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AN INTERPOLATION FORMULA IN RELATION TO A POLYNOMIAL INEQUALITY OF SCHUR

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Abstract. We study a recent interpolation formula for algebraic polynomials due to Dryanov, Fournier and Ruscheweyh and its links to a polynomial inequality of Schur.

1. Introduction

Let $\mathscr{P}_n$ denote the class of polynomials $p(z) = \sum_{k=0}^{n} a_k z^k$ with complex coefficients. We write $\mathbb{D}$ for the unit disk in the complex plane $\mathbb{C}$. Let also $\mathscr{T}_n$ be the class of trigonometric polynomials $t(\theta) := \sum_{k=-n}^{n} a_k e^{ik\theta}$ with coefficients in $\mathbb{C}$. Given a subset E of $\mathbb{C}$ and a function f defined on E, let

$$|f|_E = \sup_{z\in E} |f(z)| = |f(z)|_E.$$

The famous S. Bernstein inequality says that for $p \in \mathscr{P}_n$

$$|p'|_{\mathbb{D}} \leqslant n\,|p|_{\mathbb{D}} \tag{1}$$

while for $t \in \mathscr{T}_n$

$$|t'|_{[0,2\pi]} \leqslant n\,|t|_{[0,2\pi]}. \tag{2}$$

We refer the reader to the book [11] by Rahman and Schmeisser concerning these inequalities and their generalizations. The books by Sheil-Small [14] or Borwein and Erdélyi [1] also are very valuable sources concerning these and other polynomial inequalities. We shall also be concerned with the so-called Markov inequality for $p \in \mathscr{P}_n$ and $k \in \{1,2,\dots,n\}$

$$|p^{(k)}|_{[-1,1]} \leqslant T_n^{(k)}(1)\,|p|_{[-1,1]} \tag{3}$$

where T_n stands for the n^{th} Chebyshev polynomial and $T_n^{(k)}(1) = \frac{n^2(n^2-1^2)\cdots(n^2-(k-1)^2)}{1\cdot 3\cdots(2k-1)}$. The inequality (3) is also well-known: apart the books quoted above, we refer to a relatively recent paper by Shadrin [13] which contains a number of detailed proofs of (3) together with interesting and pertinent historical remarks. The goal of this paper is to review some consequences of an interpolation formula due to Dryanov, Fournier and Ruscheweyh in relation with variants of the above quoted inequalities.

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2. The interpolation formula

The following formula holds for all $p \in \mathscr{P}_n$ and $\theta \in [0,\pi]$:

$$\frac{p(e^{i\theta})-p(e^{-i\theta})}{e^{i\theta}-e^{-i\theta}}=\sum_{j=0}^{n} c_n(j,\theta)\frac{p\left(e^{ij\pi/n}\right)+p\left(e^{-ij\pi/n}\right)}{2} \tag{4}$$

with

$$c_n(j,\theta)=\begin{cases}\dfrac{(-1)^j\left(\cos(j\pi)-\cos(n\theta)\right)}{n\left(\cos\left(\frac{j\pi}{n}\right)-\cos\theta\right)}, & 1\leqslant j\leqslant n-1\\[2ex] \dfrac{(-1)^j\left(\cos(j\pi)-\cos(n\theta)\right)}{2n\left(\cos\left(\frac{j\pi}{n}\right)-\cos\theta\right)}, & j=0 \; or \; j=n\end{cases}$$

and $\sum_{j=0}^{n}|c_n(j,\theta)|\leqslant n$.

The formula was first obtained in [2] mainly as an application of a known quadrature formula; another proof was given in [3], based essentially on the Lagrange interpolation formula. The formula has also been studied and extended in [4], [6] and [7]. It contains in particular a refinement of (1) valid for $p\in\mathscr{P}_n$ and θ real:

$$|p'(e^{i\theta})|\leqslant n\max_{j\in\mathscr{J}_n}\left|\frac{p\left(e^{i(\theta+j\pi/n)}\right)+p\left(e^{i(\theta-j\pi/n)}\right)}{2}\right| \tag{5}$$

with $\mathscr{J}_n=\{0\}\cup\{j\,|\,1\leqslant j\leqslant n \text{ and } j \text{ is odd}\}$. This inequality is reminiscent and in some sense sharpens the Frappier-Rahman-Ruscheweyh inequality (see for example [9] or [11, p.524]). It is also true that the refinement of (3) with $k=1$ due to Duffin and Schaeffer [5], namely,

$$|p'|_{[-1,1]}\leqslant n^2\max_{0\leqslant j\leqslant n}\left|p\left(\cos\left(\frac{j\pi}{n}\right)\right)\right|, \tag{6}$$

