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SHARP ESTIMATES FOR THE ZEROS OF THE DERIVATIVE OF OSCILLATING POLYNOMIALS WITH LAGUERRE WEIGHT

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Abstract. Denote by $\mathscr{V}_n(\lambda)$ the set of all weighted polynomials of the form $f(x) = e^{-\lambda x}p(x)$ $(\lambda > 0)$, where p is an algebraic polynomial of degree n which has n simple real zeros. Given $f \in \mathscr{V}_n(\lambda)$, let $x_1 < \cdots < x_n$ and $t_1 < \cdots < t_n$ be the zeros of f and f', correspondingly. Set $h_k := x_{k+1} - x_k$, $k = 1,\ldots,n-1$. We prove sharp estimates of the forms

$$x_k + c_k h_k \leqslant t_k \leqslant x_{k+1} - d_k h_k, \quad k = 1,\ldots,n-1,$$

and

$$x_n + c_n h_{n-1} \leqslant t_n \leqslant x_n + d_n h_{n-1},$$

with explicit expressions for the coefficients, depending on λ. Known estimates of the same type for algebraic polynomials can be obtained by letting $\lambda \to 0$.

1. Introduction and statement of the results

Denote by π_n the set of all real algebraic polynomials of degree at most n. Let $\mathscr{P}_n$ be the subset of π_n which consists of the oscillating polynomials, i.e. polynomials from π_n having n simple real zeros. Various extremal problems, concerning estimation of a derivative of a function from a given class of oscillating functions were studied in the papers [1, 5, 7, 3, 4, 8, 10].

In 1918, Sz. Nagy established the following remarkable refinement of Rolle's theorem for the class $\mathscr{P}_n$ (see [11, Corollary 6.5.6]).

Theorem A. *Let $f \in \mathscr{P}_n$ has zeros $x_1 < \cdots < x_n$ and let $t_1 < \cdots < t_{n-1}$ be the zeros of f'. Then we have*

$$x_k + \frac{x_{k+1} - x_k}{n-k+1} \leqslant t_k \leqslant x_{k+1} - \frac{x_{k+1} - x_k}{k+1}, \quad k = 1,\ldots,n-1. \tag{1}$$

Another important property of the class $\mathscr{P}_n$ is given by the well known Lemma of V. Markov ([12, Lemma 2.7.1]):

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Theorem B. *Suppose that the polynomials p and q from $\mathscr{P}_n$ have zeros $x_1 < \cdots < x_n$ and $y_1 < \cdots < y_n$, respectively, which satisfy the interlacing conditions*

$$x_1 \leqslant y_1 \leqslant \cdots \leqslant x_n \leqslant y_n.$$

Then the zeros $t_1 < \cdots < t_{n-1}$ of $p'(x)$ and the zeros $\tau_1 < \cdots < \tau_{n-1}$ of $q'(x)$ interlace too, that is

$$t_1 \leqslant \tau_1 \leqslant \cdots \leqslant t_{n-1} \leqslant \tau_{n-1}. \tag{2}$$

Moreover, the inequalities (2) are strict, unless $x_i = y_i$, $i = 1, \ldots, n$.

In [9] we extended Theorem B for some Chebyshev systems on infinite intervals, including exponential polynomials, Müntz polynomials and polynomials with Laguerre weight.

A natural problem is to prove results of the type of Theorem A for other systems of functions. Note that the proof of Theorem A relies on some specific properties of algebraic polynomials and cannot be modified for systems different from $\mathscr{P}_n$.

On the other hand, Markov's interlacing property is equivalent to the fact that each zero of the derivative of a $p \in \mathscr{P}_n$ is a strictly increasing function of each zero of p, see [2]. The last observation can be used to give another proof of Theorem A.

In the present paper we shall apply this approach to prove explicit estimates for the critical points of oscillating polynomials with Laguerre weight. Let us denote

$$\mathscr{V}_n(\lambda) := \left\{ e^{-\lambda x} p(x) : \ p \in \mathscr{P}_n \right\}, \ \lambda \neq 0.$$

Our main result is the following generalization of Theorem A.

