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FRACTIONAL DIFFERENTIAL OPERATORS IN VECTOR–VALUED SPACES AND APPLICATIONS

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Abstract. Fractional differential operator equations with parameter are studied. Uniform L_p-separability properties and sharp resolvent estimates are obtained for elliptic equations in terms of fractional derivatives. Moreover, maximal regularity properties of the fractıonal abstract parabolic equation are established. Particularly, it is proven that the operators generated by these equations are positive and also are generators of analytic semigroups. As an application, the anisotropic parameter dependent fractional differential equations and the system of fractional differential equations are studied.

1. Introduction, notations and background

In the last years, the maximal regularity properties of boundary value problems (BVPs) for differential-operator equations (DOEs) have found many applications in PDE and pseudo DE with applications in physics (see [1, 5, 8-15] and the references therein). Pseudo-differential equations (PsDE) were treated e.g. in [17-18]. DOEs have found many applications in fractıonal differential equations (FDEs), pseudo-differential equations (PsDE) and PDEs (see e.g. $[1-3],[5],[11],[12],[16-19],[24]$). The regularity properties of PsDE have been studied extensively by many researchers (see e.g. [6, 10], [21-22] and the references therein). The boundedness of PsDEs in Sobolev spaces have been treated e.g. in $[10],[14],[22]$. Moreover, the smoothness of PsDE with bounded operator coefficients have been explored e.g. in $[8],[15]$. In contrast to $[8],[15]$ the FDE considered here, contain unbounded operators and parameters. In particular, the main objective of the present paper is to discuss the uniform $L_p(R^n;E)$-maximal regularity of the elliptic fractional differential operator equation (FDOE) with parameters

$$P_\varepsilon(D)u+Au+\sum_{|\alpha|<m}\varepsilon(\alpha)A_\alpha(x)D^\alpha u+\lambda u=f(x),\ x\in\mathbb{R}^n, \tag{1.1}$$

where $P_\varepsilon(D)$ is a fractıonal differential operator, A, $A_\alpha(x)$ are linear operators in a Banach space E for $\alpha_i\in[0,\infty)$ and $\alpha=(\alpha_1,\alpha_2,...,\alpha_n)$. Here, $D^\alpha=D_1^{\alpha_1}D_2^{\alpha_2},.,D_n^{\alpha_n}$

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are the Liouville derivatives, m is a positive number, ε_k are positive, λ is a complex parameter, $\varepsilon = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_n)$ and $\varepsilon(\alpha) = \prod_{k=1}^{n} \varepsilon_k^{\frac{\alpha_k}{m}}$.

Here, $L_p(\Omega; E)$ denotes the space of strongly measurable E-valued functions that are defined on the measurable subset $\Omega \subset R^n$ with the norm given by

$$\|f\|_{L_p(\Omega;E)} = \left(\int_\Omega \|f(x)\|_E^p \, dx \right)^{\frac{1}{p}}, \ 1 \leqslant p < \infty,$$

$$\|f\|_{L^\infty} = \operatorname*{ess\,sup}_{x \in \Omega} \|f(x)\|_E .$$

We prove that problem (1.1) has a maximal regular unique solution and the following uniform coercive estimate holds

$$\sum_{|\alpha| \leqslant m} \varepsilon(\alpha) |\lambda|^{1 - \frac{|\alpha|}{m}} \|D^\alpha u\|_{L_p(\mathbb{R}^n;E)} + \|Au\|_{L_p(\mathbb{R}^n;E)} \leqslant C \|f\|_{L_p(\mathbb{R}^n;E)} \tag{1.2}$$

for $f \in L_p(\mathbb{R}^n; E)$, $\lambda \in S_\varphi$, where S_φ is a set of complex numbers that is related with the spectrum of the operator A. The estimate (1.2) implies that the operator O_ε generated by (1.1) has a bounded inverse from $L_p(\mathbb{R}^n; E)$ into the space $H_p^m(\mathbb{R}^n; E(A), E)$, which will be defined subsequently. Particularly, from the estimate (1.2), we obtain that the operator O_ε is uniformly positive in $L_p(\mathbb{R}^n; E)$. By using this property we prove the uniform well posedness of the Cauchy problem for the parabolic FDOE with parameter

$$\frac{\partial u}{\partial t} + P_\varepsilon(D) u + Au = f(t,x), \ u(0,x) = 0, \tag{1.3}$$

