If $1 \leqslant p \leqslant \infty$ and $1 \leqslant q_1 \leqslant q_2 \leqslant \infty$, then

$$\|u\|_{L^p_{|x|}L^{q_1}_\theta(\mathbb{R}^n)} \leqslant C_{n,p,q_1,q_2}\|u\|_{L^p_{|x|}L^{q_2}_\theta(\mathbb{R}^n)}.$$

(c) If u is a radial function on $\mathbb{R}^n$ and $1 \leqslant p \leqslant \infty$ and $1 \leqslant q \leqslant \infty$, then

$$\|u\|_{L^p_{|x|}L^q_\theta(\mathbb{R}^n)} \simeq \|u\|_{L^p(\mathbb{R}^n)}.$$

Here and in the sequel the notation $A \simeq B$ means that there are two positive constants C, C' such that $A \leqslant CB$ and $B \leqslant C'A$.

Let $P_N(t)$ be a real polynomial on $\mathbb{R}$ of degree N satisfying $P(0) = 0$. Let Ω be a $L^1(\mathrm{S}^{n-1})$ function satisfying

$$\int_{\mathrm{S}^{n-1}} \Omega(y)d\sigma(y) = 0. \tag{2}$$

and $h \in \Delta_\gamma(\mathbb{R}_+)$ with $\mathbb{R}_+ := (0,\infty)$. Here $\Delta_\gamma(\mathbb{R}_+)$, $\gamma > 0$, is the set of all measurable functions h defined on $\mathbb{R}_+$ satisfying

$$\|h\|_{\Delta_\gamma(\mathbb{R}_+)} := \sup_{R>0}\Big(\frac{1}{R}\int_0^R |h(t)|^\gamma dt\Big)^{1/\gamma} < \infty.$$

It is clear that

$$L^\infty(\mathbb{R}_+) = \Delta_\infty(\mathbb{R}_+) \subsetneq \Delta_{\gamma_2}(\mathbb{R}_+) \subsetneq \Delta_{\gamma_1}(\mathbb{R}_+) \ \text{ for } \ 1 \leqslant \gamma_1 < \gamma_2 < \infty. \tag{3}$$

Now we define the singular integral operator T_{h,Ω,P_N} along the "polynomial curve" P_N by

$$T_{h,\Omega,P_N}f(x) = \text{p.v.}\int_{\mathbb{R}^n} f(x - P_N(|y|)y')\frac{h(|y|)\Omega(y')}{|y|^n}dy,$$

the corresponding truncated maximal singular integral operator T^*_{h,Ω,P_N} by

$$T^*_{h,\Omega,P_N}f(x) = \sup_{\varepsilon>0}\Big|\int_{|y|>\varepsilon} f(x - P_N(|y|)y')\frac{h(|y|)\Omega(y')}{|y|^n}dy\Big|,$$

and the corresponding maximal operator M_{h,Ω,P_N} by

$$M_{h,\Omega,P_N}f(x) = \sup_{r>0}\frac{1}{r^n}\int_{|y|<r} |f(x - P_N(|y|)y')||h(|y|)\Omega(y')|dy.$$

where $y' = y/|y|$ for $y \neq 0$.

For the sake of simplicity, we denote $T_{h,\Omega,P_N}=T_{\Omega,P_N}$, $T^*_{h,\Omega,P_N}=T^*_{\Omega,P_N}$ and $M_{h,\Omega,P_N}=M_{\Omega,P_N}$ if $h\equiv 1$; $T_{\Omega,P_N}=T_\Omega$ and $T^*_{\Omega,P_N}=T^*_\Omega$ if $P_N(t)=t$; $T_{h,\Omega,P_N}=T_{h,\Omega}$ if $P_N(t)=t$.

Singular integral theory was initiated in the seminal work of Calderón and Zygmund [4] and since then has been an active area of research. A celebrated work in this topic was due to Calderón and Zygmund [5] who showed that T_Ω is bounded on the Lebesgue spaces $L^p(\mathbb{R}^n)$ for $1<p<\infty$ if $\Omega\in L\log L(\mathrm{S}^{n-1})$ by the method of rotations. Here the function class $L\log L(\mathrm{S}^{n-1})$ denotes the set of all functions $\Omega:\mathrm{S}^{n-1}\to\mathbb{R}$ satisfying

$$\|\Omega\|_{L\log L(\mathrm{S}^{n-1})}:=\int_{\mathrm{S}^{n-1}}|\Omega(\theta)|\log(2+|\Omega(\theta)|)d\sigma(\theta)<\infty.$$

Subsequently, the condition was extended to the case $\Omega\in H^1(S^{n-1})$, the Hardy space on S^{n-1}, by Coifman and Weiss [6] and Connett [7] independently. In 1997, to study the L^p-boundedness of singular integrals with rough kernels, Grafakos and Stefanov [18] introduced the following function spaces:

$$\mathscr{F}_\beta(\mathrm{S}^{n-1}):=\Big\{\Omega\in L^1(\mathrm{S}^{n-1}):\ \sup_{\xi\in\mathrm{S}^{n-1}}\int_{\mathrm{S}^{n-1}}|\Omega(y')|\log^\beta\frac{2}{|\xi\cdot y'|}d\sigma(y')<\infty\Big\}\ \text{for }\beta>0,$$

and showed that

$$\mathscr{F}_{\beta_1}(\mathrm{S}^{n-1})\subsetneq\mathscr{F}_{\beta_2}(\mathrm{S}^{n-1})\ \text{for } 0<\beta_2<\beta_1,$$
$$\bigcup_{q>1}L^q(\mathrm{S}^{n-1})\subsetneq\mathscr{F}_\beta(\mathrm{S}^{n-1})\ \text{for any }\beta>0,$$

and

$$\bigcap_{\beta>1}\mathscr{F}_\beta(\mathrm{S}^{n-1})\nsubseteq L\log L(\mathrm{S}^{n-1})\subset H^1(\mathrm{S}^{n-1})\nsubseteq\bigcup_{\beta>1}\mathscr{F}_\beta(\mathrm{S}^{n-1}).$$

Moreover, Grafakos and Stefanov [18] proved that that T_Ω is of type (p,p) for $p\in(1+1/\beta,\beta+1)$ if $\Omega\in\mathscr{F}_\beta(\mathrm{S}^{n-1})$ for some $\beta>1$, and T^*_Ω is of type (p,p) for $p\in(\frac{2(\beta+1)}{2\beta-1},\frac{2(\beta+1)}{3})$ if $\Omega\in\mathscr{F}_\beta(\mathrm{S}^{n-1})$ for some $\beta>2$. Subsequently, Fan, Guo and Pan [13] improved and extended to these results as follows.

THEOREM A. ([13]) *Let $P_N(t)$ be a real polynomial on $\mathbb{R}$ of degree N and satisfy $P_N(0)=0$. Suppose that Ω satisfies (2) and $\Omega\in\mathscr{F}_\beta(\mathrm{S}^{n-1})$ for some $\beta>0$.*

(i) *If $\beta>1$, then T_{Ω,P_N} is bounded on $L^p(\mathbb{R}^n)$ for $p\in(\frac{2\beta}{2\beta-1},2\beta)$.*

(ii) *If $\beta>\frac{3}{2}$, then T^*_{Ω,P_N} is bounded on $L^p(\mathbb{R}^n)$ for $p\in(\frac{2\beta-1}{2\beta-2},2\beta-1)$.*

Here the bounds of the above operators are independent of the coefficients of P_N.

In 1979, Fefferman [16] introduced the singular integral operator $T_{h,\Omega}$ with $h\in L^\infty(\mathbb{R}_+)$ and proved that $T_{h,\Omega}$ is bounded on $L^p(\mathbb{R}^n)$ for all $p\in(1,\infty)$ if $\Omega\in\mathrm{Lip}_\alpha(\mathrm{S}^{n-1})$ for $0<\alpha\leqslant 1$ and $h\in L^\infty(\mathbb{R}_+)$. Later on, Namazi [24] improved Fefferman's result to the case $\Omega\in L^q(\mathrm{S}^{n-1})$ for some $q>1$. Subsequently, Duoandikoetxea and Rubio de Francia [12] used the Littlewood-Paley theory to improve $h\in L^\infty(\mathbb{R}_+)$ to the case $h\in\Delta_2(\mathbb{R}_+)$. Since then, the above results have been improved and extended by many authors (see [1, 14, 15, 22, 23, 25]). In particular, Fan and Sato [15] showed that $T_{h,\Omega}$ is bounded on $L^p(\mathbb{R}^n)$ for $|1/p-1/2|<\min\{1/\gamma',1/2\}-1/\beta$, provided that $h\in\Delta_\gamma(\mathbb{R}_+)$ for some $\gamma>1$ and $\Omega\in W\mathscr{F}_\beta(\mathrm{S}^{n-1})$ for some $\beta>\max\{\gamma',2\}$, where $W\mathscr{F}_\beta(\mathrm{S}^{n-1})$ for $\beta>0$ denotes the set of all functions $\Omega:\mathrm{S}^{n-1}\to\mathbb{R}$ satisfying

$$\sup_{\xi'\in\mathrm{S}^{n-1}}\iint_{\mathrm{S}^{n-1}\times\mathrm{S}^{n-1}}|\Omega(\theta)\Omega(u')|\left(\log^+\frac{1}{|(\theta-u')\cdot\xi'|}\right)^\beta d\sigma(\theta)d\sigma(u')<\infty.$$

It was pointed out in [15, 20] that

$$\mathscr{F}_\beta(\mathrm{S}^1)\subset W\mathscr{F}_\beta(\mathrm{S}^1)\ \text{ and }\ W\mathscr{F}_{2\beta}(\mathrm{S}^{n-1})\setminus\mathscr{F}_\beta(\mathrm{S}^{n-1})\neq\emptyset\ \text{ for }\beta>0.$$

$$\bigcup_{r>1}L^r(\mathrm{S}^{n-1})\subset W\mathscr{F}_{\beta_2}(\mathrm{S}^{n-1})\subset W\mathscr{F}_{\beta_1}(\mathrm{S}^{n-1})\ \text{ for } 0<\beta_1<\beta_2<\infty.$$

Afterwards, the first and third authors [23] extended the result of [15] to the singular integral along polynomial curves in mixed homogeneous setting.

THEOREM B. ([23]) *Let $P_N(t)$ be a real polynomial on $\mathbb{R}$ of degree N and satisfy $P_N(0)=0$. Suppose that $h\in\Delta_\gamma(\mathbb{R}_+)$ for some $\gamma\in(1,\infty]$ and $\Omega\in W\mathscr{F}_\beta(\mathrm{S}^{n-1})$ for some $\beta>\max\{2,\gamma'\}$ and satisfies (2). Then T_{h,Ω,P_N} is bounded on $L^p(\mathbb{R}^n)$ for $|1/p-1/2|<\min\{1/\gamma',1/2\}-1/\beta$. Here the bounds of the above operators are independent of the coefficients of P_N.*

On the other hand, the mixed radial-angular space plays an active role in singular integral theory. Córdoba [9] first proved that T_Ω is bounded on $L^p_{|x|}L^2_\theta(\mathbb{R}^n)$ for all $1<p<\infty$ if $\Omega\in\mathscr{C}^1(\mathrm{S}^{n-1})$. Later on, D'Ancona and Lucà [10] used the same argument in [9, Theorem 2.1] to extend the above results to cover the full range $1<p<\infty$ and $1<q<\infty$. The corresponding radial weighted results were established by Cacciafesta and R. Lucá [3] and Duoandikoetxea and Oruetxebarria [11]. Recently, the first author and Fan [21] extended the above result to the singular integrals along polynomial curves with rough radial kernels and improved the size condition on the sphere kernels Ω to the case $\Omega\in L^s(\mathrm{S}^{n-1})$ for $s\in(1,\infty]$, which can be stated as follows:

THEOREM C. ([21]) *Let $P_N(t)$ be a real polynomial on $\mathbb{R}$ of degree N and satisfy $P_N(0)=0$. Suppose that $\Omega\in L^s(\mathrm{S}^{n-1})$ satisfies (2) and $h\in\Delta_\gamma(\mathbb{R}_+)$ for some $s,\gamma\in(1,\infty]$.*

(i) *For $1<p<\infty$ and $1<q<\infty$, the following inequalities hold:*

$$\|T_{h,\Omega,P_N}f\|_{L^p_{|x|}L^q_\theta(\mathbb{R}^n)}\leqslant C_{h,\Omega,s,\gamma,p,q,N}\|f\|_{L^p_{|x|}L^q_\theta(\mathbb{R}^n)};$$

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}|T_{h,\Omega,P_N}f_j|^q\Big)^{1/q}\Big\|_{L^p_{|x|}L^q_\theta(\mathbb{R}^n)}\leqslant C_{h,\Omega,s,\gamma,p,q,N}\Big\|\Big(\sum_{j\in\mathbb{Z}}|f_j|^q\Big)^{1/q}\Big\|_{L^p_{|x|}L^q_\theta(\mathbb{R}^n)};$$

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}|T_{h,\Omega,P_N}f_j|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\leqslant C_{h,\Omega,s,\gamma,p,q,N}\Big\|\Big(\sum_{j\in\mathbb{Z}}|f_j|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}.$$

(ii) *For $1 < q \leqslant p < \infty$, the following inequalities hold:*

$$\|T^*_{h,\Omega,P_N} f\|_{L^p_{|x|}L^q_\theta(\mathbb{R}^n)} \leqslant C_{h,\Omega,s,\gamma,p,q,N}\|f\|_{L^p_{|x|}L^q_\theta(\mathbb{R}^n)};$$

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}|T^*_{h,\Omega,P_N} f_j|^q\Big)^{1/q}\Big\|_{L^p_{|x|}L^q_\theta(\mathbb{R}^n)} \leqslant C_{h,\Omega,s,\gamma,p,q,N}\Big\|\Big(\sum_{j\in\mathbb{Z}}|f_j|^q\Big)^{1/q}\Big\|_{L^p_{|x|}L^q_\theta(\mathbb{R}^n)};$$

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}|T^*_{h,\Omega,P_N} f_j|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)} \leqslant C_{h,\Omega,s,\gamma,p,q,N}\Big\|\Big(\sum_{j\in\mathbb{Z}}|f_j|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}.$$

Here the constants $C_{h,\Omega,s,\gamma,p,q,N} > 0$ are independent of the coefficients of P_N.

Based on Theorems A-C, it is natural to ask whether or not the conclusions in Theorem C hold under the assumption of that $\Omega \in W\mathscr{F}_\beta(\mathrm{S}^{n-1})$ for some $\beta > 1$ and $h \in \Delta_\gamma(\mathbb{R}_+)$ for some $\gamma > 1$, in particular, $\Omega \in \mathscr{F}_\beta(\mathrm{S}^{n-1})$ for some $\beta > 1$ and $h \equiv 1$.