THEOREM 1. *Let $f \in \mathscr{V}_n(\lambda)$, $\lambda > 0$ has zeros $x_1 < \cdots < x_n$ and let $t_1 < \cdots < t_n$ be the zeros of f'. Then the following estimates hold true:*

$$x_k + c_k h_k \leqslant t_k \leqslant x_{k+1} - d_k h_k, \quad k = 1, \ldots, n-1, \tag{3}$$

where

$$\begin{aligned} h_k &= x_{k+1} - x_k, \\ c_k &= \frac{2}{n-k+1+\lambda h_k + \sqrt{h_k^2 \lambda^2 + 2\lambda(n-k-1)h_k + (n-k+1)^2}}, \\ d_k &= \frac{2}{\sqrt{(k+1-\lambda h_k)^2 + 4\lambda h_k} + k + 1 - \lambda h_k}, \end{aligned}$$

and

$$x_n + c_n h_{n-1} \leqslant t_n \leqslant x_n + d_n h_{n-1}, \tag{4}$$

where

$$c_n = \frac{2}{\sqrt{h_{n-1}^2\lambda^2+4}+\lambda h_{n-1}-2},$$

$$d_n = \frac{2}{\sqrt{(n-\lambda h_{n-1})^2+4\lambda h_{n-1}}+\lambda h_{n-1}-n}.$$

In addition, the inequalities (3) *and* (4) *are sharp.*

REMARK 1. The estimates (1) follow from (3) by letting $\lambda \to 0$.

It is of interest to have simpler, rational estimates for the critical points of a polynomial from $\mathscr{V}_n(\lambda)$. We give such estimates in the following

COROLLARY 1. *Let $f \in \mathscr{V}_n(\lambda)$, $\lambda > 0$. In the notations of Theorem 1 we have:*

$$x_k + c_k' h_k \leqslant t_k \leqslant x_{k+1} - d_k' h_k, \quad k = 1,\dots,n-1, \tag{5}$$

where

$$c_k' = \frac{1}{\lambda h_k + n - k + 1}, \qquad d_k' = \frac{\lambda h_k + k + 1}{\lambda h_k + (k+1)^2},$$

and

$$x_n + \frac{1}{\lambda} \leqslant t_n \leqslant x_n + \frac{n}{\lambda}. \tag{6}$$

COROLLARY 2. *Let $D_\lambda[p] = p' - \lambda p$, $\lambda > 0$. If $p \in \pi_n$ has zeros $x_1 < \cdots < x_n$, then the zeros $t_1 < \cdots < t_n$ of $D_\lambda[p]$ satisfy the estimates* (3) *and* (4) *from Theorem* 1.

2. Proofs of the results

Let us set $X = \{(x_1,\dots,x_n) : \ x_1 < \cdots < x_n\}$. The following lemma is a particular case of [9, Lemma 2]. For reader's convenience we shall give here a direct proof.

LEMMA 1. *Let $f \in \mathscr{V}_n(\lambda)$, $\lambda > 0$ has zeros $\overline{x} = (x_1,\dots,x_n) \in X$. Denote by $t_i(\overline{x}) \in (x_i, x_{i+1})$, $i = 1,\dots,n$ $(x_{n+1} := +\infty)$ the zeros of f'. Then for all $i \in \{1,\dots,n\}$, $t_i(\overline{x})$ is a continuously differentiable function on X, which is strictly increasing with respect to x_j, $j = 1,\dots,n$.*

Proof. First we shall show that the functions $t_i(\overline{x})$, $i = 1,\dots,n$ are differentiable for $\overline{x} \in X$. Let us fix the index i. We consider the function

$$F(\overline{x};t) := f'(t) = e^{-\lambda t}[\omega'(\overline{x};t) - \lambda\,\omega(\overline{x};t)],$$

where $\omega(\overline{x};t) := (t-x_0)\cdots(t-x_n)$.