in E-valued mixed spaces $L_{\mathbf{p}}$ for $\mathbf{p} = (p, p_1)$. In other words, we show that problem (1.3) has a unique solution $u \in W_{\mathbf{p}}^{1,m}\left(\mathbb{R}_+^{n+1}; E(A), E\right)$ for $f \in L_{\mathbf{p}}\left(\mathbb{R}_+^{n+1}; E\right)$ satisfying the following uniform coercive estimate

$$\left\| \frac{\partial u}{\partial t} \right\|_{L_{\mathbf{p}}\left(\mathbb{R}_+^{n+1};E\right)} + \|P_\varepsilon(D) u\|_{L_{\mathbf{p}}\left(\mathbb{R}_+^{n+1};E\right)} + \|Au\|_{L_{\mathbf{p}}\left(\mathbb{R}_+^{n+1};E\right)}$$

$$\leqslant M \|f\|_{L_{\mathbf{p}}\left(\mathbb{R}_+^{n+1};E\right)}. \tag{1.4}$$

Note that, the constants C, M in (1.2) and (1.4) are independent of parameters. As an application, in this paper the following are established: (a) maximal regularity properties of the anisotropic elliptic FDOE in mixed $L_{\mathbf{p}}$, $\mathbf{p} = (p_1, p)$ spaces; (b) coercive properties of the system of FDOEs of infinite many order in L_p spaces.

The Banach space E is called an UMD-space if the Hilbert operator

$$(Hf)(x) = \lim_{\varepsilon \to 0} \int_{|x-y| > \varepsilon} \frac{f(y)}{x-y} dy$$

is bounded in $L^p(\mathbb{R};E)$, $p\in(1,\infty)$ (see. e.g [4]). UMD spaces include e.g. L^p, l_p and Lorentz spaces L_{pq} for p, $q\in(1,\infty)$.

Let $\mathbb{C}$ denote the set of complex numbers and

$$S_\varphi=\{\lambda;\ \lambda\in\mathbb{C},\ |\arg\lambda|\leqslant\varphi\}\cup\{0\},\ 0\leqslant\varphi<\pi.$$

A linear operator A is said to be φ-positive (or positive) in a Banach space E if $D(A)$ is dense on E and

$$\left\|(A+\lambda I)^{-1}\right\|_{B(E)}\leqslant M(1+|\lambda|)^{-1}$$

for any $\lambda\in S_\varphi$, where $\varphi\in[0\,,\pi)$, I is the identity operator in E, $B(E)$ is the space of bounded linear operators in E. Sometimes $A+\lambda I$ will be written $A+\lambda$ and will be denoted by A_λ. It is known [20, §1.15.1] that the powers A^θ, $\theta\in(-\infty,\infty)$ for a positive operator A exist.The operator $A(h)$, $h\in Q\subset\mathbb{C}$ is said to be φ-positive (or positive) in E uniformly with respect to $h\in Q$ if $D(A(h))$ is independent of h, $D(A(h))$ is dense in E and $\left\|(A(h)+\lambda)^{-1}\right\|\leqslant M(1+|\lambda|)^{-1}$ for all $\lambda\in S_\varphi$, $0\leqslant\varphi<\pi$, where M does not depend on h and λ. Let $E\left(A^\theta\right)$ denote the space $D\left(A^\theta\right)$ with the norm

$$\|u\|_{E\left(A^\theta\right)}=\left(\|u\|^p+\left\|A^\theta u\right\|^p\right)^{\frac{1}{p}},\ 1\leqslant p<\infty,\ 0<\theta<\infty.$$

A set $W\subset B(E_1,E_2)$ is called R-bounded (see e.g. [23]) if there is a constant $C>0$ such that for all $T_1,T_2,...,T_m\in W$ and $u_1,u_2,...,u_m\in E_1$, $m\in\mathbb{N}$,

$$\int_0^1\left\|\sum_{j=1}^m r_j(y)\,T_ju_j\right\|_{E_2}dy\leqslant C\int_0^1\left\|\sum_{j=1}^m r_j(y)\,u_j\right\|_{E_1}dy,$$

where $\{r_j\}$ is an arbitrary sequence of independent symmetric $\{-1,1\}$-valued random variables on $[0,1]$. The smallest C for which the above estimate holds is called an R-bound of the collection W and is denoted by $R(W)$. A set of operators $G_h\subset B(E_1,E_2)$ depending on parameter $h\in Q\subset\mathbb{C}$ is called uniformly R-bounded in h if there is a constant C independent of $h\in Q$ such that

$$\int_\Omega\left\|\sum_{j=1}^m r_j(y)\,T_j(h)\,u_j\right\|_{E_2}dy\leqslant C\int_\Omega\left\|\sum_{j=1}^m r_j(y)\,u_j\right\|_{E_1}dy$$

for all $T_1(h),T_2(h),...,T_m(h)\in G_h$ and $u_1,u_2,...,u_m\in E_1$, $m\in\mathbb{N}$. It implies that $\sup\limits_{h\in Q}R(G_h)\leqslant C$.

