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SOME NEW HERMITE–HADAMARD AND FEJER TYPE INEQUALITIES WITHOUT CONVEXITY/CONCAVITY

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Abstract. In this paper we discuss the Hermite-Hadamard and Fejer inequalities vis-a-vis the convexity concept. In particular, we derive some new theorems and examples where Hermite-Hadamard and Fejer type inequalities are satisfied without the assumptions of convexity or concavity on the actual interval $[a,b]$.

1. Introduction

The Hermite-Hadamard inequality says that for any convex function $f : I \to \mathbb{R}$, I an interval, and for $a,b \in I$

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(t)\,dt \leqslant \frac{f(a)+f(b)}{2} \tag{1.1}$$

holds, and the Fejer inequality reads

$$f\left(\frac{a+b}{2}\right)\int_a^b p(x)\,dx \leqslant \int_a^b f(t)\,p(t)dt \leqslant \frac{f(a)+f(b)}{2}\int_a^b p(x)\,dx, \tag{1.2}$$

when f is convex and $p : [a,b] \to \mathbb{R}$ is non-negative, integrable and symmetric around the midpoint $x = \frac{a+b}{2}$. If instead f is concave, then (1.1) and (1.2) hold in the reversed direction.

There have been a lot of developments and applications of these inequalities. One such development is to replace the notion of classical convexity by other variants and generalizations of convexity. An early well cited such paper is that by S. S. Dragomir et. al. [8], see also [3], [4], [7], [10] and [13]. We also mention the paper [11] and especially the book [12] by C. Niculescu and L.E. Persson, where several generalizations, variants and applications are described and put them into a more general convexity context.

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Nowadays it is well known (see [12] pages 60 and 68) that if $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$, whose restriction to each compact subinterval on $[a,b]$ satisfies the right hand-side inequality in (1.1), then f is convex. The same is true if we instead assume that the left hand-side inequality in (1.1) holds. However, if we for example assume that (1.1) holds only for $[a,b]$, this statement is not true. In fact it was recetly proved that (1.1) and (1.2) can hold even if f is not convex.

We mention in particular the papers [6] and [9] where S. S. Dragomir and A. Farissi et al. deal with a set of real functions f defined on $[a,b]$ such that its symmetrized transform f^*, defined by

$$f^*(x) := \frac{f(x) + f(a+b-x)}{2}, \qquad a \leqslant x \leqslant b,$$

is convex. In this case f is called ***symmetrized convex function*** on $[a,b]$. If f^* is concave, then f is called ***symmetrized concave function*** on $[a,b]$.

When f is convex on $[a,b]$ so is f^* but it is easy to see that the convexity of f^* does not guarantee the convexity of f as proved in [9] and in [6].

For this type of functions S. S. Dragomir and earlier A. Farissi et al., obtained Fejer and Hermite-Hadamard type inequalities as follows:

Theorem A ([6, Theorem 1] and [9, Theorem 10]). *Assume that $f : [a,b] \to \mathbb{R}$ is symmetrized (concave) convex on the interval $[a,b]$. Then, the (reverse) Hermite-Hadamard and Fejer inequalities (1.1) and (1.2) hold.*

We want to point out that also, by dealing in [5] with Hermite-Hadamard inequality for convex functions, F. Chen observed that there is no need for f to be convex on $[a,b]$ but it is sufficient that f satisfies the much weaker condition:

$$f'(x) \geqslant f'(a+b-x), \qquad x \in \left[\frac{a+b}{2}, b\right], \tag{1.3}$$

in order to get Hermite-Hadamard (1.1) and Fejer inequalities (1.2) (see Remark 2 in [5]).

These considered cases, besides being of interest by themselves, lead us to the following (probably difficult but challenging) **OPEN QUESTION**: Find necessary and sufficient conditions for the function f so that (1.1) and (1.2) hold.

In this paper we extend the definitions of f^* for symmetrically decreasing and symmetrically increasing functions defined as follows:

DEFINITION 1. A function f is called **symmetrically increasing (decreasing) function** on $[a,b]$ if it is symmetric on this interval and is increasing (decreasing) on $\left[\frac{a+b}{2}, b\right]$.

In Section 2 we show some new basic results of the considered type, namely by finding classes of functions for which the functions and their symmetrized transforms are not convex and not concave on a given interval but nevertheless satisfy some Fejer and Hermite-Hadamard type inequalities.