We fix a point $\overline{x}^0 = (x_1^0, \dots, x_n^0) \in X$ and let $t^0 := t_i(\overline{x}^0)$. Clearly, F is a continuously differentiable function in a neighborhood of $(\overline{x}^0; t^0)$. Also,

$$\frac{\partial F}{\partial t}(\overline{x}^0; t^0) = f''(t^0) = -\lambda e^{-\lambda t^0} p(t^0) + e^{-\lambda t^0} p'(t^0),$$

where $p(t) := \omega'(\overline{x}; t) - \lambda \omega(\overline{x}; t) \in \mathscr{P}_n$ has zeros $t_1(\overline{x}) < \cdots < t_n(\overline{x})$. Since $p(t^0) = 0$ and $p'(t^0) \neq 0$ we obtain $\frac{\partial F}{\partial t}(\overline{x}^0; t^0) \neq 0$.

By the implicit function theorem, there exists a neighborhood U of $\overline{x}^0$ such that the function $t_i(\overline{x})$ is continuously differentiable in U.

Next we shall compute $\frac{\partial t_i(\overline{x})}{\partial x_j}$ for $\overline{x} \in X$. If $f(t) = ce^{-\lambda t}\omega(\overline{x}; t) \in \mathscr{V}_n(\lambda)$, we have

$$\frac{f'(t)}{f(t)} = -\lambda + \frac{\omega'(\overline{x}; t)}{\omega(\overline{x}; t)}.$$

Using $f'(t_i(\overline{x})) = 0$ we get

$$-\lambda + \sum_{k=1}^{n} \frac{1}{t_i(\overline{x}) - x_k} = 0.$$

Differentiating the last identity with respect to x_j we obtain

$$\frac{\partial t_i(\overline{x})}{\partial x_j} \sum_{k=1}^{n} \frac{1}{(t_i(\overline{x}) - x_k)^2} = \frac{1}{(t_i(\overline{x}) - x_j)^2},$$

which implies $\frac{\partial t_i(\overline{x})}{\partial x_j} > 0$. Lemma 1 is proved. $\square$

Our next goal is to extend Lemma 1 to the case of multiple zeros. To this end, we need the continuity of the zeros of the derivative with respect to the zeros of the weighted polynomial, having only real zeros. We set $\overline{X} = \{(x_1, \dots, x_n) : x_1 \leqslant \cdots \leqslant x_n\}$.

LEMMA 2. *Given $\overline{x} \in \overline{X}$ and $\lambda > 0$, let $f(\overline{x}; t) = e^{-\lambda t}(t - x_1) \cdots (t - x_n)$ and $t_1(\overline{x}) \leqslant \cdots \leqslant t_n(\overline{x})$ be the zeros of $f'(\overline{x}; \cdot)$. Then for every $i = 1, \dots, n$, $t_i(\overline{x})$ is a continuous function in $\overline{X}$.*

Proof. It follows from $f'(\overline{x}; t) = e^{-\lambda t}[\omega'(\overline{x}; t) - \lambda \omega(\overline{x}; t)]$ (see the proof of Lemma 1) that $t_i(\overline{x})$, $i = 1, \dots, n$ are the zeros of $p(\overline{x}; t) := \omega'(\overline{x}; t) - \lambda \omega(\overline{x}; t) \in \pi_n$. By the formulas of Viet, the coefficients of p are continuous functions of $\overline{x}$. In addition, the leading coefficient of p is equal to $-\lambda$ and does not depend on $\overline{x}$. Now the assertion follows from a well known result for algebraic polynomials, see e.g. [6, Theorem (1,4)]. Lemma 2 is proved. $\square$

LEMMA 3. *Let f and g be two polynomials from $\mathscr{V}_n(\lambda)$, $\lambda > 0$, with zeros $\overline{x}$ and $\overline{y}$, respectively, which satisfy the conditions: $x_1 \leqslant \cdots \leqslant x_n$, $y_1 \leqslant \cdots \leqslant y_n$, and $x_i \leqslant y_i$, for $i = 1, \dots, n$. Let $t_1(\overline{x}) \leqslant \cdots \leqslant t_n(\overline{x})$ and $t_1(\overline{y}) \leqslant \cdots \leqslant t_n(\overline{y})$ be the zeros of f' and g'. Then we have $t_i(\overline{x}) \leqslant t_i(\overline{y})$, for $i = 1, \dots, n$.*

Proof. We define the vectors $\overline{x}^{\varepsilon} := (x_1 - n\varepsilon, \dots, x_n - \varepsilon)$ and $\overline{y}^{\varepsilon} := (y_1 + \varepsilon, \dots, y_n + n\varepsilon)$, where ε is a positive number. Then $x_1^{\varepsilon} < \dots < x_n^{\varepsilon}$, $y_1^{\varepsilon} < \dots < y_n^{\varepsilon}$, and $x_i^{\varepsilon} < y_i^{\varepsilon}$, for $i = 1, \dots, n$.