The operator A is said to be R-positive in a Banach space E if the set $\left\{\lambda\,(A+\lambda)^{-1}:\right.$ $\left.\lambda\in S_\varphi\right\}$ is R-bounded. A positive operator $A(h)$ is said to be uniformly R-positive in a Banach space E if there exists $\varphi\in[0,\ \pi)$ such that the set

$$\left\{\lambda\,(A(h)+\lambda)^{-1}:\lambda\in S_\varphi\right\}$$

is uniformly R-bounded. Let $S(\mathbb{R}^n;E)$ denote the E-valued Schwartz class, i.e., the space of all E-valued rapidly decreasing smooth functions on $\mathbb{R}^n$ equipped with its usual topology generated by seminorms. For $E=\mathbb{C}$ this space will be denoted by $S=S(\mathbb{R}^n)$. Here, $S'(E)=S'(\mathbb{R}^n;E)$ denotes the space of linear continuous mappings from S into E and is called E-valued Schwartz distributions. For any $\alpha=(\alpha_1,\alpha_2,...,\alpha_n)$, $\alpha_i\in[0,\infty)$ the function $(i\xi)^\alpha$ will be defined as:

$$(i\xi)^\alpha=\begin{cases}(i\xi_1)^{\alpha_1},.,(i\xi_n)^{\alpha_n}, \ \xi_1\xi_2,.,\xi_n\neq 0\\ \qquad 0,\ \xi_1,\xi_2,.,\xi_n=0,\end{cases}$$

where

$$(i\xi_k)^{\alpha_k}=\exp\left[\alpha_k\left(\ln|\xi_k|+i\pi\ \mathrm{sgn}\ \xi_k/2\right)\right], k=1,2,...,n.$$

The Liouville derivatives $D^\alpha u$ of an E-valued function u are defined similarly to the case of scalar functions [13].

$C(\Omega;E)$ and $C^{(m)}(\Omega;E)$ will denote the spaces of E-valued bounded uniformly strongly continuous and m times continuously differentiable functions on Ω, respectively. Let F and F^{-1} denote the Fourier and inverse Fourier transforms defined as

$$Fu=(2\pi)^{-\frac{n}{2}}\int\limits_{\mathbb{R}^n}[\exp(x,\xi)]\,u(x)\,dx,\ F^{-1}u=(2\pi)^{-\frac{n}{2}}\int\limits_{\mathbb{R}^n}[\exp-(x,\xi)]\,u(\xi)\,d\xi,$$

where

$$x=(x_1,x_2,...,x_n),\ \xi=(\xi_1,\xi_2,...,\xi_n)\in\mathbb{R}^n,\ (x,\xi)=\sum_{k=1}^n x_k\xi_k.$$

Through this section, the Fourier transformation of a function u will be denoted by $\hat{u}$. It is known that

$$F\left(D_x^\alpha u\right)=(i\xi_1)^{\alpha_1},.,(i\xi_n)^{\alpha_n}\hat{u},\ D_\xi^\alpha\left(F(u)\right)=F\left[(-ix_n)^{\alpha_1},.,(-ix_n)^{\alpha_n}u\ \right]$$

for all $u\in S'(\mathbb{R}^n;E)$. Let E_1 and E_2 be two Banach spaces. $B(E_1,E_2)$ denotes the space of bounded linear operators from E_1 to E_2. A function $\Psi\in C(\mathbb{R}^n;B(E_1,E_2))$ is called a Fourier multiplier from $L_p(\mathbb{R}^n;E_1)$ to $L_p(\mathbb{R}^n;E_2)$ if the map

$$u\to\Lambda u=F^{-1}\Psi(\xi)Fu,\ u\in S(\mathbb{R}^n;E_1)$$

is well defined and extends to a bounded linear operator

$$\Lambda:\ L_p(\mathbb{R}^n;E_1)\to\ L_p(\mathbb{R}^n;E_2).$$

The set of all Fourier multipliers from $L_p(\mathbb{R}^n;E_1)$ to $L_p(\mathbb{R}^n;E_2)$ will be denoted by $M_p^p(E_1,E_2)$. For $E_1=E_2=E$ it is denoted by $M_p^p(E)$. Let $\Phi_h=\left\{\Psi_h\in M_p^p(E_1,E_2),\ h\in Q\right\}$ denote a collection of multipliers depending on the parameter h. We say that W_h is a uniform collection of multipliers if there exists a positive constant M independent of $h\in Q$ such that

$$\left\|F^{-1}\Psi_hFu\right\|_{L_p(\mathbb{R}^n;E_2)}\leqslant M\left\|u\right\|_{L_p(\mathbb{R}^n;E_1)}$$

for all $h \in Q$ and $u \in S(\mathbb{R}^n; E_1)$.