Some examples, further remarks, generalizations and illustrations of the basic results are given in Section 3, which hopefully will serve as a step towards a further understanding of the open question above.

2. Some new basic results

In this section we generalize Theorem A. Our first main result reads:

THEOREM 1. *Let f be a real differentiable function on $\mathbb{R}$ that satisfies*

$$f(x) = -f(2c-x) \tag{2.1}$$

and let $f(x)$ be convex for $x \geqslant c$.

Define

$$f^*(x) := \frac{f(x)+f(a+b-x)}{2}, \qquad -\infty < a \leqslant x \leqslant b < \infty. \tag{2.2}$$

Then, for $c \geqslant \frac{a+b}{2}$, $f^*(x)$ is symmetrically decreasing on $[a,b]$, and although f and f^* are not necessarily concave we get that the reversed Fejer inequalities

$$\frac{f(a)+f(b)}{2}\int_a^b p(x)\,dx \leqslant \int_a^b f(x)\,p(x)\,dx \tag{2.3}$$

$$= \int_a^b f^*(x)\,p(x)\,dx \leqslant f\left(\frac{a+b}{2}\right)\int_a^b p(x)\,dx,$$

hold, where $p:[a,b]\to\mathbb{R}$ is non-negative, integrable and symmetric around the mid-point $x=\frac{a+b}{2}$. If $p(x)=1$ on $[a,b]$, then we get the reversed Hermite-Hadamard inequality for the function f.

Moreover, if in addition f' is concave and differentiable when $x \geqslant c$, then f^* is convex on $[c,b]$ when $b \geqslant c \geqslant \frac{a+b}{2}$ and (2.3) holds.

Proof. Case 1. Let $c > b$. This means that f is concave on $[a,b]$ and therefore for each $x \in [a,b]$, $a+b-x < c$ (which is the same as $c-b > a-x$ and this holds because $c-b > 0$ and $a-x < 0$).

Therefore, since f is concave so is $f(a+b-x)$, and the symmetric function f^* is concave too on $[a,b]$ and hence symmetrically decreasing on $[a,b]$ which means that f^* is decreasing on $[\frac{a+b}{2}, b]$.

Case 2. Let $\frac{a+b}{2} < c < b$. Then, on the interval $[a+b-c,c]$ f is concave and therefore f^* as defined in (2.2) is symmetrically decreasing and hence f^* is decreasing on $\left[\frac{a+b}{2},c\right]$.

We check now the sign of $f^{*\prime}$ on $[c,b]$:

We note that, according to (2.1), $f(a+b-x) = -f(2c-a-b+x)$. Therefore, on this interval (2.2) becomes

$$f^*(x) = \frac{f(x) - f(2c-a-b+x)}{2} \tag{2.4}$$

and

$$f^{*\prime}(x) = \frac{f'(x) - f'\left(2\left(c - \frac{a+b}{2}\right) + x\right)}{2}. \tag{2.5}$$

Since $c - \frac{a+b}{2} > 0$, we have $x < 2\left(c - \frac{a+b}{2}\right) + x$ and as $x > c$ also $2\left(c - \frac{a+b}{2}\right) + x > c$. Moreover, by (2.1) and by convexity of f for $x \geqslant c$ we find that $f'(x)$ is increasing. Hence $f'(x) \leqslant f'\left(2\left(c - \frac{a+b}{2}\right) + x\right)$, which by (2.5) means that $f^{*\prime}(x) < 0$. From this fact together with the continuity of f and the symmetrically concavity on the interval $[a+b-c,c]$ of $f^*(x)$ we conclude that $f^*(x)$ on the interval $[a,b]$ is symmetrically decreasing, which implies that

$$\frac{f(a)+f(b)}{2} = f^*(a) \leqslant f^*(x) \leqslant f^*\left(\frac{a+b}{2}\right) = f\left(\frac{a+b}{2}\right).$$

Multiplying these inequalities by the non-negative and symmetric function $p(x)$ on $[a,b]$ and integrating we get that $\int_a^b f(x)\,p(x)\,dx = \int_a^b f^*(x)\,p(x)\,dx$ and hence

$$\frac{f(a)+f(b)}{2}\int_a^b p(x)\,dx = f^*(a)\int_a^b p(x)\,dx \leqslant \int_a^b f(x)\,p(x)\,dx \tag{2.6}$$

$$= \int_a^b f^*(x)\,p(x)\,dx \leqslant f^*\left(\frac{a+b}{2}\right)\int_a^b p(x)\,dx = f\left(\frac{a+b}{2}\right)\int_a^b p(x)\,dx$$

hold and therefore (2.3) holds. For $p(x) = 1$, $x \in [a,b]$, this is Hermite-Hadamard inequality.