The main purpose of this paper is to address the above question. Our desired conclusions will directly follow from the following weighted inequalities and a criterion on the boundedness of sublinear operators on the mixed radial-angular spaces, which will be established in Section 3. Now we formulate our main results as follows.

THEOREM 1. *Let $P_N(t) = \sum_{i=1}^N b_i t^i$ with $b_i \neq 0$. Assume that Ω satisfies (2) and one of the following conditions holds:*

(a) *$h(t) \equiv 1$, $\Omega \in \mathscr{F}_\beta(\mathrm{S}^{n-1})$ for some $\beta > 1$, $\gamma' = 1$ and $\delta = \beta$;*

(b) *$h \in \Delta_\gamma(\mathbb{R}_+)$ for some $\gamma \in (1,\infty]$ and $\Omega \in W\mathscr{F}_\beta(\mathrm{S}^{n-1})$ for some $\beta > \max\{2,\gamma'\}$, $\delta = \frac{\beta}{\max\{2,\gamma'\}}$.*

Then

(i) *Let $s \in (\frac{\delta}{\delta-1},\infty)$ and $p \in [2, \frac{2\delta(\gamma'-1/s)}{1+\delta(\gamma'-1)})$. Then for any nonnegative measurable function u on $\mathbb{R}^n$,*

$$\|T_{h,\Omega,P_N} f\|_{L^p(u)} \leqslant C_{h,\Omega,\beta,\gamma',p,s,N}\|f\|_{L^p(L_{N,s}u)}. \tag{4}$$

(ii) *Let $\gamma \in (2,\infty]$, $\delta \in (\frac{2}{2-\gamma'},\infty)$, $p \in (\delta'\gamma',2]$ and $s \in (\frac{2\delta'}{p},\infty)$. Then for any nonnegative measurable function u on $\mathbb{R}^n$,*

$$\|T_{h,\Omega,P_N} f\|_{L^p(u)} \leqslant C_{h,\Omega,\beta,\gamma',p,s,N}\|f\|_{L^p(\Upsilon_{N,s}u)}. \tag{5}$$

Here $\Upsilon_{N,s}u = \mathrm{M}^N_s u + \mathrm{M}^2_s\widetilde{\mathrm{M}^N_s u} + H_{N,s}u$, $L_{N,s}u = \sum_{i=0}^{\lambda}\mathrm{M}_s^{\lambda+1-i}M^{\tilde\sigma}_{i,s}\mathrm{M}_s u$, $H_\lambda u = \sum_{i=1}^{\lambda}$ $\mathrm{M}^2 M^{\tilde\sigma}_i \mathrm{M}^{\lambda+1-i}u$, $M^{\tilde\sigma}_{\lambda,s}u = (M^{\tilde\sigma}_\lambda(u^s))^{1/s}$, $\mathrm{M}^k_s u = (\mathrm{M}^k u^s)^{1/s}$ for any $k \in \mathbb{N}$, $H_{\lambda,s}u = (H_\lambda u^s)^{1/s}$, $M^{\tilde\sigma}_\lambda$ is defined by $M^{\tilde\sigma}_\lambda f(x) = M^\sigma_\lambda \tilde f(x)$ and $M^\sigma_\lambda f(x) = \sup_{k\in\mathbb{Z}}||\sigma_{k,\lambda}| * f(x)|$, where $\sigma_{k,\lambda}$ and $|\sigma_{k,\lambda}|$ are respectively defined by

$$\int_{\mathbb{R}^n} f(x)d\sigma_{k,\lambda}(x) = \int_{2^{k-1}<|x|\leqslant 2^k} f(P_\lambda(|x|)x')\frac{h(|x|)\Omega(x)}{|x|^n}dx,$$

$$\int_{\mathbb{R}^n} f(x)d|\sigma_{k,\lambda}|(x) = \int_{2^{k-1}<|x|\leqslant 2^k} f(P_\lambda(|x|)x')\frac{|h(|x|)\Omega(x)|}{|x|^n}dx,$$

and $P_0(t)=0,\ P_\lambda(t)=\sum_{i=1}^{\lambda} b_i t^i$ *for all* $\lambda \in \{1,2,\ldots,N\}$. *The above constants* $C_{h,\Omega,\beta,\gamma',p,s,N}$ *are independent of* $\{b_\lambda\}_{\lambda=1}^N$. *The same conclusions hold for* M_{h,Ω,P_N}.

THEOREM 2. *Let* $P_N(t)=\sum_{i=1}^N b_i t^i$ *with* $b_i \neq 0$. *Assume that* Ω *satisfies* (2) *and one of the following conditions holds:*

(a) $h(t)\equiv 1,\ \Omega \in \mathscr{F}_\beta(\mathrm{S}^{n-1})$ *for some* $\beta > \frac{3}{2},\ \gamma'=1$ *and* $\delta=\beta$*;*

(b) $h\in\Delta_\gamma(\mathbb{R}_+)$ *for some* $\gamma\in(1,\infty]$ *and* $\Omega\in W\mathscr{F}_\beta(\mathrm{S}^{n-1})$ *for some* $\beta>\frac{3}{2}\max\{2,\gamma'\}$, $\delta=\frac{\beta}{\max\{2,\gamma'\}}$.

Then for any nonnegative measurable function u *on* $\mathbb{R}^n$,

(i) *for* $\delta\in(\frac{3}{2},\infty),\ s\in((\frac{\delta-1/2}{\delta-3/2})^2,\infty)$ *and* $p\in[2,\frac{\delta(2\delta-1)(1-1/\sqrt{s})(\gamma'-1/s)}{(\delta\gamma'-\delta+1)(\delta-1/2)(1-1/\sqrt{s})+(1-1/s)\delta-1})$,

$$\|T^*_{h,\Omega,P_N}f\|_{L^p(u)} \leqslant C_{h,\Omega,\beta,\gamma',p,s,N}\|f\|_{L^p(\Theta_{N,s}(\mathrm{M}_s u+\mathrm{M}_s^2 u))};\tag{6}$$

(ii) *for* $\gamma\in(2,\infty],\ \delta\in(\frac{2}{2-\gamma'},\infty),\ s\in((\frac{\delta-1/2}{\delta-3/2})^2,\infty)$ *and* $p\in(\max\{2\delta'(\frac{\delta-3/2}{\delta-1/2})^2,\frac{2\delta'\gamma'(2\delta-1)}{2\delta-1+(\delta'\gamma'-2)(\sqrt{s})'}\},2]$,

$$\|T^*_{h,\Omega,P_N}f\|_{L^p(u)} \leqslant C_{h,\Omega,\beta,\gamma',p,s,N}\|f\|_{L^p(\Upsilon_{N,s}(\mathrm{M}_s u+\mathrm{M}_s^2 u))}.\tag{7}$$

Here $\Theta_{N,s}u=\mathrm{M}_s^N u+L_{N,s}u+I_{N,s}u+J_{N,s}u,\ L_{N,s}$ *and* $\Upsilon_{N,s}$ *is given as in Theorem* 1, *where* $I_{\lambda,s}u=\sum_{i=1}^{\lambda}\mathrm{M}_s M_{i,s}^{\tilde\sigma}\mathrm{M}_s^{\lambda-i}u,\ J_{\lambda,s}u=\sum_{i=1}^{\lambda}\mathrm{M}_s^2 M_{i-1,s}^{\tilde\sigma}\mathrm{M}_s^{\lambda-i}u$ *for all* $1\leqslant\lambda\leqslant N$. *The above constants* $C_{h,\Omega,\beta,\gamma',p,s,N}$ *are independent of* $\{b_\lambda\}_{\lambda=1}^N$.

REMARK 1. In [26], Zhang established the weighted estimates for T_Ω and T_Ω^*. Theorems 1 and 2 represent an generalization of [26, Theorems 1-2].

As applications of Theorems 1 and 2, we can get the following mixed radial-angular integrability of T_{h,Ω,P_N}, T^*_{h,Ω,P_N} and M_{h,Ω,P_N}.

COROLLARY 1. *Let* $P_N(t)$ *be a real polynomial on* $\mathbb{R}$ *of degree* N *and satisfy* $P_N(0)=0$. *Assume that* Ω *satisfies* (2) *and one of the following conditions holds:*

(a) $h(t)\equiv 1,\ \Omega\in\mathscr{F}_\beta(\mathrm{S}^{n-1})$ *for some* $\beta>1,\ \gamma'=1$ *and* $\delta=\beta$*;*

(b) $h\in\Delta_\gamma(\mathbb{R}_+)$ *for some* $\gamma\in(1,\infty]$ *and* $\Omega\in W\mathscr{F}_\beta(\mathrm{S}^{n-1})$ *for some* $\beta>\max\{2,\gamma'\}$, $\delta=\frac{\beta}{\max\{2,\gamma'\}}$.

Then,

$$\|T_{h,\Omega,P_N}f\|_{L^p_{|x|}L^q_\theta(\mathbb{R}^n)} \leqslant C_{h,\Omega,\beta,\gamma',p,q,N}\|f\|_{L^p_{|x|}L^q_\theta(\mathbb{R}^n)};\tag{8}$$

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}|T_{h,\Omega,P_N}f_j|^q\Big)^{1/q}\Big\|_{L^p_{|x|}L^q_\theta(\mathbb{R}^n)} \leqslant C_{h,\Omega,\beta,\gamma',p,q,N}\Big\|\Big(\sum_{j\in\mathbb{Z}}|f_j|^q\Big)^{1/q}\Big\|_{L^p_{|x|}L^q_\theta(\mathbb{R}^n)};\tag{9}$$

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}|T_{h,\Omega,P_N}f_j|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\leqslant C_{h,\Omega,\beta,\gamma',p,q,N}\Big\|\Big(\sum_{j\in\mathbb{Z}}|f_j|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)},\tag{10}$$

provided that one of the following conditions holds:

(i) $\delta\in(1,\infty),\ s\in(\frac{\delta}{\delta-1},\infty),\ q\in[2,\frac{2(\gamma'-1/s)\delta}{1+\delta(\gamma'-1)}),\ p\in[q,\frac{qs\gamma'}{s\gamma'-1});$

(ii) $\delta\in(1,\infty),\ s\in(\frac{\delta}{\delta-1},\infty),\ q\in(\frac{2\delta(\gamma'-1/s)}{\delta(\gamma'-2/s+1)-1},2],\ p\in(\frac{qs\gamma'}{q-1+s\gamma'},q];$

(iii) $\gamma\in(2,\infty],\ \delta\in(\frac{2}{2-\gamma'},\infty),\ q\in(\delta'\gamma',2],\ p\in[q,\frac{2q\delta'\gamma'}{2\delta'\gamma'-q});$

(iv) $\gamma\in(2,\infty],\ \delta\in(\frac{2}{2-\gamma'},\infty),\ q\in[2,\frac{\delta'\gamma'}{\delta'\gamma'-1}),\ p\in(\frac{2q\delta'\gamma'}{q+2\delta'\gamma'},q].$

The above constants $C_{h,\Omega,\beta,\gamma',p,q,N}>0$ are independent of the coefficients of P_N. The same conclusions hold for M_{h,Ω,P_N} if one of the conditions (i) and (iii) holds.

REMARK 2. It should be pointed out that the range of q will be enlarged and the range of p will be shrink as s enlarges in the condition (i) of Corollary 1. Specially, the range of q is just empty set when $s=\delta'$, and the range of p is just empty set when $s=\infty$.

In particular, we can get the following conclusions.

COROLLARY 2. *Let $P_N(t)$ be a real polynomial on $\mathbb{R}$ of degree N and satisfy $P_N(0)=0$. Assume that Ω satisfies (2) and $\Omega\in\mathscr{F}_\beta(\mathrm{S}^{n-1})$ for some $\beta>1$. Then,*

$$\|T_{\Omega,P_N}f\|_{L^p_{|x|}L^q_\theta(\mathbb{R}^n)}\leqslant C_{\Omega,\beta,p,q,N}\|f\|_{L^p_{|x|}L^q_\theta(\mathbb{R}^n)};\tag{11}$$

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}|T_{\Omega,P_N}f_j|^q\Big)^{1/q}\Big\|_{L^p_{|x|}L^q_\theta(\mathbb{R}^n)}\leqslant C_{\Omega,\beta,p,q,N}\Big\|\Big(\sum_{j\in\mathbb{Z}}|f_j|^q\Big)^{1/q}\Big\|_{L^p_{|x|}L^q_\theta(\mathbb{R}^n)};\tag{12}$$

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}|T_{\Omega,P_N}f_j|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\leqslant C_{\Omega,\beta,p,q,N}\Big\|\Big(\sum_{j\in\mathbb{Z}}|f_j|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)},\tag{13}$$

provided that one of the following conditions holds:

(i) $s\in(\beta',\infty),\ q\in[2,\frac{2\beta}{s'}),\ p\in[q,qs');$

(ii) $s\in(\beta',\infty),\ q\in(\frac{2\beta}{2\beta-s'},2],\ p\in(\frac{qs}{q-1+s},q];$

(iii) $\beta\in(2,\infty),\ q\in(\beta',2],\ p\in[q,\frac{2q\beta'}{2\beta'-q});$

(iv) $\beta\in(2,\infty),\ q\in[2,\beta),\ p\in(\frac{2q\beta'}{2\beta'+q},q].$

The above constants $C_{\Omega,\beta,p,q,N}>0$ are independent of the coefficients of P_N. The same conclusions hold for M_{Ω,P_N} if one of the conditions (i) and (iii) holds.

COROLLARY 3. *Let $P_N(t)$ be a real polynomial on $\mathbb{R}$ of degree N and satisfy $P_N(0)=0$. Assume that Ω satisfies (2) and $\Omega\in\bigcap_{\beta>1}\mathscr{F}_\beta(\mathrm{S}^{n-1})$. Then the inequalities (11)-(13) hold provided that one of the following conditions holds:*

(i) $1<p,q\leqslant 2;$

(ii) $2\leqslant p,q<\infty.$

The same results hold for M_{Ω,P_N} if $1<q\leqslant p\leqslant 2$ or $2\leqslant q\leqslant p<\infty$.