Let E_0 and E be two Banach spaces and E_0 be continuously and densely embedded into E. Let $s \in \mathbb{R}$ and $\xi = (\xi_1, \xi_2, ..., \xi_n) \in \mathbb{R}^n$. Consider the following Liouville-Lions space

$$H_p^s(\mathbb{R}^n; E_0, E) = \{u \;\; u \in S'(\mathbb{R}^n; E_0), F^{-1}\left(1 + |\xi|^2\right)^{\frac{s}{2}} Fu \in L_p(\mathbb{R}^n; E),$$
$$\|u\|_{H_p^s(\mathbb{R}^n; E_0, E)} = \|u\|_{L_p(\mathbb{R}^n; E_0)} + \left\|F^{-1}\left(1 + |\xi|^2\right)^{\frac{s}{2}} Fu\right\|_{L_p(\mathbb{R}^n; E)} < \infty\Bigg\}.$$

Let $\varepsilon = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_n)$ and ε_k be positive parameters. We define the following parameterized norm in $H_p^s(\mathbb{R}^n; E_0, E)$,

$$\|u\|_{H_{p,\varepsilon}^s(\mathbb{R}^n; E_0, E)} = \|u\|_{L_p(\mathbb{R}^n; E_0)} + \left\|F^{-1}\left[1 + \left(\sum_{k=1}^n \varepsilon_k^{\frac{2}{s}} \xi_k^2\right)^{\frac{1}{2}}\right]^s Fu\right\|_{L_p(\mathbb{R}^n; E)} < \infty.$$

Sometimes we use one and the same symbol C without distinction in order to denote positive constants which may differ from each other even in a single context. When we want to specify the dependence of such a constant on a parameter, say α, we write C_α.

By using the techniques of [9, Theorem 3.7] and reasoning as in [19, Theorem A_0], we obtain the following proposition.

PROPOSITION A_0. *Let E_1 and E_2 be two UMD spaces and*

$$\Psi_h \in C^n(\mathbb{R}^n \setminus \{0\}; B(E_1, E_2)).$$

Suppose there is a positive constant K such that

$$\sup_{h \in Q} R\left(\left\{|\xi|^{|\beta|} D^\beta \Psi_h(\xi) \colon \xi \in \mathbb{R}^n \setminus \{0\},\ \beta_i \in \{0,1\}\right\}\right) \leqslant K,$$

for

$$\beta = (\beta_1, \beta_2, ..., \beta_n),\ |\beta| = \sum_{k=1}^n \beta_k.$$

Then Ψ_h is a uniform collection of multipliers from $L_p(\mathbb{R}^n; E_1)$ to $L_p(\mathbb{R}^n; E_2)$ for $p \in (1, \infty)$.

Proof. Some steps (Lemma 3.1, Proposition 3.2) of proof [9, Theorem 3.7] trivially work for the parameter dependent case. Other steps (Theorem 3.3, Lemma 3.5) can be easily shown by replacing

$$\left\{|\xi|^{|\beta|} D^\beta \Psi(\xi) \colon \xi \in \mathbb{R}^n \setminus \{0\}\right\}$$

with

$$\Sigma_h = \left\{|\xi|^{|\beta|} D^\beta \Psi_h(\xi) : \xi \in \mathbb{R}^n \setminus \{0\}\right\}$$

and by using the uniform R-boundedness of the set Σ_h. However, the parameter dependent analog of Proposition 3.4 in [9] is not straightforward. Let M_h, $M_{h,N} \in L_1^{loc}(\mathbb{R}^n, B(E_1,E_2))$ be Fourier multipliers from $L_p(\mathbb{R}^n;E_1)$ to $L_p(\mathbb{R}^n;E_2)$. Let $M_{h,N}$ converge to M_h in $L_1^{loc}(\mathbb{R}^n, B(E_1,E_2))$ and $T_{h,N} = F^{-1}M_{h,N}F$ be uniformly bounded. Then the operator $T_h = F^{-1}M_hF$ is uniformly bounded, so we obtain the assertion of Proposition A_0.

