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CHARACTERIZATION OF INNER PRODUCT SPACES AND QUADRATIC FUNCTIONS BY SOME CLASSES OF FUNCTIONS

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Abstract. We define and discuss the c-quadratic-midaffine and F-midaffine functions. Using these functions we characterize inner product spaces and quadratic functions, respectively.

1. Introduction

$$[\int_a^b x^2 dx]$$

In the literature one can find many conditions characterizing inner product spaces among normed spaces. The most common characterization of inner product spaces was given in the classical paper of Jordan and von Neumann [10]. Namely, a norm space $(X,\|\cdot\|)$ is an inner product space if and only if the following equality

$$\|x+y\|^2+\|x-y\|^2=2\|x\|^2+2\|y\|^2,$$

holds true for all $x,y\in X$. We call this equality the Jordan-von-Neumann identity or the parallelogram law. A rich collection of characterizations of inner product spaces can be found in Amir's book [7] (cf. also [[1], Chpt. 11], [4], [5], [6], [8], [18]). Using the notion of c-quadratic-affinity, given in [13], we present a new result of this type involving strongly c-quadratic-midaffine functions. This characterization will be obtained as a consequence of a characterization of quadratic functions, i.e. the functions $Q:X\to\mathbb{R}$ which satisfy the following equation

$$Q(x+y)+Q(x-y)=2Q(x)+2Q(y),$$

for all $x,y\in X$, where X is a real vector space, by F-midaffine functions (the concept of F-affinity is given in [2]).

Notice that strongly affine functions are connected with strongly convex functions introduced by Polyak [17], which play an important role in optimization theory, and they were studied by many authors (cf. [9], [12], [13], [15], [16], [19], [21]). F-affine functions are connected with F-strongly convex functions introduced in the paper [2].

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2. Main result

We start with the following definition of c-quadratic-midaffine functions.

DEFINITION 1. Let $(X, \|\cdot\|)$ be a real normed space, $D \subset X$ be a convex set and c be a nonzero real number. A function $f : D \to \mathbb{R}$ is called *c-quadratic-midaffine* if

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2} - \frac{c}{4}\|x-y\|^2,$$

for all $x, y \in D$.

Recall that if f satisfies the above equation with $c = 0$ it is called midaffine (or Jensen function). Using the methods presented in [15] or [13] we can conclude that in each inner product space there exists a c-quadratic-midaffine function and it has to be in the form of $g + c\|\cdot\|^2$ with g being a midaffine one. And the question is if in other normed spaces such functions exist and what we can say about their form. This is the first question.

Replacing in the above definition $c\|\cdot\|^2$ with a function F we introduce the following definition of F-midaffine functions.

DEFINITION 2. Let X be a real vector space, $F : X \to \mathbb{R}$ be a fixed function and $D \subset X$ be a convex set. A function $f : D \to \mathbb{R}$ is called *F-midaffine* if

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2} - \frac{1}{4}F(x-y),$$

for all $x, y \in D$.

In the light of the results presented in [2] or [3] we get that if a function F is a quadratic function, then an F-midaffine function exists and must be of the form $g+F$, where g is a midaffine function. Notice that taking F as a non-zero constant function, any F-midaffine does not exist. And the second question is when such functions exist and what their form is.

The second question can be answered in the following theorem.

THEOREM 1. *Let X be a real vector space and $F : X \to \mathbb{R}$ be a fixed function. The following conditions are equivalent:*

1. *There exists a F-midaffine function $f : X \to \mathbb{R}$,*
2. *$F : X \to \mathbb{R}$ is a quadratic function,*
3. *A function $f : X \to \mathbb{R}$ is F-midaffine if and only if the function $g = f - F$ is midaffine.*

Proof. Assuming (1) we prove (2). Let $f : X \to \mathbb{R}$ be F-midaffine. Thus it satisfies the equation

$$f(x) = 2f\left(\frac{x+z}{2}\right) - f(z) + \frac{1}{2}F(x-z), \quad x,z \in X.$$