follows from (4) without much efforts (see [11] for details). In the sequel we shall need two other known consequences of (4). The interpolation formula (4) contains the Lagrange interpolation formula for $\mathscr{P}_n$ at the modes $\left\{\cos\left(\frac{j\pi}{n}\right)\right\}_{j=0}^{n}$: let $P\in\mathscr{P}_n$; then $p\in\mathscr{P}_{2n}$ if $p(z)=z^nP\left(\frac{z+\frac{1}{z}}{2}\right)$. Applying the interpolation formula (6), with n replaced by $2n$, yields

$$\begin{aligned}\frac{\sin(n\theta)}{\sin(\theta)}P(\cos\theta)&=\sum_{\substack{j=0\\ j\text{ even}}}^{2n} c_{2n}(j,\theta)\cos\left(\frac{j\pi}{2}\right)P\left(\cos\frac{j\pi}{2n}\right)\\ &=\sum_{k=0}^{n} c_{2n}(2k,\theta)\,(-1)^kP\left(\cos\frac{k\pi}{n}\right)\end{aligned}$$

i.e.,

$$P(\cos\theta)=\sum_{k=0}^{n}\frac{c_{2n}(2k,\theta)\sin(\theta)}{\sin(n\theta)}P\left(\cos\frac{k\pi}{n}\right). \tag{7}$$

By unicity, we clearly have that (7) is the interpolation formula of Lagrange at the given modes. We shall also need the following known fact: the formula (4) contains the Marcel Riesz interpolation formula for trigonometric polynomials. Let $t\in\mathscr{T}_n$ and define $p\in\mathscr{P}_{2n}$ by $p(e^{i\theta})=e^{in\theta}t(\theta)$; it follows that

$$it'(0)=np(1)-p'(1)=\frac{1}{2n}\sum_{\substack{j=1\\ j\text{ odd}}}^{2n-1}\frac{e^{ij\pi/2}t\left(\frac{j\pi}{2n}\right)+e^{-ij\pi/2}t\left(-\frac{j\pi}{2n}\right)}{2\left(1-\cos\left(\frac{j\pi}{2n}\right)\right)} \tag{8}$$

and mild computations done in [9] lead to the Marcel Riesz formula

$$t'(\theta)=\sum_{k=1}^{2n}\frac{(-1)^{k-1}}{4n\sin^2\left(\frac{(2k-1)\pi}{4n}\right)}t\left(\theta+\frac{(2k-1)\pi}{2n}\right) \tag{9}$$

valid for any $t\in\mathscr{T}_n$. It is a well-known fact that (2) follows from (9): first setting $t(\theta)=\sin(n\theta)\in\mathscr{T}_n$ in (9) with $\theta=0$ yields

$$\begin{aligned}n&=\sum_{k=1}^{2n}\frac{(-1)^{k-1}}{4n\sin^2\left(\frac{(2k-1)\pi}{4n}\right)}\sin\left(\frac{(2k-1)\pi}{2}\right)\\&=\sum_{k=1}^{2n}\frac{1}{4n\sin^2\left(\frac{(2k-1)\pi}{4n}\right)}\end{aligned}$$

and then again by (9) for arbitrary $t\in\mathscr{T}_n$ and $\theta\in[0,2\pi]$,

$$\begin{aligned}|t'(\theta)|&=\left|\sum_{k=1}^{2n}\frac{(-1)^{k-1}}{4n\sin^2\left(\frac{(2k-1)\pi}{4n}\right)}t\left(\theta+\frac{(2k-1)\pi}{2n}\right)\right|\\&\leqslant\sum_{k=1}^{2n}\frac{1}{4n\sin^2\left(\frac{(2k-1)\pi}{4n}\right)}\left|t\left(\theta+\frac{(2k-1)\pi}{2n}\right)\right|\\&\leqslant\left(\sum_{k=1}^{2n}\frac{1}{4n\sin^2\left(\frac{(2k-1)\pi}{4n}\right)}\right)|t|_{[0,2\pi]}\\&=n\,|t|_{[0,2\pi]}.\end{aligned}$$

3. The inequality of Riesz-Schur

The following inequality is valid for all $p \in \mathscr{P}_n$ and all $\theta \in [0,\pi]$:

$$\left|\frac{p(e^{i\theta})-p(e^{-i\theta})}{e^{i\theta}-e^{-i\theta}}\right| \leqslant n \left|\frac{p(e^{i\theta})-p(e^{-i\theta})}{2}\right|_{[0,2\pi]}. \tag{10}$$