COROLLARY 4. *Let $P_N(t)$ be a real polynomial on $\mathbb{R}$ of degree N and satisfy $P_N(0)=0$. Assume that Ω satisfies (2) and one of the following conditions holds:*

(a) *$h(t)\equiv 1$, $\Omega\in\mathscr{F}_\beta(\mathrm{S}^{n-1})$ for some $\beta>\frac{3}{2}$, $\gamma'=1$ and $\delta=\beta$;*

(b) *$h\in\Delta_\gamma(\mathbb{R}_+)$ for some $\gamma\in(1,\infty]$ and $\Omega\in W\mathscr{F}_\beta(\mathrm{S}^{n-1})$ for some $\beta>\frac{3}{2}\max\{2,\gamma'\}$, $\delta=\frac{\beta}{\max\{2,\gamma'\}}$.*

Then,

$$\|T^*_{h,\Omega,P_N}f\|_{L^p_{|x|}L^q_\theta(\mathbb{R}^n)}\leqslant C_{h,\Omega,\beta,\gamma',p,q,N}\|f\|_{L^p_{|x|}L^q_\theta(\mathbb{R}^n)}; \tag{14}$$

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}|T^*_{h,\Omega,P_N}f_j|^q\Big)^{1/q}\Big\|_{L^p_{|x|}L^q_\theta(\mathbb{R}^n)}\leqslant C_{h,\Omega,\beta,\gamma',p,q,N}\Big\|\Big(\sum_{j\in\mathbb{Z}}|f_j|^q\Big)^{1/q}\Big\|_{L^p_{|x|}L^q_\theta(\mathbb{R}^n)}; \tag{15}$$

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}|T^*_{h,\Omega,P_N}f_j|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\leqslant C_{h,\Omega,\beta,\gamma',p,q,N}\Big\|\Big(\sum_{j\in\mathbb{Z}}|f_j|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}, \tag{16}$$

provided that one of the following conditions holds:

(i) *$\delta\in(\frac{3}{2},\infty)$, $s\in((\frac{\delta-1/2}{\delta-3/2})^2,\infty)$, $q\in[2,\frac{\delta(2\delta-1)(1-1/\sqrt{s})(\gamma'-1/s)}{(\delta\gamma'-\delta+1)(\delta-1/2)(1-1/\sqrt{s})+(1-1/s)\delta-1})$, $p\in[q,\frac{qs\gamma'}{s\gamma'-1})$;*

(ii) *$\gamma\in(2,\infty]$, $\delta\in(\frac{2}{2-\gamma'},\infty)$, $s\in((\frac{\delta-1/2}{\delta-3/2})^2,\infty)$, $q\in(\max\{2\delta'(\frac{\delta-3/2}{\delta-1/2})^2,\frac{2\delta'\gamma'(2\delta-1)}{2\delta-1+(\delta'\gamma'-2)(\sqrt{s})'}\},2]$, $p\in[q,\frac{2q\delta'\gamma'}{2\delta'\gamma'-q})$.*

The above constants $C_{h,\Omega,\beta,\gamma',p,q,N}>0$ are independent of the coefficients of P_N.

The rest of this paper is organized as follows. In Section 2, we shall prove Theorems Theorems 1 and 2. The proofs of Corollaries 1-4 will be given in Section 3. We would like to remark that our arguments are greatly motivated by [21], but our methods and techniques are more delicate and complex than those in [21]. The main ingredients are to establish two criterions of weighted boundedness for the operators of convolution type and the corresponding maximal operators (see Lemmas 1 and 2). The proofs of Corollaries 1-4 are based on Theorems 1 and 2 and the criterion established in Section 3 (see Proposition 1).

Throughout this paper, for any $p\in(1,\infty)$, we let p' denote the dual exponent to p defined as $1/p+1/p'=1$. In what follows, for any function f, we define $\tilde{f}$ by $\tilde{f}(x)=f(-x)$. Let $\mathbb{N}=\{1,2,\ldots\}$. We denote by M^k the Hardy-Littlewood maximal operator M iterated k times for all $k\in\mathbb{N}$. Specially, $\mathrm{M}^k=\mathrm{M}$ when $k=1$. For $s>1$ and $k\in\mathbb{N}$, we denote $\mathrm{M}_s u=(\mathrm{M}u^s)^{1/s}$ and $\mathrm{M}^k_s u=(\mathrm{M}^k u^s)^{1/s}$. For $f\in L^p(u)$, we set $\|f\|_{L^p(u)}:=\big(\int_{\mathbb{R}^n}|f(x)|^p u(x)dx\big)^{1/p}$.

2. Proofs of Theorems 1 and 2

This section is devoted to proving Theorems 1 and 2. Before presenting our proofs, let us establish two general criterions on the weighted boundedness of the convolution operators, which are the heart of our proofs.

LEMMA 1. *Let $\gamma\in[1,\infty)$, $\beta\in(1,\infty)$, $\Lambda\in\mathbb{N}\setminus\{0\}$ and $\{\sigma_{k,\lambda}:0\leqslant\lambda\leqslant\Lambda \text{ and } k\in\mathbb{Z}\}$ be a family of uniformly bounded Borel measures on $\mathbb{R}^n$. Let $\{a_\lambda:1\leqslant\lambda\leqslant\Lambda\}$ be* *a family of nonzero numbers. Suppose that there exist constants $C>0$ such that the following conditions hold for any $1 \leqslant \lambda \leqslant \Lambda$, $k \in \mathbb{Z}$ and $\xi \in \mathbb{R}^n$:*

(a) $\sigma_{k,0}(\xi)=0$ *and* $\|\sigma_{k,\lambda}\| \leqslant C$;

(b) $\max\{|\widehat{\sigma_{k,\lambda}}(\xi)|, \big|\widehat{|\sigma_{k,\lambda}|}(\xi)\big|\} \leqslant C$;

(c) $\max\{|\widehat{\sigma_{k,\lambda}}(\xi)|, \big|\widehat{|\sigma_{k,\lambda}|}(\xi)\big|\} \leqslant C(\log|2^{k\lambda}a_\lambda\xi|)^{-\beta}$ *if* $|2^{k\lambda}a_\lambda\xi|>1$;

(d) $\max\big\{|\widehat{\sigma_{k,\lambda}}(\xi)-\widehat{\sigma_{k,\lambda-1}}(\xi)|, \big|\widehat{|\sigma_{k,\lambda}|}(\xi)-\widehat{|\sigma_{k,\lambda-1}|}(\xi)\big|\big\} \leqslant C|2^{k\lambda}a_\lambda\xi|$;

(e) $M_0^\sigma f(x) \leqslant C|f(x)|$ *and* $\|M_\lambda^\sigma f\|_{L^q(\mathbb{R}^n)} \leqslant C_q\|f\|_{L^q(\mathbb{R}^n)}$ *for all* $q \in (\gamma,\infty)$, *where*

$$M_\lambda^\sigma f(x)=\sup_{k\in\mathbb{Z}}||\sigma_{k,\lambda}|*f(x)|.$$

Then for any nonnegative measurable function u on $\mathbb{R}^n$,

(i) *for* $s \in (\beta',\infty)$ *and* $p \in [2, \frac{2\beta(\gamma-1/s)}{1+\beta(\gamma-1)})$,

$$\Big\|\sum_{k\in\mathbb{Z}}\sigma_{k,\Lambda}*f\Big\|_{L^p(u)} \leqslant C\|f\|_{L^p(L_{\Lambda,s}u)},$$

where $L_{\Lambda,s}u=\sum_{i=0}^{\Lambda}\mathrm{M}_s^{\Lambda+1-i}M_{i,s}^{\tilde\sigma}\mathrm{M}_s u$, $M_{\lambda,s}^{\tilde\sigma}u=(M_\lambda^{\tilde\sigma}u^s)^{1/s}$, *and* $M_\lambda^{\tilde\sigma}f(x):=M_\lambda^\sigma\tilde f(x)$;

(ii) *for* $\gamma\in[1,2)$, $\beta\in(\frac{2}{2-\gamma},\infty)$, $p\in(\beta'\gamma,2]$ *and* $s\in(\frac{2\beta'}{p},\infty)$. *Then*

$$\Big\|\sum_{k\in\mathbb{Z}}\sigma_{k,\Lambda}*f\Big\|_{L^p(u)} \leqslant C\|f\|_{L^p(\Upsilon_{\Lambda,s}u)},$$

where $\Upsilon_{\Lambda,s}u=\mathrm{M}_s^\Lambda u+\mathrm{M}_s^2\widetilde{\mathrm{M}_s^\Lambda u}+H_{\Lambda,s}u$, $H_\lambda u=\sum_{i=1}^{\lambda}\mathrm{M}^2M_i^{\tilde\sigma}\mathrm{M}^{\lambda+1-i}u$ *and* $H_{\lambda,s}u=(H_\lambda u^s)^{1/s}$. *Here, the constants $C>0$ are independent of $\{a_\lambda\}_{\lambda=1}^{\Lambda}$, but depend on Λ.*

Proof. Let u be a nonnegative measurable function defined on $\mathbb{R}^n$. In what follows, we will prove (i) and (ii), respectively.

The proof of (i): For $1\leqslant\lambda\leqslant\Lambda$, we define the Borel measures $\{\mu_{k,\lambda}\}_{k\in\mathbb{Z}}$ on $\mathbb{R}^n$ by

$$\widehat{\mu_{k,\lambda}}(\xi)=\widehat{\sigma_{k,\lambda}}(\xi)\Phi_{\lambda+1}(\xi)-\widehat{\sigma_{k,\lambda-1}}(\xi)\Phi_\lambda(\xi),$$

where Φ_λ is defined by $\Phi_\lambda(\xi)=\prod_{j=\lambda}^{\Lambda}\phi(|2^{kj}a_j\xi|)$ and ϕ is a nonnegative Schwartz function supported in $\{|t|\leqslant 1\}$ satisfying $\phi(t)=1$ when $|t|<1/2$. It is easy to check that

$$\sigma_{k,\Lambda}=\sum_{\lambda=1}^{\Lambda}\mu_{k,\lambda}; \tag{17}$$

$$M_\lambda^\mu f(x)\leqslant \mathrm{M}^{\Lambda-\lambda}M_\lambda^\sigma|f|(x)+\mathrm{M}^{\Lambda-\lambda+1}M_{\lambda-1}^\sigma|f|(x); \tag{18}$$

$$|\widehat{\mu_{k,\lambda}}(x)| \leqslant C\min\{1, |2^{k\lambda}a_\lambda x|\}; \tag{19}$$

$$|\widehat{\mu_{k,\lambda}}(x)| \leqslant C(\log|2^{k\lambda}a_\lambda x|)^{-\beta}, \text{ if } |2^{k\lambda}a_\lambda x| > 1. \tag{20}$$

Then, by (17), we can write

$$\sum_{k\in\mathbb{Z}} \sigma_{k,\Lambda} * f(x) = \sum_{k\in\mathbb{Z}}\sum_{\lambda=1}^{\Lambda} \mu_{k,\lambda} * f(x) = \sum_{\lambda=1}^{\Lambda}\sum_{k\in\mathbb{Z}} \mu_{k,\lambda} * f(x) =: \sum_{\lambda=1}^{\Lambda} T_\lambda f(x), \tag{21}$$

and note that $u \leqslant \mathrm{M}_s u$, $\mathrm{M}_s u \in A_1$ (see [8]), it follows from (18) that

$$\sum_{\lambda=1}^{\Lambda} \mathrm{M}_s M^{\tilde{\mu}}_{\lambda,s}\mathrm{M}_s u \leqslant \sum_{\lambda=1}^{\Lambda} (\mathrm{M}_s^{\Lambda+1-\lambda} M^{\tilde{\sigma}}_{\lambda,s}\mathrm{M}_s u + \mathrm{M}_s^{\Lambda+2-\lambda} M^{\tilde{\sigma}}_{\lambda-1,s}\mathrm{M}_s u) \leqslant 2L_{\Lambda,s}u.$$

Therefore, it suffices to show that

$$\|T_\lambda f\|_{L^p(u)} \leqslant C\|f\|_{L^p(\mathrm{M}_s M^{\tilde{\mu}}_{\lambda,s}u)} \tag{22}$$

for all $1 \leqslant \lambda \leqslant \Lambda$, $p \in [2, \frac{2\beta(\gamma-1/s)}{1+\beta(\gamma-1)})$, $s \in (\beta', \infty)$ and $u \in A_1$.

We now prove (22). Fix $u \in A_1$. For $1 \leqslant \lambda \leqslant \Lambda$, let $\Psi_\lambda(t) \in \mathscr{C}_c^\infty((1/4,1))$ such that $0 \leqslant \Psi_\lambda \leqslant 1$ and $\sum_{k\in\mathbb{Z}}(\Psi_\lambda(2^{k\lambda}|a_\lambda\xi|))^3 = 1$. Define the Fourier multiplier operators $\{S_{k,\lambda}\}_{k\in\mathbb{Z}}$ by $S_{k,\lambda}f(x) = \Theta_{k,\lambda} * f(x)$, where $\widehat{\Theta_{k,\lambda}}(\xi) = \Psi_\lambda(2^{k\lambda}|a_\lambda\xi|)$. Then it follows from [19] that for $1 < p < \infty$ and $w \in A_p$,

$$\Big\|\Big(\sum_{k\in\mathbb{Z}} |S_{k,\lambda}f|^2\Big)^{1/2}\Big\|_{L^p(w)} \leqslant C_{p,w,\lambda}\|f\|_{L^p(w)} \tag{23}$$

and

$$\Big\|\sum_{k\in\mathbb{Z}} S_{k,\lambda}f_k\Big\|_{L^p(w)} \leqslant C_{p,w,\lambda}\Big\|\Big(\sum_{k\in\mathbb{Z}} |f_k|^2\Big)^{1/2}\Big\|_{L^p(w)}. \tag{24}$$

And we can write

$$T_\lambda f(x) = \sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} S^3_{j+k,\lambda}(\mu_{k,\lambda} * f)(x) = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} S^3_{j+k,\lambda}(\mu_{k,\lambda} * f)(x) =: \sum_{j\in\mathbb{Z}} T_{\lambda,j}f(x).$$

So,

$$\|T_\lambda f\|_{L^p(u)} \leqslant \sum_{j\in\mathbb{Z}} \|T_{\lambda,j}f\|_{L^p(u)}. \tag{25}$$

Now we estimate $\|T_{\lambda,j}f\|_{L^p(u)}$. By (19)-(20) and Plancherel's theorem,

$$\|\mu_{k,\lambda} * S_{j+k,\lambda}f\|_{L^2(\mathbb{R}^n)} \leqslant C(1+|j|)^{-\beta}\|f\|_{L^2(\mathbb{R}^n)}.$$

On the other hand, for $s > 1$, we have

$$\begin{aligned}\|\mu_{k,\lambda} * S_{j+k,\lambda}f\|_{L^2(u^s)} &\leqslant (\|\mu_{k,\lambda}\|\|\Theta_{j+k,\lambda}\|_{L^1(\mathbb{R}^n)})^{1/2} \\ &\quad\times\Big(\int_{\mathbb{R}^n} |\mu_{k,\lambda}| * |\Theta_{j+k,\lambda}| * |f|^2(x)u^s(x)dx\Big)^{1/2} \\ &\leqslant C\|f\|_{L^2(\mathrm{M}M^{\tilde{\mu}}_\lambda u^s)}.\end{aligned}$$

Thus, an interpolation of L^2-spaces with change of measure ([2, Theorem 5.4.1]) implies that

$$\|\mu_{k,\lambda} * S_{j+k,\lambda} f\|_{L^2(u)} \leqslant C(1+|j|)^{-\beta(1-1/s)} \|f\|_{L^2(\mathrm{M}_s M^{\tilde{\mu}}_{\lambda,s} u)}. \tag{26}$$

This combing with (23) yields that

$$\begin{aligned}
\|T_{\lambda,j} f\|_{L^2(u)} &= \Big\| \sum_{k\in\mathbb{Z}} S^3_{j+k,\lambda} \mu_{k,\lambda} * f \Big\|_{L^2(u)} \\
&\leqslant C_\lambda \Big(\sum_{k\in\mathbb{Z}} \|\mu_{k,\lambda} * S^2_{j+k,\lambda} f\|^2_{L^2(u)} \Big)^{1/2} \\
&\leqslant C(1+|j|)^{-\beta(1-1/s)} \Big\| \Big(\sum_{k\in\mathbb{Z}} |S_{j+k,\lambda} f|^2 \Big)^{1/2} \Big\|_{L^2(\mathrm{M}_s M^{\tilde{\mu}}_{\lambda,s} u)} \\
&\leqslant C(1+|j|)^{-\beta(1-1/s)} \|f\|_{L^2(\mathrm{M}_s M^{\tilde{\mu}}_{\lambda,s} u)},
\end{aligned} \tag{27}$$

since $\mathrm{M}_s M^{\tilde{\mu}}_{\lambda,s} u \in A_p$.