Now we calculate the expression $f\left(\frac{x+y}{2}\right)$ in two ways. On the one hand

$$\begin{aligned} f\left(\frac{x+y}{2}\right) &= f\left(\frac{f(x)+(y)}{2}\right) - \frac{1}{4}F(x-y) \\ &= \frac{2f\left(\frac{x+z}{2}\right) - f(z) + \frac{1}{2}F(x-z) + 2f\left(\frac{y+z}{2}\right) - f(z) + \frac{1}{2}F(y-z)}{2} - \frac{1}{4}F(x-y) \\ &= f\left(\frac{x+z}{2}\right) + f\left(\frac{y+z}{2}\right) - f(z) + \frac{1}{4}\left(F(x-z) + F(y-z) - F(x-y)\right), \end{aligned}$$

on the other hand

$$\begin{aligned} f\left(\frac{x+y}{2}\right) &= 2f\left(\frac{\frac{x+y}{2}+z}{2}\right) - f(z) + \frac{1}{2}F\left(\frac{x+y}{2} - z\right) \\ &= 2\left(\frac{f\left(\frac{x+z}{2}\right) + f\left(\frac{y+z}{2}\right)}{2} - \frac{1}{4}F\left(\frac{x-y}{2}\right)\right) - f(z) + \frac{1}{2}F\left(\frac{x+y}{2} - z\right) \\ &= f\left(\frac{x+z}{2}\right) + f\left(\frac{y+z}{2}\right) - f(z) + \frac{1}{4}\left(2F\left(\frac{x+y}{2} - z\right) - 2F\left(\frac{x-y}{2}\right)\right). \end{aligned}$$

Thus

$$F(x-z) + F(y-z) - F(x-y) = 2F\left(\frac{x+y}{2} - z\right) - 2F\left(\frac{x-y}{2}\right)$$

for all $x, y \in X$. Taking $z = 0$ we get

$$\begin{aligned} F(x) + F(y) - F(x-y) &= 2F\left(\frac{x+y}{2}\right) - 2F\left(\frac{x-y}{2}\right) \\ &= -\left(2F\left(\frac{x-y}{2}\right) - 2F\left(\frac{x+y}{2}\right)\right) = -F(x) - F(-y) + F(x+y). \end{aligned}$$

Moreover, from F-midaffinity of the function f we conclude that the function F must by even. Hence

$$F(x+y) + F(x-y) = 2F(x) + 2F(y)$$

for all $x, y \in X$.

Now we prove the implication (2) $\Rightarrow$ (3). Assuming that F is a quadratic function it is also F-midaffine, and it is sufficient to observe the following equivalent equations (with g=f-F).

$$
\begin{aligned}
f\left(\frac{x+y}{2}\right) &= \frac{f(x)+f(y)}{2}-\frac{1}{4}F(x-y),\\
f\left(\frac{x+y}{2}\right)-F\left(\frac{x+y}{2}\right) &= \frac{f(x)-F(x)+f(y)-F(y)}{2},\\
g\left(\frac{x+y}{2}\right) &= \frac{g(x)+g(y)}{2}
\end{aligned}
$$

for all $x,y\in X$. Finally, we prove the implication (3) $\Rightarrow$ (1). Taking $g=0$ we conclude that $f:=F$ is F-midaffine. The proof is finished.

Because a quadratic function F is also F-midaffine, note that a similar characterization as in point (3) of Theorem 1 can be also obtained for F-midconvex and F-midconcave functions.

DEFINITION 3. Let X be a real vector space, $F:X\to\mathbb{R}$ be a fixed function and $D\subset X$ be a convex set. A function $f:D\to\mathbb{R}$ is called *F-midconvex* if

$$f\left(\frac{x+y}{2}\right)\leqslant\frac{f(x)+f(y)}{2}-\frac{1}{4}F(x-y),$$

for all $x,y\in D$.

DEFINITION 4. Let X be a real vector space, $F:X\to\mathbb{R}$ be a fixed function and $D\subset X$ be a convex set. A function $f:D\to\mathbb{R}$ is called *F-midconcave* if

$$f\left(\frac{x+y}{2}\right)\geqslant\frac{f(x)+f(y)}{2}-\frac{1}{4}F(x-y),$$

for all $x,y\in D$.

A characterization of such functions is given in Lemma 1.