Let $\overline{z}^{\varepsilon}(s) := (1-s)\overline{x}^{\varepsilon} + s\overline{y}^{\varepsilon}$, $s \in [0,1]$. By Lemma 1, $t_i(\overline{z}^{\varepsilon}(s))$, $i = 1, \dots, n$ are strictly increasing functions of s. This implies

$$t_i(\overline{x}^{\varepsilon}) < t_i(\overline{y}^{\varepsilon}), \text{ for } i = 1, \dots, n. \tag{7}$$

The proof is completed by letting ε to 0 in (7) and using Lemma 2. □

Proof of Theorem 1.

If $n = 2$ the zeros $t_1 < t_2$ of f' can be computed in explicit form and it can be checked that (3) and (4) are satisfied as equalities. Thus, we can suppose that $n \geqslant 3$.

We begin with the proof of the upper bound in (3). Let us consider first the general case $k \in \{2, \dots, n-3\}$ for $n \geqslant 5$. Without loss of generality we can assume that $f(x) = e^{-\lambda x}(x - x_1) \cdots (x - x_n)$. We define the auxiliary polynomial

$$g_k(\overline{y}; x) = e^{-\lambda x}(x - y_1) \cdots (x - y_n),$$

where $y_1 < \dots < y_n$ satisfy the conditions:

$$\begin{aligned} y_i &\nearrow x_k, \;\; i = 1, \dots, k-1, \;\; y_i \in [x_i, x_k), \\ y_i &= x_i, \;\; i = k, k+1, \\ y_i &\nearrow x_n, \;\; i = k+2, \dots, n-1, \;\; y_i \in [x_i, x_n), \\ y_n &= x_n. \end{aligned} \tag{8}$$

(As usual, notation $x \nearrow c$ means that x is strictly increasing and tends to c.)

We denote the zeros of $g_k'(\overline{y}; x)$ by $\tau_{1,k}(\overline{y}) < \dots < \tau_{n,k}(\overline{y})$. According to Lemma 1 $\tau_{i,k}(\overline{y})$, $i = 1, \dots, n$ are strictly increasing when $\overline{y} \to \overline{z} := ((x_k, k), x_{k+1}, (x_n, n-k-1))$ as in (8). Let $\overline{t}_{i,k}$, $i = 1, \dots, n$ be the zeros of the derivative of

$$\overline{g}_k(x) := g_k(\overline{z}; x) = e^{-\lambda x}(x - x_k)^k (x - x_{k+1})(x - x_n)^{n-k-1}.$$

By Lemma 2, $\tau_{i,k}(\overline{y}) \to \overline{t}_{i,k}$, $i = 1, \dots, n$. In particular, it follows that

$$t_k = \tau_{k,k}(\overline{x}) < \overline{t}_{k,k}. \tag{9}$$

Furthermore, we introduce the polynomials

$$\overline{g}_k(b; x) := e^{-\lambda x}(x - x_k)^k (x - x_{k+1})(x - b)^{n-k-1}, \text{ for } b \geqslant x_n.$$