Next we will prove

$$\|T_{\lambda,j} f\|_{L^p(u)} \leqslant C \|f\|_{L^p(\mathrm{M}_s M^{\tilde{\mu}}_{\lambda,s} u)}, \quad p \in (2, \frac{2(\gamma - 1/s)}{\gamma - 1}). \tag{28}$$

Fix $p \in (2, \frac{2(\gamma-1/s)}{\gamma-1})$, and choose a function $v \in L^{(p/2)'}(u)$ with unit norm such that

$$\Big\| \Big(\sum_{k\in\mathbb{Z}} |\mu_{k,\lambda} * g_k|^2 \Big)^{1/2} \Big\|^2_{L^p(u)} = \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} |\mu_{k,\lambda} * g_k(x)|^2 \cdot v(x) u(x) dx,$$

which together with the fact that $\|\mu_{k,\lambda}\| \leqslant C$ leads to

$$\Big\| \Big(\sum_{k\in\mathbb{Z}} |\mu_{k,\lambda} * g_k|^2 \Big)^{1/2} \Big\|^2_{L^p(u)} \leqslant C \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} |g_k(x)|^2 \big| |\tilde{\mu}_{k,\lambda}| * (vu)(x) \big| dx.$$

And for $r := \frac{ps}{2}$, the Hölder inequality tells us that

$$\big| |\tilde{\mu}_{k,\lambda}| * (vu) \big| \leqslant (|\tilde{\mu}_{k,\lambda}| * u^s)^{1/r} (|\tilde{\mu}_{k,\lambda}| * (u^{r'/(p/2)'} v^{r'}))^{1/r'}.$$

Hence, by Hölder's inequality with exponents $\frac{p}{2}$ and $(\frac{p}{2})'$ again, we get

$$\begin{aligned}
&\Big\| \Big(\sum_{k\in\mathbb{Z}} |\mu_{k,\lambda} * g_k|^2 \Big)^{1/2} \Big\|^2_{L^p(u)} \\
&\quad \leqslant C \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} |g_k(x)|^2 (M^{\tilde{\mu}}_\lambda u^s)^{1/r} (M^{\tilde{\mu}}_\lambda (u^{r'/(p/2)'} v^{r'}))^{1/r'}(x) dx \\
&\quad \leqslant C \Big\| \Big(\sum_{k\in\mathbb{Z}} |g_k|^2 \Big)^{1/2} \Big\|^2_{L^p(M^{\tilde{\mu}}_{\lambda,s} u)} \|M^{\tilde{\mu}}_\lambda (u^{r'/(p/2)'} v^{r'})\|^{1/r'}_{L^{(p/2)'/r'}(\mathbb{R}^n)}.
\end{aligned}$$

Also, it follows from our assumptions (e) and (18) that

$$\|M^{\tilde{\mu}}_\lambda f\|_{L^t(\mathbb{R}^n)} \leqslant C \|f\|_{L^t(\mathbb{R}^n)}, \quad \forall t \in (\gamma, \infty),$$

which leads to

$$\|M_{\lambda}^{\tilde{\mu}}(u^{r'/(p/2)'}v^{r'})\|_{L^{(p/2)'/r'}(\mathbb{R}^n)}^{1/r'} \leqslant C\|u^{r'/(p/2)'}v^{r'}\|_{L^{(p/2)'/r'}(\mathbb{R}^n)}^{1/r'} \leqslant C,$$

since $(p/2)' > r'\gamma$. Consequently, for $p \in (2, \frac{2(\gamma-1/s)}{\gamma-1})$ and $s \in (1,\infty)$,

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}|\mu_{k,\lambda} * g_k|^2\Big)^{1/2}\Big\|_{L^p(u)} \leqslant C\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^2\Big)^{1/2}\Big\|_{L^p(M_{\lambda,s}^{\tilde{\mu}}u)}.$$

Noticing that $M_{\lambda,s}^{\tilde{\mu}}u \leqslant \mathrm{M}_s M_{\lambda,s}^{\tilde{\mu}}u$, and invoking (23)-(24), we deduce that

$$\begin{aligned}\|T_{\lambda,j}f\|_{L^p(u)} &= \Big\|\sum_{k\in\mathbb{Z}} S_{j+k,\lambda}^3 \mu_{k,\lambda} * f\Big\|_{L^p(u)}\\ &\leqslant C\Big\|\Big(\sum_{k\in\mathbb{Z}}|\mu_{k,\lambda} * S_{j+k,\lambda}^2 f|^2\Big)^{1/2}\Big\|_{L^p(u)}\\ &\leqslant C\Big\|\Big(\sum_{k\in\mathbb{Z}}|S_{j+k,\lambda}^2 f|^2\Big)^{1/2}\Big\|_{L^p(M_{\lambda,s}^{\tilde{\mu}}u)}\\ &\leqslant C\|f\|_{L^p(\mathrm{M}_s M_{\lambda,s}^{\tilde{\mu}}u)}\end{aligned}$$

for all $p \in (2, \frac{2(\gamma-1/s)}{\gamma-1})$. This proves (28).

Since $\beta/s' > 1$, for $p \in [2, \frac{2\beta(\gamma-1/s)}{1+\beta(\gamma-1)})$, there exist $p_1 \in [2, \frac{2(\gamma-1/s)}{\gamma-1})$ and $\theta \in (s'/\beta, 1]$ such that $1/p = \theta/2 + (1-\theta)/p_1$. Then interpolating between (27) and (28) yields that

$$\|T_{\lambda,j}f\|_{L^p(u)} \leqslant C(1+|j|)^{-\theta\beta(1-1/s)}\|f\|_{L^p(\mathrm{M}_s M_{\lambda,s}^{\tilde{\mu}}u)}.$$

This together with (25) yields (22) and completes the proof of (i).

The proof of (ii): Let $\gamma \in [1,2)$ and $\beta \in (\frac{2}{2-\gamma}, \infty)$. Employing the notation in the proof of (i), we need to show that

$$\|T_\lambda f\|_{L^p(u)} \leqslant C\|f\|_{L^p(\Upsilon_{\Lambda,s}u)} \tag{29}$$

for all $1 \leqslant \lambda \leqslant \Lambda$, $p \in (\beta'\gamma, 2]$ and $s \in (2\beta'/p, \infty)$. Note that

$$(\mathrm{M}^\Lambda u^s + \mathrm{M}^2\widetilde{\mathrm{M}^\Lambda u^s} + H_\Lambda u^s)^{1/s} \leqslant \mathrm{M}_s^\Lambda u + \mathrm{M}_s^2\widetilde{\mathrm{M}_s^\Lambda} u + H_{\Lambda,s}u = \Upsilon_{\Lambda,s}u.$$

It suffices to prove that

$$\|T_\lambda f\|_{L^p(u^{1/s})} \leqslant C\|f\|_{L^p((\mathrm{M}^\Lambda u + \mathrm{M}^2\widetilde{\mathrm{M}^\Lambda u} + H_\Lambda u)^{1/s})} \tag{30}$$

for all $1 \leqslant \lambda \leqslant \Lambda$, $p \in (\beta'\gamma, 2]$ and $s \in (2\beta'/p, \infty)$.

We now prove (30). Define the family of Borel measures $\{\omega_{k,\lambda}\}_{k\in\mathbb{Z}}$ on $\mathbb{R}^n$ by

$$\widehat{\omega_{k,\lambda}}(\xi) = |\widehat{\sigma_{k,\lambda}}|(\xi) - \psi_{k,\lambda}(\xi)|\widehat{\sigma_{k,\lambda-1}}|(\xi), \tag{31}$$

where $\psi_{k,\lambda}$ is defined by $\widehat{\psi_{k,\lambda}}(\xi) = \phi(2^{k\lambda}|a_\lambda \xi|)$. One can easily verify that

$$|\widehat{\omega_{k,\lambda}}(x)| \leqslant C\min\{1, |2^{k\lambda}a_\lambda x|\}; \tag{32}$$

$$|\widehat{\omega_{k,\lambda}}(x)| \leqslant C(\log|2^{k\lambda}a_\lambda x|)^{-\beta}, \ \text{ if } |a_\lambda x| > 1; \tag{33}$$

$$M_\lambda^\omega f(x) \leqslant M_\lambda^\sigma |f|(x) + \mathrm{M}M_{\lambda-1}^\sigma |f|(x); \tag{34}$$

$$M_\lambda^\sigma f(x) \leqslant \mathrm{M}M_{\lambda-1}^\sigma |f|(x) + G_\lambda^\omega f(x), \tag{35}$$

where

$$M_\lambda^\omega f(x) := \sup_{k\in\mathbb{Z}} ||\omega_{k,\lambda}| * f(x)| \ \text{ and } \ G_\lambda^\omega f(x) := \Big(\sum_{k\in\mathbb{Z}} |\omega_{k,\lambda} * f(x)|^2\Big)^{1/2}.$$

Then for $s > 1$, it follows from (35) that

$$\|M_\lambda^\sigma f\|_{L^p(u^{1/s})} \leqslant \|\mathrm{M}M_{\lambda-1}^\sigma |f|\|_{L^p(u^{1/s})} + \|G_\lambda^\omega f\|_{L^p(u^{1/s})}, \quad 1 < p < \infty. \tag{36}$$

And the well-known Fefferman-Stein inequality for M (see [17]) tells us that

$$\|\mathrm{M}f\|_{L^p(u)} \leqslant C_p \|f\|_{L^p(\mathrm{M}u)}, \quad 1 < p < \infty, \tag{37}$$

which deduces that

$$\|\mathrm{M}M_{\lambda-1}^\sigma |f|\|_{L^p(u^{1/s})} \leqslant C\|M_{\lambda-1}^\sigma |f|\|_{L^p(\mathrm{M}u^{1/s})} \leqslant C\|M_{\lambda-1}^\sigma |f|\|_{L^p((\mathrm{M}u)^{1/s})}, \quad 1 < p < \infty. \tag{38}$$

For $G_\lambda^\omega f$, by Minkowski's inequality, we have

$$\begin{aligned} G_\lambda^w f(x) &= \Big(\sum_{k\in\mathbb{Z}} \Big|\omega_{k,\lambda} * \sum_{j\in\mathbb{Z}} S_{j+k,\lambda}^3 f(x)\Big|^2\Big)^{1/2} \leqslant \sum_{j\in\mathbb{Z}} \Big(\sum_{k\in\mathbb{Z}} |\omega_{k,\lambda} * S_{j+k,\lambda}^3 f(x)|^2\Big)^{1/2} \\ &=: \sum_{j\in\mathbb{Z}} G_{\lambda,j} f(x). \end{aligned}$$

Consequently,

$$\|G_\lambda^w f\|_{L^p(u^{1/s})} \leqslant \sum_{j\in\mathbb{Z}} \|G_{\lambda,j} f\|_{L^p(u^{1/s})}.$$

In what follows, we estimate $\|G_{\lambda,j} f\|_{L^p(u^{1/s})}$. It is not difficult to see that

$$\|\omega_{k,\lambda} * f\|_{L^\infty(\mathbb{R}^n)} \leqslant C\|f\|_{L^\infty(\mathbb{R}^n)},$$

and

$$\|\omega_{k,\lambda} * f\|_{L^1(u)} \leqslant C\|f\|_{L^1(M_\lambda^{\tilde{\sigma}} u + M_{\lambda-1}^{\tilde{\sigma}} \mathrm{M}u)} \leqslant C\|f\|_{L^1(\mathrm{M}M_\lambda^{\tilde{\sigma}} u + \mathrm{M}M_{\lambda-1}^{\tilde{\sigma}} \mathrm{M}u)}.$$

An interpolation gives

$$\|\omega_{k,\lambda} * f\|_{L^p(u)} \leqslant C\|f\|_{L^p(\mathrm{M}M_\lambda^{\tilde{\sigma}} u + \mathrm{M}M_{\lambda-1}^{\tilde{\sigma}} \mathrm{M}u)}, \quad 1 < p < \infty,$$

which implies that

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}|\omega_{k,\lambda}*f_k|^p\Big)^{1/p}\Big\|_{L^p(u)}\leqslant C\Big\|\Big(\sum_{k\in\mathbb{Z}}|f_k|^p\Big)^{1/p}\Big\|_{L^p(\mathrm{M}M_\lambda^{\tilde{\sigma}}u+\mathrm{M}M_{\lambda-1}^{\tilde{\sigma}}\mathrm{M}u)},\quad 1<p<\infty. \tag{39}$$