If in addition f' is concave on $x \geqslant c$, then f'' is decreasing and from (2.5) we can conclude that

$$f^{*\prime\prime}(x) = \frac{f''(x) - f''\left(2\left(c - \frac{a+b}{2}\right) + x\right)}{2} \geqslant 0$$

and we get a symmetrically decreasing f^* and therefore (2.6) holds although f^* is convex on $[c,b]$ when $\frac{a+b}{2} < c < b$. The proof is complete.

We illustrate Theorem 1 with the following example:

EXAMPLE 1. Choose $f(x) = x^{\frac{5}{3}}$ and let $a = -3$, $b = 1$ and $c = 0$. Then on the interval $[-3,1]$ with the mid-point at $\frac{-3+1}{2} = -1$ we get that the symmetrized transform $f^*(x)$ of $f(x)$ on this interval is

$$f^*(x) = \frac{f(x) + f(-3+1-x)}{2}, \qquad -3 \leqslant x \leqslant 1,$$

that is in our case

$$f^*(x) = \frac{x^{\frac{5}{3}} + (-2-x)^{\frac{5}{3}}}{2} = \frac{x^{\frac{5}{3}} - (2+x)^{\frac{5}{3}}}{2}$$

and

$$f^{*\prime\prime}(x) = \frac{10}{9}\left(x^{-\frac{1}{3}} - (2+x)^{-\frac{1}{3}}\right), \qquad x \neq 0.$$

On the interval $[0,1]$ both $x \geqslant 0$ and $(2+x) > 0$. Therefore we get that the second derivative of f^* is greater than zero on $x > 0$.

On the interval $[-3,-2]$ both $x < 0$ and $(2+x) < 0$, and putting $-x = y$ we get that

$$f^{*\prime\prime}(x) = \frac{10}{9}\left(-\left(y^{-\frac{1}{3}}\right) + (y-2)^{-\frac{1}{3}}\right) > 0$$

because $2 \leqslant y \leqslant 3$ and $0 < y-2 \leqslant 1$ and also $y-2 < y$.

Similarly, we see that $f^{*\prime\prime}(x) \leqslant 0$ on the interval $[-2,0]$.

We conclude that f^* is symmetrically decreasing on $[-3,1]$ although f and f^* are not concave on $[-3,1]$ and $f(x) = x^{\frac{5}{3}}$ satisfies the reversed Fejer inequality (2.6) on the interval $[-3,1]$.

Moreover, our $f(x) = x^{\frac{5}{3}}$ satisfies all the conditions of Theorem 1 including the fact that $f'''(x) < 0$ when $x > 0$ that is f' is concave on $x > 0$. This guaranties that f^* is not convex or concave on the whole interval $[-3,1]$ but nevertheless is symmetrically decreasing and satisfies (2.3).

Theorem 1 can be complemented, for the same function f, in the following way:

THEOREM 2. *Let f be a real differentiable function on $\mathbb{R}$ that satisfies*

$$f(x) = -f(2c-x)$$

and let $f(x)$ be convex for $x \geqslant c$.

Define

$$f^*(x) := \frac{f(x) + f(a+b-x)}{2}, \quad x \in [a,b], \quad -\infty < a < b < \infty.$$

Then, for $c \leqslant \frac{a+b}{2}$, $f^*(x)$ is symmetrically increasing on $[a,b]$, and although f and f^* are not necessarily convex we get that the Fejer inequalities

$$\frac{f(a)+f(b)}{2}\int_a^b p(x)\,dx \geqslant \int_a^b f(x)\,p(x)\,dx \tag{2.7}$$

$$= \int_a^b f^*(x)\,p(x)\,dx \geqslant f\left(\frac{a+b}{2}\right)\int_a^b p(x)\,dx,$$

hold, where $p : [a,b] \to \mathbb{R}$ is non-negative, integrable and symmetric around the mid-point $x = \frac{a+b}{2}$. If $p(x) = 1$ on $[a,b]$, then we get the Hermite-Hadamard inequality for the function f.

Moreover, if f' is also concave when $x \geqslant c$, then f^* is concave on $[a,c]$ when $a < c < \frac{a+b}{2}$ and (2.7) holds.