Clearly, $\overline{g}_k(x_n; x) = \overline{g}_k(x)$. Lemma 3 implies that the zeros $\overline{t}_{1,k}(b) \leqslant \dots \leqslant \overline{t}_{n,k}(b)$ of $\overline{g}_k'(b; x)$ are increasing as $b \nearrow +\infty$. By Rolle's theorem, $\overline{t}_{k,k}(b) \in (x_k, x_{k+1})$ hence there exists $l_k := \lim_{b \to +\infty} \overline{t}_{k,k}(b)$. Consequently,

$$\overline{t}_{k,k} = \overline{t}_{k,k}(x_n) \leqslant l_k. \tag{10}$$

It follows from (9) and (10) that

$$t_k < l_k. \tag{11}$$

Our next goal is to find l_k. We have

$$\overline{g}_k'(b;x) = \overline{g}_k(b;x)\overline{h}_k(b;x),$$

where

$$\overline{h}_k(b;x) := -\lambda + \frac{k}{x-x_k} + \frac{1}{x-x_{k+1}} + \frac{n-k-1}{x-b}.$$

The definition of $\overline{g}_k(b;x)$ and the theorem of Rolle imply that $\overline{h}_k(b;x)$ has exactly three real zeros: $\overline{t}_{k,k}(b) \in (x_k, x_{k+1})$, $\overline{t}_{k+1,k}(b) \in (x_{k+1}, b)$, and $\overline{t}_{n,k}(b) \in (b, +\infty)$.

Letting $b \to +\infty$ in the equality

$$\overline{h}_k(b;\overline{t}_{k,k}(b)) = 0, \tag{12}$$

we get

$$-\lambda + \frac{k}{l_k - x_k} + \frac{1}{l_k - x_{k+1}} = 0, \tag{13}$$

where we have used that l_k is different from x_k and x_{k+1}. Indeed, if $l_k = x_k$ then $\frac{k}{\overline{t}_{k,k}(b)-x_k}$ would tend to $+\infty$, which contradicts (12). The proof of $l_k \neq x_{k+1}$ is similar. In fact, the location of $\overline{t}_{k,k}(b)$ gives

$$x_k < l_k < x_{k+1}. \tag{14}$$

Now, (13) is equivalent to $p(l_k) = 0$, where

$$p(x) = -\lambda(x-x_k)(x-x_{k+1}) + k(x-x_{k+1}) + x - x_k.$$

Since the leading coefficient of p is negative, $p(x_k) < 0$, and $p(x_{k+1}) > 0$, we conclude by (14) that l_k is equal to the smaller root of the equation $p(x) = 0$, i.e.

$$l_k = \frac{k+1+\lambda(x_k+x_{k+1}) - \sqrt{D}}{2\lambda},$$

where $D = [k+1+\lambda(x_k+x_{k+1})]^2 - 4\lambda(x_k + kx_{k+1} + \lambda x_k x_{k+1})$. It can be verified that $l_k = x_{k+1} - d_k h_k$, which in view of (10) completes the proof of the upper bound in (3) for $k \in \{2, \ldots, n-3\}$.

Let us consider the case $k = 1$. We keep the introduced notations. Then the first row in (8) is missing. If $n \geqslant 4$ then at least y_3 is strictly increasing from x_3 to x_n, which ensures the validity of (9). The remaining part of the proof needs no changes. If $n = 3$ conditions (8) reduce to $y_i = x_i$ for $i = 1,2,3$, hence $t_1 = \overline{t}_{1,1}$. Now (10) holds as an strict inequality since Lemma 1 can be applied instead of Lemma 3 and the proof can be completed as in the general case.

The case $k=n-2$ is similar to that for $k=1$, now we have

$$\begin{aligned} y_i &\nearrow x_{n-2}, \quad i=1,\dots,n-3, \quad y_i \in [x_i, x_{n-2}), \\ y_i &= x_i, \quad i=n-2,\dots,n. \end{aligned}$$

Finally, let $k=n-1$. The conditions (8) have to be replaced with

$$\begin{aligned} y_i &\nearrow x_{n-1}, \quad i=1,\dots,n-2, \quad y_i \in [x_i, x_{n-1}), \\ y_i &= x_i, \quad i=n-1,n. \end{aligned} \tag{15}$$

Using Lemmas 1 and 2 we get $t_{n-1} < \bar{t}_{n-1,n-1}$, where $\bar{t}_{n-1,n-1}$ is the $(n-1)$-st zero of the derivative of

$$\overline{g}_{n-1}(x) = e^{-\lambda x}(x-x_{n-1})^{n-1}(x-x_n). \tag{16}$$

It is seen that $\bar{t}_{n-1,n-1}$ is the smaller root of the quadratic equation

$$-\lambda(x-x_{n-1})(x-x_n) + (n-1)(x-x_n) + x - x_{n-1} = 0. \tag{17}$$

Then $\bar{t}_{n-1,n-1}$ can be found explicitly, which gives the desired result.