On the other hand, by (34) and our assumption (e), we have

$$\Big\|\sup_{k\in\mathbb{Z}}|\omega_{k,\lambda}*f_k|\Big\|_{L^p(\mathbb{R}^n)}\leqslant C\Big\|\sup_{k\in\mathbb{Z}}|f_k|\Big\|_{L^p(\mathbb{R}^n)} \tag{40}$$

for all $p\in(\gamma,2]$. Interpolating between (39) and (40) gives

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}|\omega_{k,\lambda}*f_k|^2\Big)^{1/2}\Big\|_{L^p(u^{1/t_1})}\leqslant C\Big\|\Big(\sum_{k\in\mathbb{Z}}|f_k|^2\Big)^{1/2}\Big\|_{L^p((\mathrm{M}M_\lambda^{\tilde{\sigma}}u+\mathrm{M}M_{\lambda-1}^{\tilde{\sigma}}\mathrm{M}u)^{1/t_1})}$$

for all $p\in(\gamma,2]$, where $t_1=2/p$. This leads to

$$\begin{aligned}\Big\|\Big(\sum_{k\in\mathbb{Z}}|\omega_{k,\lambda}*f_k|^2\Big)^{1/2}\Big\|_{L^p(u)}&\leqslant C\Big\|\Big(\sum_{k\in\mathbb{Z}}|f_k|^2\Big)^{1/2}\Big\|_{L^p((\mathrm{M}M_\lambda^{\tilde{\sigma}}u^{t_1}+\mathrm{M}M_{\lambda-1}^{\tilde{\sigma}}\mathrm{M}u^{t_1})^{1/t_1})}\\&\leqslant C\Big\|\Big(\sum_{k\in\mathbb{Z}}|f_k|^2\Big)^{1/2}\Big\|_{L^p((\mathrm{M}_{t_1}M_{\lambda,t_1}^{\tilde{\sigma}}u+\mathrm{M}_{t_1}M_{\lambda-1,t_1}^{\tilde{\sigma}}\mathrm{M}_{t_1}u))}.\end{aligned} \tag{41}$$

Hence,

$$\begin{aligned}\|G_{\lambda,j}f\|_{L^p(u)}&=\Big\|\Big(\sum_{k\in\mathbb{Z}}|\omega_{k,\lambda}*S_{j+k,\lambda}^3f|^2\Big)^{1/2}\Big\|_{L^p(u)}\\&\leqslant C\Big\|\Big(\sum_{k\in\mathbb{Z}}|S_{j+k,\lambda}^3f|^2\Big)^{1/2}\Big\|_{L^p(\mathrm{M}_{t_1}M_{\lambda,t_1}^{\tilde{\sigma}}u+\mathrm{M}_{t_1}M_{\lambda-1,t_1}^{\tilde{\sigma}}\mathrm{M}_{t_1}u)}\\&\leqslant C\|f\|_{L^p(\mathrm{M}_{t_1}M_{\lambda,t_1}^{\tilde{\sigma}}u+\mathrm{M}_{t_1}M_{\lambda-1,t_1}^{\tilde{\sigma}}\mathrm{M}_{t_1}u)},\quad \gamma<p\leqslant 2,\end{aligned}$$

since $\mathrm{M}_{t_1}M_{\lambda,t_1}^{\tilde{\sigma}}u+\mathrm{M}_{t_1}M_{\lambda-1,t_1}^{\tilde{\sigma}}\mathrm{M}_{t_1}u\in A_1$, and the weighted Littlewood-Paley theory and (41). Substituting u^{1/t_1} for u, we get

$$\|G_{\lambda,j}f\|_{L^p(u^{1/t_1})}\leqslant C\|f\|_{L^p((\mathrm{M}M_\lambda^{\tilde{\sigma}}u+\mathrm{M}M_{\lambda-1}^{\tilde{\sigma}}\mathrm{M}u)^{1/t_1})},\quad \gamma<p\leqslant 2. \tag{42}$$

By (32)-(33) and the arguments similar to those used in deriving (26), we can obtain that for $s>1$,

$$\|\omega_{k,\lambda}*S_{j+k,\lambda}f\|_{L^2(u)}\leqslant C(1+|j|)^{-\beta(1-1/s)}\|f\|_{L^2(\mathrm{M}_sM_{\lambda,s}^{\tilde{\omega}}u)}.$$

This together with (24) deduces that

$$\begin{aligned}\|G_{\lambda,j}f\|_{L^2(u)}&=\Big\|\Big(\sum_{k\in\mathbb{Z}}|\omega_{k,\lambda}*S_{j+k,\lambda}^3f|^2\Big)^{1/2}\Big\|_{L^2(u)}\\&\leqslant\Big\|\Big(\sum_{k\in\mathbb{Z}}|\omega_{k,\lambda}*S_{j+k,\lambda}^2f|^2\Big)^{1/2}\Big\|_{L^2(u)}\\&\leqslant C(1+|j|)^{-\beta(1-1/s)}\Big\|\Big(\sum_{k\in\mathbb{Z}}|S_{j+k,\lambda}f|^2\Big)^{1/2}\Big\|_{L^2(\mathrm{M}_sM_{\lambda,s}^{\tilde{\omega}}u)}\\&\leqslant C(1+|j|)^{-\beta(1-1/s)}\|f\|_{L^2(\mathrm{M}_sM_{\lambda,s}^{\tilde{\omega}}u)}.\end{aligned} \tag{43}$$

Take $s=t_1$ and substitute u^{1/t_1} for u in (43), we obtain

$$\|G_{\lambda,j}f\|_{L^2(u^{1/t_1})} \leqslant C(1+|j|)^{-\beta(1-1/t_1)}\|f\|_{L^2((\mathrm{M}M_\lambda^{\tilde{\omega}}u)^{1/t_1})}. \tag{44}$$

Note that by (34)

$$\mathrm{M}M_\lambda^{\tilde{\omega}}u \leqslant \mathrm{M}M_\lambda^{\tilde{\sigma}}u + \mathrm{M}^2M_{\lambda-1}^{\tilde{\sigma}}u \leqslant \mathrm{M}M_\lambda^{\tilde{\sigma}}\mathrm{M}u + \mathrm{M}^2M_{\lambda-1}^{\tilde{\sigma}}\mathrm{M}u.$$

It follows from (44) that

$$\|G_{\lambda,j}f\|_{L^2(u^{1/t_1})} \leqslant C(1+|j|)^{-\beta(1-1/t_1)}\|f\|_{L^2((\mathrm{M}M_\lambda^{\tilde{\sigma}}\mathrm{M}u+\mathrm{M}^2M_{\lambda-1}^{\tilde{\sigma}}\mathrm{M}u)^{1/t_1})}. \tag{45}$$

Also, for $\beta\in(2,\infty)$, $p\in(\gamma\beta',2]$ and $s\in(\frac{2\beta'}{p},\infty)$, there exists $q\in(\gamma,2)$ such that $p\in(q\beta',2]$, $s=2/q$ and $\theta\in(s'/\beta,1]$ satisfying $1/p=\theta/2+(1-\theta)/q$. An interpolation between (42) and (45) leads to

$$\|G_{\lambda,j}f\|_{L^p(u^{1/s})} \leqslant CA(1+|j|)^{-\theta\beta/s'}\|f\|_{L^p((\mathrm{M}M_\lambda^{\tilde{\sigma}}\mathrm{M}u+\mathrm{M}^2M_{\lambda-1}^{\tilde{\sigma}}\mathrm{M}u)^{1/s})}.$$

So,

$$\|G_\lambda^{\omega}f\|_{L^p(u^{1/s})} \leqslant \sum_{j\in\mathbb{Z}}\|G_{\lambda,j}f\|_{L^p(u^{1/s})} \leqslant C\|f\|_{L^p((\mathrm{M}M_\lambda^{\tilde{\sigma}}\mathrm{M}u+\mathrm{M}^2M_{\lambda-1}^{\tilde{\sigma}}\mathrm{M}u)^{1/s})}$$

for all $\beta>2$, $p\in(\beta'\gamma,2]$ and $s\in(2\beta'/p,\infty)$. This together with (36) and (38) implies that

$$\|M_\lambda^{\sigma}f\|_{L^p(u^{1/s})} \leqslant C(\|M_{\lambda-1}^{\sigma}|f|\|_{L^p((\mathrm{M}u)^{1/s})} + \|f\|_{L^p((\mathrm{M}M_\lambda^{\tilde{\sigma}}\mathrm{M}u+\mathrm{M}^2M_{\lambda-1}^{\tilde{\sigma}}\mathrm{M}u)^{1/s})}) \tag{46}$$

for all $p\in(\beta'\gamma,2]$ and $s\in(2\beta'/p,\infty)$.

We now prove that

$$\|M_\lambda^{\sigma}f\|_{L^p(u^{1/t_1})} \leqslant C\|f\|_{L^p((\mathrm{M}^\lambda u+\mathrm{M}^2\widetilde{\mathrm{M}^\lambda} u+H_\lambda u)^{1/t_1})} \tag{47}$$

for all $1\leqslant\lambda\leqslant\Lambda$, $p\in(\beta'\gamma,2]$, $s\in(2\beta'/p,\infty)$ and $t_1=2/p$.

When $\lambda=1$, we get from our assumption (e) and (46) that

$$\begin{aligned}\|M_1^{\sigma}f\|_{L^p(u^{1/s})} &\leqslant C(\|M_0^{\sigma}|f|\|_{L^p((\mathrm{M}u)^{1/s})} + \|f\|_{L^p((\mathrm{M}M_1^{\tilde{\sigma}}\mathrm{M}u+\mathrm{M}^2M_0^{\tilde{\sigma}}\mathrm{M}u)^{1/s})})\\ &\leqslant C\|f\|_{L^p((\mathrm{M}u+\mathrm{M}^2\widetilde{\mathrm{M}}u+\mathrm{M}M_1^{\tilde{\sigma}}\mathrm{M}u)^{1/s})}\\ &\leqslant C\|f\|_{L^p((\mathrm{M}u+\mathrm{M}^2\widetilde{\mathrm{M}}u+H_1u)^{1/s})}\end{aligned}$$

for any $p\in(\beta'\gamma,2]$, which proves (47) for $\lambda=1$. Assume that (47) holds for $\lambda=\iota-1$ with $\iota\in\{2,\dots,\Lambda\}$. Combining this assumption with (46) yields that

$$\begin{aligned}\|M_\iota^{\sigma}f\|_{L^p(u^{1/s})} &\leqslant C(\|M_{\iota-1}^{\sigma}|f|\|_{L^p((\mathrm{M}u)^{1/s})} + \|f\|_{L^p((\mathrm{M}M_\iota^{\tilde{\sigma}}\mathrm{M}u+\mathrm{M}^2M_{\iota-1}^{\tilde{\sigma}}\mathrm{M}u)^{1/s})})\\ &\leqslant C(\|f\|_{L^p((\mathrm{M}^{\iota-1}\mathrm{M}u+\mathrm{M}^2\widetilde{\mathrm{M}^{\iota-1}\mathrm{M}}u+H_{\iota-1}\mathrm{M}u)^{1/s})} + \|f\|_{L^p((\mathrm{M}M_\iota^{\tilde{\sigma}}\mathrm{M}u+\mathrm{M}^2M_{\iota-1}^{\tilde{\sigma}}\mathrm{M}u)^{1/s})}\\ &\leqslant C\|f\|_{L^p((\mathrm{M}^\iota u+\mathrm{M}^2\widetilde{\mathrm{M}^\iota}u+H_\iota\mathrm{M}u)^{1/s})}\end{aligned}$$

for all $p\in(\beta'\gamma,2]$. This yields (47) for $\lambda=\iota$. Then (47) is proved.

Using (18), (47) and (37), we have

$$\begin{aligned}\|M_{\lambda}^{\mu}f\|_{L^p(u^{1/s})}&\leqslant\|\mathrm{M}^{\Lambda-\lambda}M_{\lambda}^{\sigma}|f|\|_{L^p(u^{1/s})}+\|\mathrm{M}^{\Lambda-\lambda+1}M_{\lambda-1}^{\sigma}|f|\|_{L^p(u^{1/s})}\\&\leqslant C(\|M_{\lambda}^{\sigma}|f|\|_{L^p((\mathrm{M}^{\Lambda-\lambda}u)^{1/s})}+\|M_{\lambda-1}^{\sigma}|f|\|_{L^p((\mathrm{M}^{\Lambda-\lambda+1}u)^{1/s})})\\&\leqslant C(\|f\|_{L^p((\mathrm{M}^{\lambda}(\mathrm{M}^{\Lambda-\lambda}u)+\mathrm{M}^2\mathrm{M}^{\lambda}\widetilde{(\mathrm{M}^{\Lambda-\lambda}u)}+H_{\lambda}(\mathrm{M}^{\Lambda-\lambda}u))^{1/s})}\\&\quad+\|f\|_{L^p((\mathrm{M}^{\lambda-1}(\mathrm{M}^{\Lambda-\lambda+1}u)+\mathrm{M}^2\mathrm{M}^{\lambda-1}\widetilde{(\mathrm{M}^{\Lambda-\lambda+1}u)}+H_{\lambda-1}(\mathrm{M}^{\Lambda-\lambda+1}u))^{1/s})})\\&\leqslant C\|f\|_{L^p((\mathrm{M}^{\Lambda}u+\mathrm{M}^2\widetilde{\mathrm{M}^{\Lambda}u}+H_{\Lambda}u)^{1/s})}\end{aligned}\tag{48}$$

for all $1\leqslant\lambda\leqslant\Lambda$, $p\in(\beta'\gamma,2]$ and $s\in(2\beta'/p,\infty)$. Then (30) follows from (48) and Lemma in [26, p.1574]. Lemma 1 is proved.