In the case that $c = \frac{a+b}{2}$, $f^*(x) = 0$ on $[a,b]$.

Proof. The proof is similar to that of Theorem 1, so we omit the details.

The following result deals with functions that are neither symmetric nor antisymmetric on any interval but nevertheless satisfies (1.1) and (1.2), in particular, it generalizes Theorem A. It reads:

THEOREM 3. *Let $m = 1,2,\ldots$. Let $f(x)$ be defined by*

$$f(x) = \begin{cases} (4m+1)(x-1)^{2m+1}, & x \leqslant 1, \\ (2m+1)(x-1)^{4m+1}, & x \geqslant 1. \end{cases}$$

Then,

$$f^*(x) = \frac{f(x) + f(-x)}{2}, \qquad -3 \leqslant x \leqslant 3$$

is symmetrically decreasing on $[-3,3]$ and Fejer and Hermite-Hadamard inequalities for concave functions hold for f, although neither f nor f^* are concave on $[-3,3]$.

Proof. The symmetrized function f^* for $x \geqslant 1$ is

$$f^*(x) = \frac{(2m+1)(x-1)^{4m+1} + (4m+1)(-x-1)^{2m+1}}{2}$$

$$= \frac{(2m+1)(x-1)^{4m+1} - (4m+1)(x+1)^{2m+1}}{2}, \qquad x \geqslant 1$$

and its derivative is

$$f^{*\prime}(x) = \frac{(2m+1)(4m+1)(x-1)^{4m} - (4m+1)(2m+1)(x+1)^{2m}}{2}, \quad x \geqslant 1.$$

We check when $f^{*\prime}(x) \leqslant 0$ and when $f^{*\prime}(x) \geqslant 0$. Obviously,

$$f^{*\prime}(x) = 0 \Rightarrow (x-1)^{4m} = (x+1)^{2m} \Rightarrow \left((x-1)^2\right)^{2m} = (x+1)^{2m},$$

which leads to

$$(x+1) = \pm(x-1)^2$$

$$x+1 = (x-1)^2 \Rightarrow x = 0, \quad x = 3,$$

and the case

$$x+1 = -(x-1)^2,$$

never holds for any real x.

As we deal now with $x \geqslant 1$, this means that

$$f^{*\prime}(x) = \begin{cases} < 0, & 1 < x < 3 \\ > 0 & x > 3 \end{cases}.$$

Now we check the interval $-1 \leqslant x \leqslant 1$ and find that

$$f^{*}(x) = \frac{(4m+1)(x-1)^{2m+1} + (4m+1)(-x-1)^{2m+1}}{2}.$$

Since $f(x)$ is concave on $-1 \leqslant x \leqslant 1$, so is $f^{*}(x)$, and as $f^{*}(x)$ is symmetrically concave it is symmetrically decreasing on $[-1,1]$ and $f^{*}(x)$ is also decreasing on $[1,3]$ and is continuous. We conclude that $f^{*}(x)$ is symmetrically decreasing on $[-3,3]$. Now we show that on the interval $[2,3]$ $f^{*}(x)$ changes from concave to convex: indeed, because

$$f^{*\prime\prime}(x) = \frac{(2m+1)(4m+1)2m\left(2(x-1)^{4m-1} - (x+1)^{2m-1}\right)}{2},$$

we have that $f^{*\prime\prime}(2) < 0$, $f^{*\prime\prime}(3) > 0$, which show that even if $f(x)$ and $f^{*}(x)$ are not concave on the interval $[-3,3]$ the Fejer and the Hermite-Hadamard inequalities as for concave functions hold. The proof is complete.

Our next result reads:

THEOREM 4. *Let f be differentiable on $[-a,a]$, decreasing on $[-a,-c]$, $0 \leqslant c < a$, increasing on $[c,a]$ and let f be convex on $[-c,c]$. Then the Fejer and Hermite-Hadamard inequalities for convex functions hold for f on the interval $[-a,a]$ although f and f^{*} are not necessarily convex on $[-a,a]$.*

Proof. The proof follows from the fact that f as well as $f^{*}(x) = \frac{f(x)+f(-x)}{2}$, $-c \leqslant x \leqslant c$, are convex and because of f^{*} symmetry on $[-c,c]$ it is symmetrically increasing on $[-c,c]$ and therefore increasing on $[0,c]$. Moreover, on the interval $[c,a]$ $f^{*\prime}(x) = \frac{f'(x)-f'(-x)}{2} \geqslant 0$ because $f'(x) \geqslant 0 \geqslant f'(-x)$ on $[c,a]$. Therefore, Fejer and Hermite-Hadamard inequalities as for convex functions hold for f.