Now we shall prove the lower bound in (3). First we suppose that $k \in \{3,\dots,n-2\}$ for $n \geqslant 5$. We consider the polynomial $g_k(\overline{y};x) = e^{-\lambda x}(x-y_1)\cdots(x-y_n)$, where $\overline{y} \in X$ satisfy the conditions:

$$\begin{aligned} y_1 &= x_1, \\ y_i &\searrow x_1, \quad i=2,\dots,k-1, \quad y_i \in (x_1, x_i], \\ y_i &= x_i, \quad i=k,k+1, \\ y_i &\searrow x_{k+1}, \quad i=k+2,\dots,n, \quad y_i \in (x_{k+1}, x_i]. \end{aligned} \tag{18}$$

It follows from Lemma 1 that the zeros $\tau_{1,k}(\overline{y}) < \cdots < \tau_{n,k}(\overline{y})$ of $g_k'(\overline{y};x)$ are strictly decreasing when $\overline{y} \to \overline{z} := ((x_1,k-1), x_k, (x_{k+1}, n-k))$ as in (18). By Lemma 2, $\tau_{i,k}(\overline{y}) \to \underline{t}_{i,k}$, $i=1,\dots,n$, where $\{\underline{t}_{i,k}\}_1^n$ are the zeros of the derivative of

$$\underline{g}_k(x) := g_k(\overline{z};x) = e^{-\lambda x}(x-x_1)^{k-1}(x-x_k)(x-x_{k+1})^{n-k}.$$

Since $\tau_{k,k}(\overline{y})$ strictly decreases from t_k to $\underline{t}_{k,k}$, we conclude that

$$\underline{t}_{k,k} < \tau_{k,k}(\overline{x}) = t_k. \tag{19}$$

For $a \leqslant x_1$ we define the polynomials

$$\underline{g}_k(a;x) := e^{-\lambda x}(x-a)^{k-1}(x-x_k)(x-x_{k+1})^{n-k},$$

and let $\underline{t}_{1,k}(a) \leqslant \cdots \leqslant \underline{t}_{n,k}(a)$ be the zeros of $\underline{g}_k'(a;x)$. By Lemma 3 each of $\{\underline{t}_{i,k}(a)\}_1^n$ decreases as $a \searrow -\infty$. The theorem of Rolle implies $\underline{t}_{k,k}(a) \in (x_k, x_{k+1})$ and there exists $l_k := \lim_{a\to-\infty} \underline{t}_{k,k}(a)$. Therefore,

$$l_k \leqslant \underline{t}_{k,k}(x_1) = \underline{t}_{k,k}, \tag{20}$$

which gives

$$l_k < t_k. \tag{21}$$

We have

$$\underline{g}_k'(a;x) = \underline{g}_k(a;x)\left[-\lambda + \frac{k-1}{x-a} + \frac{1}{x-x_k} + \frac{n-k}{x-x_{k+1}}\right] =: \underline{g}_k(a;x)\underline{h}_k(a;x). \tag{22}$$

It is seen that $\underline{h}_k(a;x)$ has exactly three real zeros: $\underline{t}_{k-1,k}(a) \in (a, x_k)$, $\underline{t}_{k,k}(a) \in (x_k, x_{k+1})$, and $\underline{t}_{n,k}(a) \in (x_{k+1}, +\infty)$. As in the proof of the upper bound, we obtain that $x_k < l_k < x_{k+1}$ and l_k is a solution of the equation $\underline{h}_k(-\infty;x) = 0$, which is equivalent to

$$-\lambda(x-x_k)(x-x_{k+1}) + x - x_{k+1} + (n-k)(x-x_k) = 0.$$

In fact, l_k is the smaller root of the above equation, i.e.

$$l_k = \frac{n+1-k+\lambda(x_k+x_{k+1}) - \sqrt{D}}{2\lambda},$$

where $D := [n+1-k+\lambda(x_k+x_{k+1})]^2 - 4\lambda[(n-k)x_k + x_{k+1} + \lambda x_k x_{k+1}]$. It can be shown, that the last expression is equal to $x_k + c_k h_k$, which completes the proof of the lower bound in (3) for $3 \leqslant k \leqslant n-2$.