LEMMA 2. *Let γ, β, Λ, $\{\sigma_{k,\lambda}\}_k$, $\{a_\lambda\}_{\lambda=1}^{\Lambda}$, M_{λ}^{σ}, $L_{\Lambda,s}$, $\Upsilon_{N,s}$ and a_1,a_2 be given as in Lemma 1.*

(i) *Let $\beta\in(1,\infty)$, $s\in(\beta',\infty)$ and $p\in[2,\frac{2\beta(\gamma-1/s)}{1+\beta(\gamma-1)})$. Then for any nonnegative measurable function u on $\mathbb{R}^n$,*

$$\|M_{\Lambda}^{\sigma}f\|_{L^p(u)}\leqslant C\|f\|_{L^p(\Theta_{\Lambda,s}\mathrm{M}_s u)};$$

(ii) *Let $\gamma\in[1,2)$, $\beta\in(\frac{2}{2-\gamma},\infty)$, $p\in(\beta'\gamma,2]$ and $s\in(\frac{2\beta'}{p},\infty)$. Then for any nonnegative measurable function u on $\mathbb{R}^n$,*

$$\|M_{\Lambda}^{\sigma}f\|_{L^p(u)}\leqslant C\|f\|_{L^p(\Upsilon_{\Lambda,s}\mathrm{M}_s u)};$$

(iii) *Let $\beta\in(\frac{3}{2},\infty)$, $s\in((\frac{\beta-1/2}{\beta-3/2})^2,\infty)$ and $p\in[2,\frac{\beta(2\beta-1)(1-1/\sqrt{s})(\gamma-1/s)}{(\beta\gamma-\beta+1)(\beta-1/2)(1-1/\sqrt{s})+(1-1/s)\beta-1})$. Then for any nonnegative measurable function u on $\mathbb{R}^n$,*

$$\Big\|\sup_{k\in\mathbb{Z}}\Big|\sum_{j=k}^{\infty}\sigma_{j,\Lambda}*f\Big|\Big\|_{L^p(u)}\leqslant C\|f\|_{L^p(\Theta_{\Lambda,s}(\mathrm{M}_s u+\mathrm{M}_s^2 u))};$$

(iv) *Let $\gamma\in[1,2)$, $\beta\in(\frac{2}{2-\gamma},\infty)$, $s\in((\frac{\beta-1/2}{\beta-3/2})^2,\infty)$ and*

$$p\in(\max\{2\beta'(\frac{\beta-3/2}{\beta-1/2})^2,\frac{2\beta'\gamma(2\beta-1)}{2\beta-1+(\beta'\gamma-2)(\sqrt{s})'}\},2].$$

Then for any nonnegative measurable function u on $\mathbb{R}^n$,

$$\Big\|\sup_{k\in\mathbb{Z}}\Big|\sum_{j=k}^{\infty}\sigma_{j,\Lambda}*f\Big|\Big\|_{L^p(u)}\leqslant C\|f\|_{L^p(\Upsilon_{\Lambda,s}(\mathrm{M}_s u+\mathrm{M}_s^2 u))}.$$

Here $\Theta_{\Lambda,s}u=\mathrm{M}_s^{\Lambda}u+L_{\Lambda,s}u+I_{\Lambda,s}u+J_{\Lambda,s}u$, where

$$I_{\lambda,s}u=\sum_{i=1}^{\lambda}\mathrm{M}_s M_{i,s}^{\tilde{\sigma}}\mathrm{M}_s^{\lambda-i}u,\quad J_{\lambda,s}u=\sum_{i=1}^{\lambda}\mathrm{M}_s^2 M_{i-1,s}^{\tilde{\sigma}}\mathrm{M}_s^{\lambda-i}u,\ \ \forall 1\leqslant\lambda\leqslant\Lambda.$$

The constants $C>0$ are independent of $\{a_\lambda\}_{\lambda=1}^{\Lambda}$, but depend on Λ.

Proof. Let u be a nonnegative measurable function defined on $\mathbb{R}^n$. In what follows, we will prove (i)–(iv), respectively.

The proof of (i): Employing the notation in the proof of Lemma 1, by the arguments similar to those used in deriving (2), we can get

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}|\omega_{k,\lambda}*g_k|^2\Big)^{1/2}\Big\|_{L^p(u)}\leqslant C\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^2\Big)^{1/2}\Big\|_{L^p(M^{\tilde\omega}_{\lambda,s}u)},1<s<\infty,2<p<\frac{2(\gamma-1/s)}{\gamma-1}.$$

Applying the weighted Littlewood-Paley theory and the fact that $\mathrm{M}_sM^{\tilde\omega}_{\lambda,s}u\in A_1$, we get

$$\begin{aligned}\|G_{\lambda,j}f\|_{L^p(u)}&=\Big\|\Big(\sum_{k\in\mathbb{Z}}|\omega_{k,\lambda}*S^3_{j+k}f|^2\Big)^{1/2}\Big\|_{L^p(u)}\\&\leqslant C\Big\|\Big(\sum_{k\in\mathbb{Z}}|S^3_{j+k}f|^2\Big)^{1/2}\Big\|_{L^p(\mathrm{M}_sM^{\tilde\omega}_{\lambda,s}u)}\\&\leqslant C\|f\|_{L^p(\mathrm{M}_sM^{\tilde\omega}_{\lambda,s}u)},\qquad 1<s<\infty,\ 2<p<\frac{2(\gamma-1/s)}{\gamma-1}.\end{aligned}\tag{49}$$

On the other hand, similarly to the arguments in proving (43), we can deduce that

$$\|G_{\lambda,j}f\|_{L^2(u)}\leqslant C(1+|j|)^{-\beta/s'}\|f\|_{L^2(M_sM^{\tilde\mu}_{\lambda}u)}.\tag{50}$$

Note that $\beta/s'>1$, for $p\in[2,\frac{2(\gamma-1/s)\beta}{1+\beta(\gamma-1)})$, there exist $p_1\in(2,\frac{2(\gamma-1/s)}{\gamma-1})$ and $\theta\in(s'/\beta,1]$ such that $1/p=\theta/2+(1-\theta)/p_1$. An interpolation between (49) and (50) implies that

$$\|G_{\lambda,j}f\|_{L^p(u)}\leqslant C(1+|j|)^{-\theta\beta/s'}\|f\|_{L^p(\mathrm{M}_sM^{\tilde\mu}_{\lambda,s}u)}.$$

So,

$$\|G^{\omega}_{\lambda}f\|_{L^p(u)}\leqslant\sum_{j\in\mathbb{Z}}\|G_{\lambda,j}f\|_{L^p(u)}\leqslant C\|f\|_{L^p(\mathrm{M}_sM^{\tilde\omega}_{s}u)},\quad \beta'<s<\infty,\ 2\leqslant p<\frac{2\beta(\gamma-1/s)}{1+\beta(\gamma-1)}.$$

This together with (34) deduces that

$$\|G^{\omega}_{\lambda}f\|_{L^p(u)}\leqslant C\|f\|_{L^p(\mathrm{M}_sM^{\tilde\sigma}_{\lambda,s}u+\mathrm{M}^2_sM^{\tilde\sigma}_{\lambda-1,s}u)},\quad \beta'<s<\infty,\ 2\leqslant p<\frac{2\beta(\gamma-1/s)}{1+\beta(\gamma-1)}.$$

Therefore, by (35) and (37) we get

$$\begin{aligned}\|M^{\sigma}_{\lambda}f\|_{L^p(u)}&\leqslant\|\mathrm{M}M^{\sigma}_{\lambda-1}|f|\|_{L^p(u)}+\|G^{\omega}_{\lambda}f\|_{L^p(u)}\\&\leqslant C_p\|M^{\sigma}_{\lambda-1}|f|\|_{L^p(\mathrm{M}u)}+C\|f\|_{L^p(\mathrm{M}_sM^{\tilde\sigma}_{\lambda,s}u+\mathrm{M}^2_sM^{\tilde\sigma}_{\lambda-1,s}u)}\end{aligned}\tag{51}$$

for $s\in(\beta',\infty)$ and $p\in[2,\frac{2\beta(\gamma-1/s)}{1+\beta(\gamma-1)})$. This together with an induction argument and our assumption (e) deduces that

$$\|M^{\sigma}_{\lambda}f\|_{L^p(u)}\leqslant C\|f\|_{L^p(\mathrm{M}^{\lambda}u+I_{\lambda,s}u+J_{\lambda,s}u)},\ \forall 1\leqslant\lambda\leqslant\Lambda,$$

which leads to

$$\|M_\lambda^\sigma f\|_{L^p(u)} \leqslant C\|f\|_{L^p(\mathrm{M}_s^{\lambda+1}u+I_{\lambda,s}\mathrm{M}_s u+J_{\lambda,s}\mathrm{M}_s u)}, \quad \beta' < s < \infty,\ 2 \leqslant p < \frac{2\beta(\gamma-1/s)}{1+\beta(\gamma-1)}, \tag{52}$$

since $u \leqslant \mathrm{M}_s u$ and $\mathrm{M}_s u \leqslant A_1$. This proves (i).

The proof of (ii): By (48), we have

$$\|M_\lambda^\sigma f\|_{L^p(u)} \leqslant C\|f\|_{L^p(\mathrm{M}_s^\lambda u+\mathrm{M}_s^2\widetilde{\mathrm{M}_s^\lambda u}+H_{\lambda,s}u)} \tag{53}$$

holds for all $1 \leqslant \lambda \leqslant \Lambda$, $\beta \in (\frac{2}{2-\gamma},\infty)$, $p \in (\beta'\gamma,2]$ and $s \in (2\beta'/p,\infty)$. (ii) is proved.

The proof of (iii): By (17), we can write

$$\sup_{k\in\mathbb{Z}}\Big|\sum_{j=k}^\infty \sigma_{j,\Lambda}*f(x)\Big| \leqslant \sum_{\lambda=1}^\Lambda \sup_{k\in\mathbb{Z}}\Big|\sum_{j=k}^\infty \mu_{j,\lambda}*f(x)\Big|,$$

and

$$\begin{aligned}
&\sup_{k\in\mathbb{Z}}\Big|\sum_{j=k}^\infty \mu_{j,\lambda}*f(x)\Big| \\
&\quad= \sup_{k\in\mathbb{Z}}\Big|\psi_{k,\lambda}*T_\lambda f(x)-\psi_{k,\lambda}*\sum_{j=-\infty}^k \mu_{j,\lambda}*f(x)+(\delta-\psi_{k,\lambda})*\sum_{j=k+1}^\infty \mu_{j,\lambda}*f(x)\Big| \\
&\quad\leqslant \sup_{k\in\mathbb{Z}}|\psi_{k,\lambda}*T_\lambda f(x)|+\sup_{k\in\mathbb{Z}}\Big|\psi_{k,\lambda}*\sum_{j=-\infty}^k \mu_{j,\lambda}*f(x)\Big| \\
&\qquad+\sup_{k\in\mathbb{Z}}\Big|(\delta-\psi_{k,\lambda})*\sum_{j=k+1}^\infty \mu_{j,\lambda}*f(x)\Big| \\
&\quad=: A_{1,\lambda}f(x)+A_{2,\lambda}f(x)+A_{3,\lambda}f(x),
\end{aligned}$$

where $\psi_{k,\lambda}$ is given as in (31), T_λ is given as in (21) and δ is the Dirac-Delta. Therefore, we need only to estimate $\|A_{i,\lambda}f\|_{L^p(u)}$, $i = 1,2,3$.

For $A_{1,\lambda}f$, noting that $\mathrm{M}u \leqslant \mathrm{M}_s u \in A_1$, by (37) and (22), we obtain

$$\begin{aligned}
\|A_{1,\lambda}f\|_{L^p(u)} &\leqslant \|\mathrm{M}(T_\lambda f)\|_{L^p(u)} \leqslant C_p\|T_\lambda f\|_{L^p(\mathrm{M}u)} \leqslant C_p\|T_\lambda f\|_{L^p(\mathrm{M}_s u)} \\
&\leqslant C\|f\|_{L^p(\Upsilon_{\Lambda,s}\mathrm{M}_s u)} \leqslant C\|f\|_{L^p(\Theta_{\Lambda,s}\mathrm{M}_s u)}
\end{aligned}$$

for all $p \in [2,\frac{2\beta(\gamma-1/s)}{1+\beta(\gamma-1)})$ and $s \in (\beta',\infty)$.

For $A_{2,\lambda}f$, we write

$$A_{2,\lambda}f(x) = \sup_{k\in\mathbb{Z}}\Big|\sum_{j=0}^\infty \psi_{k,\lambda}*\mu_{k-j,\lambda}*f(x)\Big| \leqslant \sum_{j=0}^\infty \sup_{k\in\mathbb{Z}}|\psi_{k,\lambda}*\mu_{k-j,\lambda}*f(x)| =: \sum_{j=0}^\infty I_j f(x).$$

Consequently,

$$\|A_{2,\lambda}f\|_{L^p(u)} \leqslant \sum_{j=0}^\infty \|I_j f\|_{L^p(u)}, \quad 1<p<\infty.$$

By (37), (18) and (52), we obtain

$$\begin{aligned}\|I_j f\|_{L^p(u)} &\leqslant \|\mathrm{M}M_\lambda^\mu|f|\|_{L^p(u)} \leqslant C_p\|M_\lambda^\mu|f|\|_{L^p(\mathrm{M}u)}\\ &\leqslant C_p\big(\|M_\lambda^\sigma|f|\|_{L^p(\mathrm{M}^{\Lambda-\lambda+1}u)} + \|M_{\lambda-1}^\sigma|f|\|_{L^p(\mathrm{M}^{\Lambda-\lambda+2}u)}\big)\\ &\leqslant C\|f\|_{L^p(\mathrm{M}_s^{\Lambda+2}u+I_{\lambda,s}\mathrm{M}_s^{\Lambda-\lambda+2}u+I_{\lambda,s}\mathrm{M}_s^{\Lambda-\lambda+3}u+J_{\lambda,s}\mathrm{M}_s^{\Lambda-\lambda+2}u+J_{\lambda-1,s}\mathrm{M}_s^{\Lambda-\lambda+3}u)}\\ &\leqslant C\|f\|_{L^p(\mathrm{M}_s^{\Lambda+2}u+I_{\Lambda,s}\mathrm{M}_s^2u+J_{\Lambda,s}\mathrm{M}_s^2u)} \leqslant C\|f\|_{L^p(\Theta_{\Lambda,s}\mathrm{M}_s^2u)}\end{aligned} \tag{54}$$

for all $p \in [2, \frac{2\beta(\gamma-1/s)}{1+\beta(\gamma-1)})$ and $s \in (\beta', \infty)$. Also, by (19) and Plancherel's theorem, we have

$$\begin{aligned}\|I_j f\|_{L^2(\mathbb{R}^n)}^2 &\leqslant \Big\|\Big(\sum_{k\in\mathbb{Z}}|\psi_{k,\lambda}*\mu_{k-j,\lambda}*f|^2\Big)^{1/2}\Big\|_{L^2(\mathbb{R}^n)}^2\\ &\leqslant \sum_{k\in\mathbb{Z}}\int_{\{|a_\lambda\xi|\leqslant 2^{-k\lambda}\}}|\widehat{\mu_{k-j,\lambda}}(\xi)|^2|\hat{f}(\xi)|^2d\xi\\ &\leqslant C\int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}}|\widehat{\mu_{k-j,\lambda}}(\xi)|^2\chi_{\{|a_\lambda\xi|\leqslant 2^{-k\lambda}\}}|\hat{f}(\xi)|^2d\xi\\ &\leqslant C\sup_{\xi\in\mathbb{R}^n}\sum_{k\in\mathbb{Z}}|a_\lambda 2^{\lambda(k-j)}\xi|^2\chi_{\{|a_\lambda\xi|\leqslant 2^{-k\lambda}\}}\|f\|_{L^2(\mathbb{R}^n)}^2\\ &\leqslant C2^{-2\lambda j}\sup_{\xi\in\mathbb{R}^n}\sum_{k\in\mathbb{Z}}|2^{k\lambda}a_\lambda\xi|^2\chi_{\{|a_\lambda\xi|\leqslant 2^{-k\lambda}\}}\|f\|_{L^2(\mathbb{R}^n)}^2\\ &\leqslant C2^{-2\lambda j}\|f\|_{L^2(\mathbb{R}^n)}^2,\end{aligned}$$

where in the last inequality we have used the properties of lacunary sequence. It follows that

$$\|I_j f\|_{L^2(\mathbb{R}^n)} \leqslant C2^{-\lambda j}\|f\|_{L^2(\mathbb{R}^n)}.$$

On the other hand, by (54) with $p = 2$ and replacing u by u^s, we get

$$\|I_j f\|_{L^2(u^s)} \leqslant C\|f\|_{L^2(\Theta_{\Lambda,s}\mathrm{M}_s^2u^s)}, \quad s > \beta'.$$

Thus, an interpolation leads to

$$\|I_j f\|_{L^2(u)} \leqslant C2^{-(1-1/s)\lambda j}\|f\|_{L^2((\Theta_{\Lambda,s}\mathrm{M}_s^2u^s)^{1/s})} \leqslant C2^{-(1-1/s)\lambda j}\|f\|_{L^2(\Theta_{\Lambda,s^2}\mathrm{M}_{s^2}^2u)}, \quad s > \beta',$$

which implies that

$$\|I_j f\|_{L^2(u)} \leqslant C2^{-(1-1/\sqrt{s})\lambda j}\|f\|_{L^2(\Theta_{\Lambda,s}\mathrm{M}_s^2u)}, \quad \sqrt{s} > \beta'. \tag{55}$$

Interpolating between (55) and (54) yields that

$$\|I_j f\|_{L^p(u)} \leqslant C2^{-\varsigma(p,s)j}\|f\|_{L^p(\Theta_{\Lambda,s}\mathrm{M}_s^2u)},$$

for all $p \in [2, \frac{2\beta(\gamma-1/s)}{1+\beta(\gamma-1)})$ and $s > (\frac{\beta}{\beta-1})^2$, where $\varsigma(p,s) > 0$. Then,

$$\|A_{2,\lambda}f\|_{L^p(u)} \leqslant \sum_{j=0}^{\infty}\|I_j f\|_{L^p(u)} \leqslant C\|f\|_{L^p(\Theta_{\Lambda,s}\mathrm{M}_s^2u)}$$

for all $p \in [2, \frac{2\beta(\gamma-1/s)}{1+\beta(\gamma-1)})$ and $s \in ((\beta')^2, \infty)$.