LEMMA 1. *Let X be a real vector space, $F:X\to\mathbb{R}$ be a quadratic function. Then*

1. *A function $f:X\to\mathbb{R}$ is F-midconvex if and only if the function $f-F$ is midconvex,*
2. *A function $f:X\to\mathbb{R}$ is F-midconcave if and only if the function $f-F$ is midconcave.*

Proof. Using the fact that a quadratic function F is also F-midaffine i.e.

$$F\left(\frac{x+y}{2}\right)=\frac{F(x)+F(y)}{2}-\frac{1}{4}F(x-y),\quad x,y\in X.$$

It is enough to subtract side be side this equation from the inequalities defining F-midconvex and F-midconcave functions, respectively.

The assumption in the above lemma that F is a quadratic function is essential. An example proving this statement can be found in [15]. Using this lemma we are able to get some counterparts of the classical separation an support results.

COROLLARY 1. (Separation property) *Let X be a real vector space. The following conditions are equivalent:*

1. *If a function $f : X \to \mathbb{R}$ is F-midconvex, $g : X \to \mathbb{R}$ is a F-midconcave function and $g \leqslant f$, then there exists a F-midaffine function $h : X \to \mathbb{R}$ such that $g \leqslant h \leqslant f$,*
2. *F is a quadratic function.*

Proof. The second point is derived from the first point immediately from Theorem 1. If we assume that F is a quadratic function, then from Lemma 1 $f - F$ is midconvex, $g - f$ is midconcave and, of course $f - F \leqslant g - F$. Now, using the Hahn-Banach type theorem due to Rode, König or Nikodem (cf.[20], [11], [14]) we get that there exists a midaffine function h^* such that $f - F \leqslant h^* \leqslant g - F$. Hence $f \leqslant h^* + F \leqslant g$ and moreover, from Lemma 1, the function $h := h^* + F$ is F-midaffine.

Also, as a consequence of the Hahn-Banach type theorem and Lemma 1 we get the next result.

COROLLARY 2. *(Support property) Let X be a real vector space. The following conditions are equivalent:*

1. *A function $f : X \to \mathbb{R}$ is F-midconvex if and only if f has a F-midaffine support in each point of X,*
2. *F is a quadratic function.*

Proof. The proof is analogous as the proof of Corrolary 1 but instead of the classical separation type theorem we use the classical support theorem for midconvex functions (see [14]).

Let us focus now on the c-quadratic-midaffinity and characterizations of inner product spaces. Characterizations of inner product spaces can be obtained as a consequence of the previous results substituting the function F with the function $c\|\cdot\|^2$. It should only be mentioned that we get the definitions of c-quadratic-midconvex and c-quadratic-midconcave functions if we replace in Definition (1) equality ”=” by inequality ”$\leqslant$” and ”$\geqslant$”, respectively.

THEOREM 2. *Let $(X, \|\cdot\|)$ be a real normed space. The following conditions are equivalent:*

1. *There exists a c-quadratic-midaffine function $f : X \to \mathbb{R}$,*

2. $(X,\|\cdot\|)$ *is an inner product space,*

3. *A function* $f : X \to \mathbb{R}$ *is* c*-quadratic-midaffine if and only if the function* $g = f - c\|\cdot\|$ *is midaffine.*

COROLLARY 3. *(Separation property) Let* $(X,\|\cdot\|)$ *be a real normed space. The following conditions are equivalent:*

1. *If a function* $f : X \to \mathbb{R}$ *is* c*-quadtatic-midconvex,* $g : X \to \mathbb{R}$ *is a* c*-quadtratic-midconcave function and* $g \leqslant f$*, then there exists a* c*-quadratic-midaffine function* $h : X \to \mathbb{R}$ *such that* $g \leqslant h \leqslant f$*,*

2. $(X,\|\cdot\|)$ *is an inner product space.*

COROLLARY 4. *(Support property) Let* $(X,\|\cdot\|)$ *be a real normed space. The following conditions are equivalent:*

1. *A function* $f : X \to \mathbb{R}$ *is* c*-quadratic-midconvex if and only if* f *has a* c*-quadratic-midaffine support in each point of* X*,*

2. $(X,\|\cdot\|)$ *is an inner product space.*

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