Next we consider the case $k = 1$. Then the conditions (18) are replaced with

$$\begin{aligned} y_i &= x_i, \ \ i = 1,2, \\ y_i &\searrow x_2, \ \ i = 3,\dots,n, \ \ y_i \in (x_2, x_i]. \end{aligned}$$

Since at least y_3 strictly decreases from x_3 to x_2, by Lemmas 1 and 2 we get $\underline{t}_{1,1} < t_1$. Now, $\underline{t}_{1,1}$ is the smallest zero of the derivative of $\underline{g}_1(x) = e^{-\lambda x}(x-x_1)(x-x_2)^{n-1}$ which can be computed explicitly and is equal to $x_1 + c_1 h_1$.

Suppose now that $k = 2$. The second row in (18) is missing. This leads to $\underline{t}_{2,2} < t_2$, provided $n \geqslant 4$. If $n = 3$ then $y_1 = x_1$, $y_2 = x_2$, and $y_3 = x_3$ hence $\underline{t}_{2,2} = t_2$. Then, studying the limit behavior of $\underline{g}_2'(a;x)$ as $a \to -\infty$, gives $l_2 \leqslant \underline{t}_{2,2}$. Note that if $n = 3$ the last inequality is strict due to the applicability of Lemma 1. As a consequence, $l_2 < t_2$ and l_2 is found as in the general case.

The case $k = n-1$ is similar to the previous one, the conditions (18) are substituted with

$$\begin{aligned} y_1 &= x_1, \\ y_i &\searrow x_1, \ \ i = 2,\dots,n-2, \ \ y_i \in (x_1, x_i], \\ y_i &= x_i, \ \ i = n-1, n. \end{aligned} \tag{23}$$

The proof of (3) is completed.

Proof of (4). Recall that we can assume $n \geqslant 3$. We consider the polynomial $g_{n-1}(\overline{y};x) = e^{-\lambda x}(x-y_1)\cdots(x-y_n)$, where $\overline{y} \in X$ satisfies (15). Then the reasonings used in the proof of the upper bound in (3) show that $\tau_{n,n-1}(\overline{y}) \nearrow \overline{t}_{n,n-1}$, which is the largest zero of the $\overline{g}'_{n-1}$, where $\overline{g}_{n-1}$ is given by (16). Consequently $\overline{t}_{n,n-1}$ is the largest root of the equation (17), which leads to the upper estimate in (4).

Next we shall prove the lower estimate in (4). Now we take the polynomials $g_{n-1}(\overline{y};x)$ with $\overline{y}$ as in (23). Let $n \geqslant 4$. Then $\tau_{n,n-1}(\overline{y}) \searrow \underline{t}_{n,n-1}$, which is the largest zero of the derivative of $\underline{g}_{n-1}(x) = e^{-\lambda x}(x-x_1)^{n-2}(x-x_{n-1})(x-x_n)$. This implies $\underline{t}_{n,n-1} < t_n$. Next we introduce the polynomials

$$\underline{g}_{n-1}(a;x) := e^{-\lambda x}(x-a)^{n-2}(x-x_{n-1})(x-x_n), \tag{24}$$

for $a \leqslant x_1$ and let $\underline{t}_{1,n-1}(a) \leqslant \cdots \leqslant \underline{t}_{n,n-1}(a)$ be the zeros of $\underline{g}'_{n-1}(a;x)$. The largest zero $\underline{t}_{n,n-1}(a) \in (x_n, +\infty)$ is decreasing as $a \searrow -\infty$, hence there exists the limit $l_n := \lim_{a\to-\infty} \underline{t}_{n,n-1}(a)$ and $l_n \leqslant \underline{t}_{n,n-1}(x_1) = \underline{t}_{n,n-1}$. Thus $l_n < t_n$. The same conclusion holds true also for $n = 3$, since by Lemma 1 we have $l_3 < \underline{t}_{3,2} = t_3$.