Fore $A_{3,\lambda} f$, we write

$$A_{3,\lambda} f(x) = \sup_{k\in\mathbb{Z}} \Big| \sum_{j=1}^{\infty} (\delta - \psi_{k,\lambda}) * \mu_{k+j,\lambda} * f(x) \Big| \leqslant \sum_{j=1}^{\infty} \sup_{k\in\mathbb{Z}} |(\delta - \psi_{k,\lambda}) * \mu_{k+j,\lambda} * f(x)| =: \sum_{j=1}^{\infty} J_j f(x).$$

It follows that

$$\|A_{3,\lambda} f\|_{L^p(u)} \leqslant \sum_{j=1}^{\infty} \|J_j f\|_{L^p(u)}, \quad 1 < p < \infty.$$

By the argument similar to those used in deriving (54), we get

$$\|J_j f\|_{L^p(u)} \leqslant C \|f\|_{L^p(\Theta_{\Lambda,s} \mathrm{M}_s^2 u)} \tag{56}$$

for all $p \in [2, \frac{2\beta(\gamma-1/s)}{1+\beta(\gamma-1)})$ and $s \in (\beta', \infty)$.

On the other hand, by (20) and Plancherel's theorem, we have

$$\begin{aligned}
\|J_j f\|^2_{L^2(\mathbb{R}^n)} &\leqslant \Big\| \Big(\sum_{k\in\mathbb{Z}} |(\delta - \psi_{k,\lambda}) * \mu_{j+k,\lambda} * f|^2 \Big)^{1/2} \Big\|^2_{L^2(\mathbb{R}^n)} \\
&\leqslant \sum_{k\in\mathbb{Z}} \int_{\{2^{k\lambda} a_\lambda \xi| \geqslant 1\}} |\widehat{\mu_{j+k,\lambda}}(\xi)|^2 |\hat{f}(\xi)|^2 d\xi \\
&\leqslant \sum_{k\in\mathbb{Z}} \sum_{i=-k}^{\infty} \int_{\{2^{\lambda i} \leqslant |a_\lambda \xi| < 2^{\lambda(i+1)}\}} |\widehat{\mu_{j+k,\lambda}}(\xi)|^2 |\hat{f}(\xi)|^2 d\xi \\
&\leqslant C \sum_{k\in\mathbb{Z}} \sum_{i=-k}^{\infty} (k+j+i)^{-2\beta} \int_{\{2^{\lambda i} \leqslant |a_\lambda \xi| < 2^{\lambda(i+1)}\}} |\hat{f}(\xi)|^2 d\xi \\
&\leqslant C \sum_{k\in\mathbb{Z}} \sum_{i=0}^{\infty} (i+j)^{-2\beta} \int_{\{2^{\lambda(i-k)} \leqslant |a_\lambda \xi| < 2^{\lambda(i-k+1)}\}} |\hat{f}(\xi)|^2 d\xi \\
&\leqslant C \sum_{i=0}^{\infty} (i+j)^{-2\beta} \|f\|^2_{L^2(\mathbb{R}^n)} \\
&\leqslant C j^{1-2\beta} \|f\|^2_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

Hence,

$$\|J_j f\|_{L^2(\mathbb{R}^n)} \leqslant C(1+j)^{1/2-\beta} \|f\|_{L^2(\mathbb{R}^n)}.$$

Also, by (56) with $p = 2$ and replacing u by u^s, we get

$$\|J_j f\|_{L^2(u^s)} \leqslant C \|f\|_{L^2(\Theta_{\Lambda,s} \mathrm{M}_s^2 u^s)}, \quad s > \beta'.$$

Then, an interpolation yields that for $s > \beta'$,

$$\|J_j f\|_{L^2(u)} \leqslant C j^{-(\beta-1/2)(1-1/s)} \|f\|_{L^2((\Theta_{\Lambda,s} \mathrm{M}_s^2 u^s)^{1/s})} \leqslant C j^{-(\beta-1/2)(1-1/s)} \|f\|_{L^2(\Theta_{\Lambda,s^2} \mathrm{M}_{s^2}^2 u)},$$

which leads to

$$\|J_j f\|_{L^2(u)} \leqslant C j^{-(\beta-1/2)(1-1/\sqrt{s})} \|f\|_{L^2(\Theta_{\Lambda,s} \mathrm{M}_s^2 u)}, \quad \sqrt{s} > \beta'. \tag{57}$$

Note that $\beta \in (\frac{3}{2},\infty)$ and $s \in ((\frac{\beta-1/2}{\beta-3/2})^2$, we know that $(\beta-1/2)(1-1/\sqrt{s}) > 1$. Therefore, for

$$p \in \Big[2, \frac{\beta(2\beta-1)(1-1/\sqrt{s})(\gamma-1/s)}{(\beta\gamma-\beta+1)(\beta-1/2)(1-1/\sqrt{s})+(1-1/s)\beta-1}\Big),$$

there exist $p_1 \in (2, \frac{2\beta(\gamma-1/s)}{1+\beta(\gamma-1)})$ and $\theta \in (\frac{1}{(\beta-1/2)(1-1/\sqrt{s})}, 1]$ such that $1/p = \theta/2 + (1-\theta)/p_1$. Interpolation between (57) and (56) gives

$$\|J_j f\|_{L^p(u)} \leqslant C j^{-\theta(\beta-1/2)(1-1/\sqrt{s})} \|f\|_{L^p(\Theta_{\Lambda,s}\mathrm{M}_s^2 u)}.$$

Consequently,

$$\|A_{3,\lambda} f\|_{L^p(u)} \leqslant \sum_{j=1}^{\infty} \|J_j f\|_{L^p(u)} \leqslant C\|f\|_{L^p(\Theta_{\Lambda,s}\mathrm{M}_s^2 u)}$$

for all $p \in [2, \frac{\beta(2\beta-1)(1-1/\sqrt{s})(\gamma-1/s)}{(\beta\gamma-\beta+1)(\beta-1/2)(1-1/\sqrt{s})+(1-1/s)\beta-1})$, $\beta \in (\frac{3}{2},\infty)$ and $s \in ((\frac{\beta-1/2}{\beta-3/2})^2,\infty)$. This completes the proof of (iii).

The proof of (iv): Employing the notation in the proof of (iii), we need only to estimate $\|A_{i,\lambda} f\|_{L^p(u)}$, $i = 1, 2, 3$.

For $A_{1,\lambda} f$, by (37) and (29), we have

$$\|A_{1,\lambda} f\|_{L^p(u)} \leqslant C\|\mathrm{M}(T_\lambda f)\|_{L^p(u)} \leqslant C_p \|T_\lambda f\|_{L^p(\mathrm{M}u)} \leqslant C\|f\|_{L^p(\Upsilon_{\Lambda,s}\mathrm{M}u)}$$

for any $1 \leqslant \lambda \leqslant \Lambda$, $\beta \in (\frac{2}{2-\gamma},\infty)$, $p \in (\beta'\gamma, 2]$ and $s \in (\frac{2\beta'}{p},\infty)$.

For $A_{2,\lambda} f$, it follows from (37), (18) and (53) that

$$\begin{aligned}
\|I_j f\|_{L^p(u)} &\leqslant C\|\mathrm{M}M_\lambda^\mu f\|_{L^p(u)} \leqslant C_p \|M_\lambda^\mu f\|_{L^p(\mathrm{M}u)} \\
&\leqslant C_p(\|M_\lambda^\sigma |f|\|_{L^p(\mathrm{M}^{\Lambda-\lambda+1}u)} + \|M_{\lambda-1}^\sigma |f|\|_{L^p(\mathrm{M}^{\Lambda-\lambda+2}u)}) \\
&\leqslant C\|f\|_{L^p(\mathrm{M}_s^\Lambda M u + \mathrm{M}_s^2 \widetilde{\mathrm{M}_s^\Lambda \mathrm{M}} u + H_{\lambda,s}\mathrm{M}^{\Lambda-\lambda+1}u + H_{\lambda-1,s}\mathrm{M}^{\Lambda-\lambda+2}u)} \\
&\leqslant C\|f\|_{L^p(\mathrm{M}_s^\Lambda \mathrm{M}u + \mathrm{M}_s^2 \widetilde{\mathrm{M}_s^\Lambda \mathrm{M}} u + H_{\Lambda,s}\mathrm{M}u)} \\
&\leqslant C\|f\|_{L^p(\Upsilon_{\Lambda,s}\mathrm{M}_s^2 u)}
\end{aligned} \tag{58}$$

for $1 \leqslant \lambda \leqslant \Lambda$, $\beta \in (\frac{2}{2-\gamma},\infty)$, $p \in (\beta'\gamma, 2]$ and $s \in (2\beta'/p,\infty)$. Also, similarly to (55), we can get

$$\|J_i f\|_{L^2(u)} \leqslant C 2^{-(1-1/\sqrt{s})\lambda j} \|f\|_{L^2(\Upsilon_{\Lambda,s}\mathrm{M}_s^2 u)}.$$

Therefore, interpolation theorem tells us that

$$\|I_j f\|_{L^p(u)} \leqslant C 2^{-\delta(p,s)j} \|f\|_{L^p(\Upsilon_{\Lambda,s}\mathrm{M}_s^2 u)}$$

for all $1 \leqslant \lambda \leqslant \Lambda$, $\beta \in (\frac{2}{2-\gamma},\infty)$, $s \in (\max\{\frac{2\beta'}{p}, (\frac{\beta}{\beta-1})^2\},\infty)$ and $p \in (\beta'\gamma, 2]$, were $\delta(p,s) > 0$. So,

$$\|A_{2,\lambda} f\|_{L^p(u)} \leqslant \sum_{j=1}^{\infty} \|J_j f\|_{L^p(u)} \leqslant C\|f\|_{L^p(\Upsilon_{\Lambda,s}\mathrm{M}_s^2 u)}$$

for all $1 \leqslant \lambda \leqslant \Lambda$, $\beta \in (\frac{2}{2-\gamma},\infty)$, $p \in (\beta'\gamma,2]$ and $s \in (\max\{2\beta'/p,(\beta')^2\},\infty)$.

For $A_{3,\lambda}f$, by the argument similar to those used to derive (58), we get

$$\|J_j f\|_{L^p(u)} \leqslant C\|f\|_{L^p(\Upsilon_{\Lambda,s}\mathrm{M}_s^2 u)} \tag{59}$$

holds for all $1 \leqslant \lambda \leqslant \Lambda$, $\beta \in (\frac{2}{2-\gamma},\infty)$, $p \in (\beta'\gamma,2]$ and $s \in (\frac{2\beta'}{p},\infty)$. And similarly to the arguments in deriving (57), we have

$$\|J_i f\|_{L^p(u)} \leqslant C j^{-(\beta-1/2)(1-1/\sqrt{s})}\|f\|_{L^2(\Upsilon_{\Lambda,s}\mathrm{M}_s^2 u)}.$$

Note that $\beta \in (\frac{2}{2-\gamma},\infty)$, and $s \in (\max\{(\frac{\beta-1/2}{\beta-3/2})^2,\frac{2\beta'}{p}\},\infty)$, then $(\beta-1/2)(1-1/\sqrt{s}) > 1$. Thus, for $p \in (\frac{2\beta'\gamma(2\beta-1)}{2\beta-1+(\beta'\gamma-2)(\sqrt{s})'},2]$ there exist $p_1 \in (\beta'\gamma,2]$ and $\theta \in (\frac{1}{(\beta-1/2)(1-1/\sqrt{s})},1]$ such that $\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{p_1}$. Interpolation between (57) and (59) yields that

$$\|J_j f\|_{L^p(u)} \leqslant C j^{-\theta(\beta-1/2)(1-1/\sqrt{s})}\|f\|_{L^p(\Upsilon_{\Lambda,s}\mathrm{M}_s^2 u)}$$

Consequently,

$$\|A_{3,\lambda}f\|_{L^p(u)} \leqslant \sum_{j=1}^{\infty}\|J_j f\|_{L^p(u)} \leqslant C\|f\|_{L^p(\Upsilon_{\Lambda,s}\mathrm{M}_s^2 u)}$$

for $\beta \in (\frac{2}{2-\gamma},\infty)$, $s \in (\max\{(\frac{\beta-1/2}{\beta-3/2})^2,\frac{2\beta'}{p}\},\infty)$ and $p \in (\frac{2\beta'\gamma(2\beta-1)}{2\beta-1+(\beta'\gamma-2)(\sqrt{s})'},2]$. Summing up the estimates of $\|A_{i,\lambda}f\|_{L^p(u)}$ $(i=1,2,3)$, we completes the proof of (iv). Lemma 2 is proved.