As explained by Paul Nevai in his very charming paper [10], there are problems of "priority" concerning this inequality and also the inequality (2) of Bernstein for trigonometric prolynomials. We refer to [10] for the full story and shall content ourselves with attributing (10) to both Marcel Riesz and Isai Schur. In his paper [10], Nevai remarked that (10) is in fact equivalent with

$$\left|\frac{t(\theta)}{\sin(\theta)}\right|_{[0,2\pi]} \leqslant n\,|t(\theta)|_{[0,2\pi]} \tag{11}$$

for any odd polynomial $t \in \mathscr{T}_n$. It is also equivalent with

$$|p(x)|_{[-1,1]} \leqslant n \left|\sqrt{1-x^2}\, p(x)\right|_{[-1,1]} \tag{12}$$

for any polynomial $p \in \mathscr{P}_{n-1}$. Moreover, Nevai also observed that (10), (11) and (12) are in fact equivalent with (2). It therefore follows from our remarks above that the following holds:

THEOREM 1. *The inequality* (10) *of Riesz and Schur follows from the interpolation formula* (4).

4. Comparision of two inequalities

The following inequality

$$\left|\frac{p(e^{i\theta})-p(e^{-i\theta})}{e^{i\theta}-e^{-i\theta}}\right|_{[0,2\pi]} \leqslant n \max_{0\leqslant j\leqslant n}\left|\frac{p(e^{ij\pi/n})+p(e^{-ij\pi/n})}{2}\right| \tag{13}$$

holds for all polynomials $p \in \mathscr{P}_n$. It is clearly a simple consequence of (4). Of course, it is rather similar to (10) and these two inequalities should be in some sense or another compared. We first note that sometimes (10) is stronger than (13), i.e., there are polynomials $p \in \mathscr{P}_n$ such that

$$\left|p(e^{i\theta})-p(e^{-i\theta})\right|_{[0,2\pi]} \leqslant \max_{0\leqslant j\leqslant n}\left|p(e^{ij\pi/n})+p(e^{-ij\pi/n})\right|, \tag{14}$$

it suffices clearly to consider polynomials $p(z)=A+Bz^n$ since in that case

$$|p(e^{i\theta})-p(e^{-i\theta})|_{[0,2\pi]} = 2|B|$$

and

$$\max_{0\leqslant j\leqslant n}\left|p(e^{ij\pi/n})+p(e^{-ij\pi/n})\right| = \max_{0\leqslant j\leqslant n}\left|2A+2B\cos\left(\frac{j\pi}{n}\right)\right|$$
$$\geqslant 2|A|-2|B|.$$

It follows that (14) holds whenever $|A|$ is large enough compared to $|B|$, that is $|A| > 2|B|$. To give explicit examples of polynomials $p\in\mathscr{P}_n$ such that

$$\max_{0\leqslant j\leqslant n}|p(e^{ij\pi/n})+p(e^{-ij\pi/n})| < |p(e^{i\theta})-p(e^{-i\theta})|_{[0,2\pi]} \tag{15}$$

is perhaps not that easy but the following construction seems to work. Let f be a conformal map of $\mathbb{D}$ onto the rectangle $\mathscr{R}$ in the plane with vertices $(\pm\alpha,\pm 5\alpha)$ where $\alpha>0$. Let also $Q_n(z)=\sum_{k=0}^{n+1}\left(1-\frac{k}{n+1}\right)z^k$; then, as well known [12], $\operatorname{Re}Q_n(z)>\frac{1}{2}$ if $z\in\mathbb{D}$ and Q_n admits a representation

$$Q_n(z)=\int_{\partial\mathbb{D}}\frac{1}{1-\zeta z}\mathrm{d}\mu_n(\zeta)$$

where μ_n is a probability measure. We set

$$p_n(z)=Q_n(z)*f(z)=f_{n+1}(z)-\frac{zf'_{n+1}(z)}{n+1} \tag{16}$$

where $*$ denotes the Hadamard product of two functions analytic in $\mathbb{D}$ and f_{n+1} is the $(n+1)^{th}$ section of the conformal map f. Then

$$p_n(z)=Q_n*f(z)=\int_{\partial\mathbb{D}}f(\zeta z)\mathrm{d}\mu_n(\zeta), \qquad z\in\mathbb{D},$$

and $p_n(\mathbb{D})\subset\mathscr{R}$. It follows from (16) that $p_n\to f$ uniformly on compact subsets of $\mathbb{D}$ and the polynomial $p_n\in\mathscr{P}_n$ has real Taylor coefficients because Q_n has real coefficients and f has real coefficients because $\mathscr{R}$ is symmetrical with respect to the real axis. We have

$$\max_{0\leqslant j\leqslant n}\left|\frac{p_n(e^{ij\pi/n})+p_n(e^{-ij\pi/n})}{2}\right| = \max_{0\leqslant j\leqslant n}\left|\frac{p_n(e^{ij\pi/n})+\overline{p_n(e^{ij\pi/n})}}{2}\right| \tag{17}$$
$$=\max_{0\leqslant j\leqslant n}|\operatorname{Re}p_n(e^{ij\pi/n})|$$
$$\leqslant \alpha.$$