The proof is complete.

We conclude this section by emphasizing that there are cases which apriori we can verify that f^{*} is not symmetrically increasing or symmetrically decreasing:

REMARK 1. Let f be continuous decreasing on $[-a,-c]$, $0 < c < a$, increasing on $[c,a]$ and let f be concave on $[-c,c]$. Then, $f^{*}(x) = \frac{f(x)+f(-x)}{2}$, is not symmetrically increasing and not symmetrically decreasing on $[-a,a]$. Indeed, on the interval $[-c,c]$ f^{*} is evidently symmetrically decreasing because f is concave on $[-c,c]$. On $[c,a]$ it yields that $f'(x) \geqslant 0 \geqslant f'(-x)$ and therefore $f^{*\prime}(x) \geqslant 0$. Therefore, in this case f^{*} is not symmetrically increasing or symmetrically decreasing on the whole interval $[-a,a]$.

3. Further examples, remarks and generalizations

In this section we present some generalizations and examples of the results discussed in Section 2.

First we discuss the possibility that a given continuous real function f defined on $[a,b]$, $-\infty < a < b < \infty$, and its symmetrized transform

$$f^*(x) = \frac{f(x) + f(a+b-x)}{2}, \quad a \leqslant x \leqslant b, \tag{3.1}$$

are not symmetrical increasing and not symmetrical decreasing but f^{**} defined as

$$f^{**}(x) = \begin{cases} \frac{f^*(x) + f^*\left(\frac{a+3b}{2} - x\right)}{2}, & \frac{a+b}{2} \leqslant x \leqslant b \\ f^{**}(a+b-x), & a \leqslant x \leqslant \frac{a+b}{2}, \end{cases} \tag{3.2}$$

is symmetrically increasing or symmetrically decreasing on $\left[\frac{a+b}{2}, b\right]$ and on $\left[a, \frac{a+b}{2}\right]$.

THEOREM 5. *Let $f : [a,b]$ be a continuous function. If f^{**} is symmetrically increasing on $\left[\frac{a+b}{2}, b\right]$, then, the Hermite-Hadamard type inequality*

$$\begin{aligned} &\frac{f\left(\frac{a+3b}{4}\right) + f\left(\frac{3a+b}{4}\right)}{2} \\ &\leqslant \frac{1}{b-a}\int_a^b f(x)\,dx \leqslant \frac{1}{2}\left(\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right), \end{aligned} \tag{3.3}$$

holds.

Proof. After some straightforward manipulations on (3.2) we get that

$$f^{**}(x) = \frac{f(x) + f(a+b-x) + f\left(\frac{a+3b}{2} - x\right) + f\left(\frac{a-b}{2} + x\right)}{4}, \quad x \in \left[\frac{a+3b}{4}, b\right]. \tag{3.4}$$

If $f^{**}(x)$ is symmetrically increasing on $\left[\frac{a+b}{2}, b\right]$, then, since f^{**} is increasing on $\left[\frac{a+3b}{4}, b\right]$, we find that

$$f^{**}\left(\frac{a+3b}{4}\right) \leqslant f^{**}(x) \leqslant f^{**}(b) \tag{3.5}$$

holds and therefore

$$\frac{b-a}{4} f^{**}\left(\frac{a+3b}{4}\right) \leqslant \int_{\frac{a+3b}{4}}^{b} f^{**}(x)dx \leqslant \frac{b-a}{4} f^{**}(b). \tag{3.6}$$

By using (3.1) and (3.2) we obtain that

$$\int_{\frac{a+3b}{4}}^{b} f^{**}(x)dx = \frac{1}{4}\int_a^b f(x)\,dx. \tag{3.7}$$

Moreover, by using (3.4) we see that

$$f^{**}(b)=\frac{1}{2}\left(\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right) \tag{3.8}$$

and

$$f^{**}\left(\frac{a+3b}{4}\right)=\frac{f\left(\frac{a+3b}{4}\right)+f\left(\frac{3a+b}{4}\right)}{2} \tag{3.9}$$

hold and, thus, according to (3.7), (3.8) and (3.9), we conclude that (3.3) holds. The proof is complete.