We now turn to prove Theorems 1 and 2.

Proof of Theorems 1 *and* 2. Let $P_0(t)=0$ and $\{P_\lambda\}_{\lambda=1}^N$ be given as in Theorem 1. Let $\sigma_{k,\lambda}$, $|\sigma_{k,\lambda}|$, $\{M_\lambda^\sigma\}_{\lambda=1}^N$ be defined as in Theorem 1 and δ,γ be given as in Theorem 1. One can easily check that

$$T_{h,\Omega,P_N}f(x) = \sum_{k\in\mathbb{Z}}\sigma_{k,N} * f(x);$$

$$T^*_{h,\Omega,P_N}f(x) \leqslant M_N^\sigma f(x) + \sup_{k\in\mathbb{Z}}\Big|\sum_{j=k}^{\infty}\sigma_{j,N} * f(x)\Big|;$$

$$M_{h,\Omega,P_N}f(x) \leqslant C\sup_{k\in\mathbb{Z}}||\sigma_{k,N}| * f(x)|;$$

$$\sigma_{k,0}(\xi) = 0;$$

$$M_0^\sigma f(x) \leqslant C|f(x)|;$$

$$\max\{|\widehat{\sigma_{k,\lambda}}(\xi)|,|\widehat{|\sigma_{k,\lambda}|}(\xi)|,\|\sigma_{k,\lambda}\|\} \leqslant C;$$

$$\max\big\{|\widehat{\sigma_{k,\lambda}}(\xi)-\widehat{\sigma_{k,\lambda-1}}(\xi)|,|\widehat{|\sigma_{k,\lambda}|}(\xi)-\widehat{|\sigma_{k,\lambda-1}|}(\xi)|\big\} \leqslant C|2^{k\lambda}b_\lambda\xi|.$$

By the arguments similar to those used in deriving [13, Lemma 2.2] and [23, Lemma 2.2], we can get

$$\max\{|\widehat{\sigma_{k,\lambda}}(\xi)|, \big|\widehat{|\sigma_{k,\lambda}|}(\xi)\big|\} \leqslant C(\log|2^{k\lambda}b_\lambda\xi|)^{-\delta}, \text{ if } |2^{k\lambda}b_\lambda\xi| > 1.$$

And, the arguments similar to those used in deriving [23, Lemma 2.5] can deduces that

$$\|M_\lambda^\sigma f\|_{L^q(\mathbb{R}^n)} \leqslant C_q\|f\|_{L^q(\mathbb{R}^n)}, \quad q \in (\gamma', \infty). \tag{60}$$

Therefore, applying Lemmas 1 and 2 with the estimates above, we can obtain the desired conclusions of Theorems 1 and 2 and complete our proofs.

3. Proofs of Corollaries 1-4

Before proving Corollaries 1-4, let us introduce an useful proposition, which is a variant of [21, Proposition 2.1].

PROPOSITION 1. *Let $1 < q < \infty$, $\delta \in [1,\infty)$ and $s_0 \in [1,\infty)$. Let T be a sublinear operator such that*

$$\|Tf\|_{L^q(u)} \leqslant C_{q,s,s_0}\|f\|_{L^q(\Theta_s(u))} \tag{61}$$

for all $s \in (s_0,\infty)$ and any nonnegative measurable function u on $\mathbb{R}^n$, where the operator Θ_s satisfies

$$\|\Theta_s(f)\|_{L^r(\mathbb{R}^n)} \leqslant C_r\|f\|_{L^r(\mathbb{R}^n)} \tag{62}$$

for all $r \in (s\delta,\infty)$ and all radial functions f. Then for any fixed $s \in [s_0,\infty)$ and $p \in (q, \frac{q\delta s}{\delta s-1})$, the following inequalities hold:

$$\|Tf\|_{L^p_{|x|}L^q_\theta(\mathbb{R}^n)} \leqslant C_{p,q}\|f\|_{L^p_{|x|}L^q_\theta(\mathbb{R}^n)}; \tag{63}$$

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}|Tf_j|^q\Big)^{1/q}\Big\|_{L^p_{|x|}L^q_\theta(\mathbb{R}^n)} \leqslant C_{p,q}\Big\|\Big(\sum_{j\in\mathbb{Z}}|f_j|^q\Big)^{1/q}\Big\|_{L^p_{|x|}L^q_\theta(\mathbb{R}^n)}; \tag{64}$$

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}|Tf_j|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)} \leqslant C_{p,q}\Big\|\Big(\sum_{j\in\mathbb{Z}}|f_j|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}. \tag{65}$$

Proof. We only prove (63) since (64) and (65) can be obtained similarly. The argument is essentially same as in the proof of [21, Proposition 2.1]. Fix $s \in [s_0,\infty)$. Let $p \in (q, \frac{q\delta s}{\delta s-1})$. We write $r = \frac{p}{p-q}$ and fix $\tau \in (s, \frac{r}{\delta})$. It is clear that $r > \delta\tau$. Let X denote the set of all functions $g \in \mathscr{S}(\mathbb{R})$ with $\int_0^\infty g^r(\rho)\rho^{n-1}d\rho \leqslant 1$. By changes of variables, one has

$$\begin{aligned}\|Tf\|^q_{L^p_{|x|}L^q_\theta(\mathbb{R}^n)} &= \Big(\int_0^\infty\Big(\int_{S^{n-1}}|Tf(\rho\theta)|^q d\sigma(\theta)\Big)^{p/q}\rho^{n-1}d\rho\Big)^{q/p} \\ &= \sup_{g\in X}\int_0^\infty\int_{S^{n-1}}|Tf(\rho\theta)|^q g(\rho)\rho^{n-1}d\sigma(\theta)d\rho \\ &= \sup_{g\in X}\int_{\mathbb{R}^n}|Tf(x)|^q g(|x|)dx.\end{aligned} \tag{66}$$

Fix $g \in X$. Let $I(g) := \int_{\mathbb{R}^n} |Tf(x)|^p g(|x|)dx$ and $h(x) = g(|x|)$. By (61)-(62), Hölder's inequality and changes of variables, we have

$$\begin{aligned} I(g) &\leqslant C_{q,s,s_0} \int_{\mathbb{R}^n} |f(x)|^q \Theta_s(h)(x)dx \\ &\leqslant C_{q,s,s_0} \int_0^\infty \int_{S^{n-1}} |f(\rho\theta)|^q d\sigma(\theta)\Theta_s(g)(\rho)\rho^{n-1}d\rho \\ &\leqslant C_{q,s,s_0} \int_0^\infty \Big(\int_{S^{n-1}} |f(\rho\theta)|^q d\sigma(\theta)\Big)^{p/q}\rho^{n-1}d\rho\Big)^{q/p}\Big(\int_0^\infty (\Theta_s(g)(\rho))^r \rho^{n-1}d\rho\Big)^{1/r} \\ &\leqslant C_{p,q}\|f\|^q_{L^p_{|x|}L^q_\theta(\mathbb{R}^n)}\|\Theta_s(h)\|_{L^r(\mathbb{R}^n)} \\ &\leqslant C_{p,q}\|f\|^q_{L^p_{|x|}L^q_\theta(\mathbb{R}^n)}, \end{aligned}$$

which together with (66) leads to (63).

We now prove Corollaries 1-4.

Proof of Corollary 1. We only prove Corollary 1 for the operator T_{h,Ω,P_N} since the conclusions for M_{h,Ω,P_N} can be obtained similarly.

(i) By (60), we have

$$\|L_{N,s}f\|_{L^r(\mathbb{R}^n)} \leqslant C\|f\|_{L^r(\mathbb{R}^n)}$$

for any $s \in (\delta',\infty)$ and $r \in (s\gamma',\infty)$. This together with (4) and Proposition 1, we have that (8)-(10) hold for $s \in (\delta',\infty)$, $q \in [2, \frac{2\delta(\gamma'-1/s)}{1+\delta(\gamma'-1)})$ and $p \in (q, \frac{qs\gamma'}{s\gamma'-1})$.

When the condition (a) holds, we have $\delta = \beta$ and $\gamma' = 1$. By Theorem A, (1) and the fact that $2\beta(1-1/s) \leqslant 2\beta$, we have that (8)-(10) hold for $s \in (\beta',\infty)$, $q \in [2, 2\beta(\gamma'-1/s))$ and $p = q$.

When the condition (b) holds, we have $\delta = \frac{\beta}{\max\{2,\gamma'\}}$. By Theorem B we have that T_{h,Ω,P_N} is bounded on $L^p(\mathbb{R}^n)$ for $p \in (\frac{2\max\{2,\gamma'\}\delta}{(\max\{2,\gamma'\}+2)\delta-2}, \frac{2\max\{2,\gamma'\}\delta}{(\max\{2,\gamma'\}-2)\delta+2})$. This together with (1) and the fact that $\frac{2\delta(\gamma'-1/s)}{1+\delta(\gamma'-1)} \leqslant \frac{2\max\{2,\gamma'\}\delta}{(\max\{2,\gamma'\}-2)\delta+2}$ yields that (8)-(10) hold for $s \in (\delta',\infty)$, $q \in [2, \frac{2\delta(\gamma'-1/s)}{1+\delta(\gamma'-1)})$ and $p = q$.

By duality we have that (8)-(10) hold for $s \in (\delta',\infty)$, $q \in (\frac{2\delta(\gamma'-1/s)}{\delta(\gamma'-2/s+1)-1}, 2]$ and $p \in (\frac{qs\gamma'}{q-1+s\gamma'}, q]$. This proves (i).

(ii) Let $\delta \in (\frac{2}{2-\gamma'},\infty)$ and $q \in (\delta'\gamma', 2]$. By (60), we have

$$\|\Upsilon_{N,s}f\|_{L^r(\mathbb{R}^n)} \leqslant C\|f\|_{L^r(\mathbb{R}^n)}, \quad 2\delta'/p < s < \infty, s\gamma' < r < \infty,$$

which together with (5) and Proposition 1 implies that (8)-(10) hold for all $q \in (\delta'\gamma', 2]$, $p \in (q, \frac{2q\delta'\gamma'}{2\delta'\gamma'-q})$.

When the condition (a) holds. Then we have $\delta'\gamma' = \beta'$. Hence we have that (8)-(10) hold for all $q \in (\delta'\gamma', 2]$, and $p = q$ by Theorem A and (1).

When the condition (b) holds and $\gamma \in (2,\infty]$. Then $\delta = \frac{\beta}{2}$. By Theorem B we have that T_{h,Ω,P_N} is bounded on $L^p(\mathbb{R}^n)$ for $p \in (\frac{\beta}{\beta-1}, \beta)$. This together with (1) and the fact that $(\frac{\beta}{2})'\gamma' \geqslant \frac{\beta}{\beta-1}$ yields that (8)-(10) hold for $q \in (\delta'\gamma', 2]$ and $p = q$.

By duality, we can obtain (8)-(10) hold for $\delta \in (\frac{2}{2-\gamma'},\infty)$, $q \in [2,\frac{\delta'\gamma'}{\delta'\gamma'-1})$, $p \in (\frac{2q\delta'\gamma'}{2\delta'\gamma'+q},q]$. This proves Corollary 1.

Proof of Corollary 2. Taking $\gamma=\infty$, Corollary 2 follows easily from Corollary 1.

Proof of Corollary 3. We only consider the operator T_{Ω,P_N} since the corresponding results for M_{Ω,P_N} can be proved similarly.

Let $s=\frac{\sqrt{\beta}}{\sqrt{\beta}-1}$. Corollary 2 implies that (11)-(13) hold for $q\in[2,2\sqrt{\beta})$ and $p\in[q,q\sqrt{\beta})$.

Let $2\leqslant q\leqslant p<\infty$. There exists $\beta\in(1,\infty)$ such that $q\in[2,2\sqrt{\beta})$ and $p\in[q,q\sqrt{\beta})$. This proves (11)-(13) for the case $2\leqslant q\leqslant p<\infty$. By duality we have that (11)-(13) hold for the case $1<p\leqslant q\leqslant 2$.

On the other hand, let $q\in(1,2]$ and $p\in[q,2]$, there exists $\beta>\max\{\big(\frac{1}{2(\frac{1}{q}-\frac{1}{p})}\big)',q',2\}$ such that $q\in(\beta',2]$ and $p\in[q,\frac{2\beta'q}{2\beta'-q})$. This together with Corollary 2 implies that (11)-(13) for the case $1<q\leqslant p\leqslant 2$. By duality, we have that (11)-(13) hold for the case $2\leqslant p\leqslant q<\infty$. This finishes the proof of Corollary 3.

Proof of Corollary 4. (i) By (60), we have

$$\|\Theta_{N,s}(\mathrm{M}_s f+\mathrm{M}_s^2 f)\|_{L^r(\mathbb{R}^n)}\leqslant C\|f\|_{L^r(\mathbb{R}^n)}$$

for any $s\in((\frac{\delta-1/2}{\delta-3/2})^2,\infty)$ and $r\in(s\gamma',\infty)$. This together with (6) and Proposition 1 implies that (14)-(16) hold for $s\in((\frac{\delta-1/2}{\delta-3/2})^2,\infty)$, $q\in[2,\frac{\delta(2\delta-1)(1-1/\sqrt{s})(\gamma'-1/s)}{(\delta\gamma'-\delta+1)(\delta-1/2)(1-1/\sqrt{s})+(1-1/s)\delta-1})$ and $p\in(q,\frac{qs\gamma'}{s\gamma'-1})$.

(ii) Let $\gamma\in(2,\infty]$, $\delta\in(\frac{2}{2-\gamma'},\infty)$, $s\in((\frac{\delta-1/2}{\delta-3/2})^2,\infty)$ and $q\in(\max\{2\delta'(\frac{\delta-3/2}{\delta-1/2})^2,\frac{2\delta'\gamma'(2\delta-1)}{2\delta-1+(\delta'\gamma'-2)(\sqrt{s})'}\},2]$. It follows from (60) that

$$\|\Upsilon_{N,s}(\mathrm{M}_s f+\mathrm{M}_s^2 f)\|_{L^r(\mathbb{R}^n)}\leqslant C\|f\|_{L^r(\mathbb{R}^n)},\quad s\gamma'<r<\infty,$$

which together with (7) and Proposition 1 deduces that (14)-(16) hold for $s\in((\frac{\delta-1/2}{\delta-3/2})^2,\infty)$, $q\in(\max\{2\delta'(\frac{\delta-3/2}{\delta-1/2})^2,\frac{2\delta'\gamma'(2\delta-1)}{2\delta-1+(\delta'\gamma'-2)(\sqrt{s})'}\},2]$ and $p\in[q,\frac{2q\delta'\gamma'}{2\delta'\gamma'-q})$. Corollary 4 is proved.

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