It remains to find l_n. As in the proof of the lower bound in (3) for $k = n-1$ (see (22)), l_n is a solution of the equation

$$-\lambda(x-x_{n-1})(x-x_n) + 2x - x_{n-1} - x_n = 0.$$

Since $l_n > x_n$, it is the largest root of the above equation. It can be verified that $l_n = x_n + c_n h_{n-1}$, which completes the proof of (4).

It remains to explain the sharpness of the estimates (3) and (4). If $n = 2$ both (3) and (4) are fulfilled as equalities. Let $n \geqslant 3$. It follows from the the proof of (3) that the upper bound in (3) is attained asymptotically for the polynomials $\overline{g}_k(b;x) = e^{-\lambda x}(x - x_k)^k(x-x_{k+1})(x-b)^{n-k-1}$, as $b \to +\infty$ and it is equal to $l_k = x_{k+1} - d_k h_k$. Note also that $\overline{g}_k(b;\cdot)$ can be approximated arbitrarily closely by polynomials from $\mathscr{V}_n(\lambda)$. Similarly, the polynomials $\overline{g}_k(a;\cdot)$ $(a \to -\infty)$ can be used to prove the sharpness of the lower bound in (3).

Furthermore, the upper bound in (4) is attained for $\overline{g}_{n-1}$ given by (16). The polynomials (24) provide an example that the lower bound in (4) cannot be improved.

The proof of Theorem 1 is completed. $\square$

Proof of Corollary 1. In order to prove (5) it is sufficient to show that $c'_k \leqslant c_k$ and $d'_k \leqslant d_k$, for $k = 1, \ldots, n-1$. Let us set $t := \lambda h_k$. Then the inequality $c'_k \leqslant c_k$ is equivalent to

$$\sqrt{t^2 + 2(n-k-1)t + (n-k+1)^2} \leqslant t + n - k + 1. \tag{25}$$

Squaring both sides of (25) we obtain $4t \geqslant 0$, which is true since $t > 0$.

Similarly, $d'_k \leqslant d_k$ is equivalent to

$$(t+k+1)\sqrt{(k+1-t)^2 + 4t} \leqslant t^2 + 2t + (k+1)^2,$$

which is reduced to the obvious inequality $4k^2t^2 \geqslant 0$.

Next we shall prove (6). By (4), the right inequality in (6) would follow from $d_n h_{n-1} \leqslant \frac{n}{\lambda}$, which is equivalent to

$$\frac{2t}{\sqrt{(n-t)^2+4t}+t-n} \leqslant n,$$

where $t := \lambda h_{n-1} > 0$. The denominator is positive hence the above inequality is equivalent to

$$2t + n(n-t) \leqslant n\sqrt{(n-t)^2+4t}. \tag{26}$$

This is satisfied if $2t + n(n-t) \leqslant 0$. Otherwise, squaring both sides of (26) we get $4(n-1)t^2 \geqslant 0$, which is true.

For the left inequality in (6) it is sufficient to prove that $\frac{1}{\lambda} \leqslant c_n h_{n-1}$. Replacing the explicit value of c_n and noticing that $\sqrt{h_{n-1}^2\lambda^2+4} + \lambda h_{n-1} - 2 > 0$ for $\lambda > 0$, we get the equivalent inequality $2 + \lambda h_{n-1} \geqslant \sqrt{h_{n-1}^2\lambda^2+4}$, which is fulfilled for every $\lambda > 0$. Corollary 1 is proved. □

Proof of Corollary 2. Let us consider the weighted polynomial $f(x) := e^{-\lambda x}p(x)$, which belongs to $\mathscr{V}_n(\lambda)$. We have $D_\lambda[p](x) = e^{\lambda x} f'(x)$ hence the zeros of $D_\lambda[p]$ and f' coincide. Now Corollary 2 is obtained by applying Theorem 1 to f. □

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