EXAMPLE 2. In the special case $-a=b>0$ we get that

$$f^{**}(x)=\frac{1}{2^2}\left[f(x)+f(-x)+f(b-x)+f(x-b)\right],$$

and

$$\frac{1}{2}\left(f\left(\frac{b}{2}\right)+f\left(-\frac{b}{2}\right)\right)\leqslant\frac{1}{2b}\int_{-b}^{b}f(x)\,dx\leqslant\frac{1}{2}\left(\frac{f(b)+f(-b)}{2}+f(0)\right).$$

REMARK 2. Similarly we can prove that if $f^{**}(x)$ on $\left[\frac{a+b}{2},b\right]$ is symmetrically decreasing, then we get the reverse inequality of (3.3).

REMARK 3. We emphasize that (3.3) holds in particular if f is convex on $[a,b]$ and then

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leqslant \frac{f\left(\frac{a+3b}{4}\right)+f\left(\frac{3a+b}{4}\right)}{2} \\ &\leqslant \frac{1}{b-a}\int_a^b f(x)\,dx \\ &\leqslant \frac{1}{2}\left(\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right)\leqslant\frac{f(a)+f(b)}{2}. \end{aligned}$$

Moreover, we know that (3.3) holds if the symmetrized transform of f is convex on $\left[\frac{a+b}{2},b\right]$ and on $\left[a,\frac{a+b}{2}\right]$.

REMARK 4. In [6, Theorem 4] the same inequality (3.3) was proved for a function f defined on $[a,b]$, where its symmetrized transform is convex on $\left[\frac{a+b}{2},b\right]$ and on $\left[a,\frac{a+b}{2}\right]$ but not necessarily on the whole inteval $[a,b]$. Our result is more general and includes this theorem as a special case since it includes not necessarily convex functions on $\left[\frac{a+b}{2},b\right]$.

In order to find further generalizations we define $f^{(n*)}$, $n = 1, 2, ...$ by:

$$f^{(n*)}(x) = \frac{f^{((n-1)*)}(x) + f^{((n-1)*)}\left(\frac{a+(2^n-1)b}{2^{n-1}} - x\right)}{2}, \quad \frac{a+\left(2^{n-1}-1\right)b}{2^{n-1}} \leqslant x \leqslant b, \tag{3.10}$$

$$f^{(n*)}(x) = f^{(n*)}\left(x + k\frac{b-a}{2^{n-1}}\right), \quad \frac{a+\left(2^{n-1}-1\right)b}{2^{n-1}} - k\frac{b-a}{2^{n-1}} \leqslant x \leqslant b - k\frac{b-a}{2^{n-1}},$$

$$k = 1, 2, ..., \left(2^{n-1}-1\right),$$

where $f^{(0*)}(x) = f(x)$, $f^{(1*)}(x) = f^*(x)$ and $f^{(2*)} = f^{**}(x)$.

If $f^{(2*)}(x)$ is not symmetrically increasing or symmetrically decreasing on $\left[\frac{a+b}{2}, b\right]$, then we can continue the procedure and check if $f^{(3*)}$ is symmetrically increasing or symmetrically decreasing on $\left[\frac{a+3b}{4}, b\right]$, where

$$f^{(3*)}(x) = \frac{f^{(2*)}(x) + f^{(2*)}\left(\frac{a+7b}{4} - x\right)}{2}, \quad \frac{a+3b}{4} \leqslant x \leqslant b, \tag{3.11}$$

$$f^{(3*)}(x) = f^{(3*)}\left(x + k\frac{b-a}{4}\right), \quad \frac{a+3b}{4} - k\frac{b-a}{4} \leqslant x \leqslant b - k\frac{b-a}{4},$$

$$k = 1, 2, 3,$$

and we continue inductively the same procedure and get:

THEOREM 6. *Let $f : [a,b] \to \mathbb{R}$ be a continuous function. For $n = 2, 3, ...,$ let $f^{(n*)}$ be defined by (3.10) where $f^{(0*)}(x) = f(x)$ and $f^*\ (x) = f^{(1*)}(x)$. If $f^{(n*)}$ is symmetrically increasing on $\left[\frac{a+\left(2^{n-1}-1\right)b}{2^n}, b\right]$, then*

$$\frac{b-a}{2^n} f^{(n*)}\left(\frac{a+(2^n-1)b}{2^n}\right) \leqslant \int_{\frac{a+(2^n-1)b}{2^n}}^{b} f^{(n*)}(x)\,dx \leqslant \frac{b-a}{2^n} f^{(n*)}(b), \tag{3.12}$$

i.e.,

$$f^{(n*)}\left(\frac{a+(2^n-1)b}{2^n}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)\,dx \leqslant f^{(n*)}(b) \tag{3.13}$$

holds.