Let also $0 < r < 1$. Since $p_n \to f$ locally uniformly on $\mathbb{D}$, we shall have for r close enough to 1 and n large,

$$\begin{aligned} 4\alpha &\leqslant \max_{|z|\leqslant r} |\operatorname{Im} p_n(z)| \\ &\leqslant \max_{|z|\leqslant 1} |\operatorname{Im} p_n(z)| \\ &= \max_{|z|=1} |\operatorname{Im} p_n(z)| \\ &= \left| \frac{p_n(e^{i\theta}) - \overline{p_n(e^{i\theta})}}{2} \right|_{[0,2\pi]} \\ &= \left| \frac{p_n(e^{i\theta}) - p_n(e^{-i\theta})}{2} \right|_{[0,2\pi]} \end{aligned} \tag{18}$$

It clearly follows from (17) and (18) that (15) holds for $p = p_n$ with n large enough and we have obtained

THEOREM 2. *The inequalities* (10) *and* (13) *are not comparable.*

Note: The above argument is perhaps a bit tricky but it has at least some geometric content and it is self-contained. A more direct reasoning is available. Notice first that it follows from the work of Nevai [10] and from the known cases of equality in (2) that the equality

$$\left| \frac{p(e^{i\theta}) - p(e^{-i\theta})}{e^{i\theta} - e^{-i\theta}} \right|_{[0,2\pi]} = n \left| \frac{p(e^{i\theta}) - p(e^{-i\theta})}{2} \right|_{[0,2\pi]}$$

can hold for some $p \in \mathscr{P}_n$ if and only if $p(z) = A + Bz^n$ for some complex numbers A, B. It is also known [4] that for $n > 1$, there exist polynomials $p \in \mathscr{P}_n$ which are not binomials and such that

$$\left| \frac{p(e^{i\theta}) - p(e^{-i\theta})}{e^{i\theta} - e^{-i\theta}} \right|_{[0,2\pi]} = n \max_{0\leqslant j\leqslant n} \left| \frac{p(e^{ij\pi/n}) + p(e^{-ij\pi/n})}{2} \right|.$$

Therefore any such p satisfies

$$\begin{aligned} \left| p(e^{i\theta}) - p(e^{-i\theta}) \right|_{[0,2\pi]} &> \frac{2}{n} \left| \frac{p(e^{i\theta}) - p(e^{-i\theta})}{e^{i\theta} - e^{-i\theta}} \right|_{[0,2\pi]} \\ &= \max_{0\leqslant j\leqslant n} |p(e^{ij\pi/n}) + p(e^{-ij\pi/n})| \end{aligned}$$

5. Conclusion

We would like to end this paper with some remarks concerning higher order analogues of (4). It has been obtained in [3] that by defining for a given $p \in \mathscr{P}_n$

$$p_0 = p \quad \text{and} \quad p_{\ell+1}(z) = zp'_\ell(z), \quad \ell \geqslant 0,$$

we have

$$\left|\frac{p_\ell(e^{i\theta})-p_\ell(e^{-i\theta})}{e^{i\theta}-e^{-i\theta}}\right|_{[0,2\pi]} \leqslant n^{1+\ell} \max_{0\leqslant j\leqslant n}\left|\frac{p(e^{ij\pi/n})+p(e^{-ij\pi/n})}{2}\right|.$$

Of course, the case $\ell=0$ is just (13) and it has been observed that the case $\ell=1$ is equivalent with the Duffin and Schaeffer inequality (6). It is indeed a consequence of (4) that for any $p\in\mathscr{P}_n$ and any real θ

$$p'(\cos\theta)=\sum_{j,k=0}^{n} c_n(j,0)c_n(k,\theta)\,\frac{p\left(\cos\left(\frac{j+k}{n}\right)\pi\right)+p\left(\cos\left(\frac{j-k}{n}\right)\pi\right)}{2}$$

and (6) follows. We even have for $\ell\geqslant 1$

$$p^{(\ell)}(\cos\theta)=\sum_{j,k=0}^{n} c_n(j,0)(-1)^{\ell-1}D_\ell(c_n(k,\theta))\frac{p\left(\cos\left(\frac{j+k}{n}\right)\pi\right)+p\left(\cos\left(\frac{j-k}{n}\right)\pi\right)}{2}$$

where the differential operator D_ℓ is defined recursively by

$$D_1(C)=\frac{1}{\sin\theta}\frac{\mathrm{d}C}{\mathrm{d}\theta},\qquad D_\ell(C)=\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}D_{\ell-1}(C)$$

for any polynomial C in $\cos\theta$. It would be of interest to decide if the full Markov inequality (3) or the analogue statement by Duffin and Schaeffer follow more or less directly from (4).

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