Proof. If for a specific n, $f^{(n*)}$ is symmetrically increasing on $\left[\frac{a+\left(2^{n-1}-1\right)b}{2^{n-1}}, b\right]$, then it is increasing on $\left[\frac{a+(2^n-1)b}{2^n}, b\right]$ and therefore

$$f^{(n*)}\left(\frac{a+(2^n-1)b}{2^n}\right) \leqslant f^{(n*)}(x) \leqslant f^{(n*)}(b).$$

Hence

$$\frac{b-a}{2^n} f^{(n*)}\left(\frac{a+(2^n-1)b}{2^n}\right) \leqslant \int_{\frac{a+(2^n-1)b}{2^n}}^{b} f^{(n*)}(x)\,dx \leqslant \frac{b-a}{2^n} f^{(n*)}(b),$$

and since as a result of the periodicity of $f^{(n*)}$ and its relation to $f(x)$

$$2^n \int_{\frac{a+(2^n-1)b}{2^n}}^{b} f^{(n*)}(x)\,dx = \int_a^b f(x)\,dx,$$

we find that

$$f^{(n*)}\left(\frac{a+(2^n-1)b}{2^n}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)\,dx \leqslant f^{(n*)}(b),$$

and the proof follows.

We finish the paper by further discussing and illustrating the function f considered in Theorem 3 for $m=1$ which is an example of Theorem 6 for $n=3$. In this example f and f^* on $[-4,4]$ and $f^{(2*)}$ on $[0,4]$ are not symmetrically increasing but $f^{(3*)}$ is symmetrically increasing on $[2,4]$:

EXAMPLE 3. We show the results of calculation that manifest the behavior of the graphs of $f^{(n*)}$, $n=0,1,2,3$ in Figure 1 (see it in the end of the article) of the functions

$$f(x) = \begin{cases} 5(x-1)^3, & -4 \leqslant x \leqslant 1 \quad, \\ 3(x-1)^5, & 1 \leqslant x \leqslant 4 \quad, \end{cases}$$

$$f^*(x) = \frac{f(x)+f(-x)}{2} = \begin{cases} \frac{5(x-1)^3-5(x+1)^3}{2}, & -1 \leqslant x \leqslant 1, \\ \frac{3(x-1)^5-5(x+1)^3}{2}, & 1 \leqslant x \leqslant 4, \\ \frac{5(x-1)^3-3(x+1)^5}{2}, & -4 \leqslant x \leqslant -1, \end{cases}$$

and of $f^{(2*)}(x)$ and $f^{(3*)}(x)$ as are defined on the inteval $[a,b]$ by using (3.2), and (3.11) on the interval $[-4,4]$. We have that

$$f^{(2*)}(x) = \frac{f^*(x)+f^*(4-x)}{2}, \qquad 0 \leqslant x \leqslant 4,$$

and

$$f^{(3*)}(x) = \frac{f^{(2*)}(x)+f^{(2*)}(6-x)}{2}, \quad 2 \leqslant x \leqslant 4.$$

By using (3.10) we see that on the interval $[2,4]$

$$\begin{aligned} f^{(3*)}(x) = \frac{1}{2^3}&[f(x)+f(-x)+f(4-x)+f(x-4) \\ &+f(6-x)+f(x-6)+f(x-2)+f(2-x)], \end{aligned}$$

is symmetrically increasing and is increasing on $[3,4]$. Hence,

$$f^{(3*)}(3) \leqslant f^{(3*)}(x) \leqslant f^{(3*)}(4),$$

and, therefore,

$$f^{(3*)}(3) \leqslant \int_3^4 f^{(3*)}(x)\,dx \leqslant f^{(3*)}(4)$$

so we find that

$$f^{(3*)}(3) \leqslant \int_{3}^{4} f^{(3*)}(x)\,dx = \frac{1}{8}\int_{-4}^{4} f(x)\,dx \leqslant f^{(3*)}(4).$$

Concluding Note. We hope that this paper will serve as a step towards finding necessary and sufficient conditions for (1.1) and (1.2) to be satisfied.

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