The embedding theorems in vector valued spaces play a key role in the theory of DOEs. For estimating lower order derivatives in terms of interpolation spaces we use following embedding theorems from [17].

THEOREM A_1. *Suppose E is an UMD space, $0 < \varepsilon_k \leqslant \varepsilon_0 < \infty$, $1 < p \leqslant q < \infty$ and A is an R-positive operator in E. Then for $s \in (0,\infty)$ with $\varkappa = |\alpha| + n\left(\frac{1}{p} - \frac{1}{q}\right) \leqslant s$, $0 \leqslant \mu \leqslant 1 - \varkappa$ the embedding*

$$D^{\alpha}H_p^s(\mathbb{R}^n;E(A),E) \subset L_q\left(\mathbb{R}^n;E\left(A^{1-\varkappa-\mu}\right)\right)$$

is continuous and there exists a constant $C_\mu > 0$, depending only on μ such that

$$\varepsilon(\alpha)\left\|D^{\alpha}u\right\|_{L_q\left(\mathbb{R}^n;E\left(A^{1-\varkappa-\mu}\right)\right)} \leqslant C_\mu\left[h^{\mu}\left\|u\right\|_{H^s_{p,\varepsilon}(\mathbb{R}^n;E(A),E)} + h^{-(1-\mu)}\left\|u\right\|_{L_p(\mathbb{R}^n;E)}\right]$$

for all $u \in H_p^s(\mathbb{R}^n;E(A),E)$ and $0 < h \leqslant h_0 < \infty$.

2. FDOE with parameters in Banach spaces

Consider the principal part of the problem (1.1),

$$(L_\varepsilon + \lambda)u = P_\varepsilon(D)u + Au + \lambda u = f(x),\ x \in \mathbb{R}^n, \tag{2.1}$$

where $P_\varepsilon(D)$ is the fractional differential operator defined by

$$P_\varepsilon(D)u = F^{-1}P_\varepsilon(\xi)\hat{u}(\xi) = (2\pi)^{-\frac{n}{2}}\int\limits_{\mathbb{R}^n} e^{i(x,\xi)}P_\varepsilon(\xi)\hat{u}(\xi)\,d\xi. \tag{2.2}$$

CONDITION 2.1.Assume $P_\varepsilon(\xi) \in S^m$ for some positive number m, i.e.,

$$\left|D_\xi^{\alpha}P_\varepsilon(\xi)\right| \leqslant C_\alpha\left[1 + \left(\sum_{k=1}^n \varepsilon_k^{\frac{2}{m-|\alpha|}}\xi_k^2\right)^{\frac{1}{2}}\right]^{m-|\alpha|}$$

for all $\xi \in \mathbb{R}^n$, $\alpha_k > 0$, $\alpha = (\alpha_1,\alpha_2,...,\alpha_n)$ with $|\alpha| \leqslant m$ and $\varepsilon_k \in (0, \varepsilon_0]$. Suppose $P_\varepsilon(\xi) \in S_{\varphi_1}$ for all $\xi \in \mathbb{R}^n$, $0 \leqslant \varphi_1 < \pi$ and there is a constant $\gamma > 0$ such that $|P_\varepsilon(\xi)| \geqslant \gamma\sum\limits_{k=1}^n \varepsilon_k|\xi_k|^m$.

Let

$$X = L_p(\mathbb{R}^n;E),\ Y = H_p^m(\mathbb{R}^n;E(A),E).$$

In this section we prove the following:

THEOREM 2.1. *Assume the Condition 2.1 holds. Suppose E is an UMD space, $p \in (1,\infty)$ and A is an R-positive operator in E with respect to $\varphi \in (0\,,\pi]$. Then for $f \in X$, $\lambda \in S_{\varphi_2}$, $0 \leqslant \varphi_1 < \pi - \varphi_2$ and $\varphi_1 + \varphi_2 \leqslant \varphi$ there is a unique solution u of the equation (2.1) belonging to Y and the following coercive uniform estimate holds*

$$\sum_{|\alpha| \leqslant m} \varepsilon(\alpha) |\lambda|^{1-\frac{|\alpha|}{m}} \|D^\alpha u\|_X + \|Au\|_X \leqslant C \|f\|_X . \tag{2.3}$$

Proof. By applying the Fourier transform to equation (2.1) we obtain

$$[P_\varepsilon(\xi) + A + \lambda]\hat{u}(\xi) = \hat{f}(\xi). \tag{2.4}$$

By construction $\lambda + P_\varepsilon(\xi) \in S_\varphi$, for all $t_k \in (0\,, t_0]$, $\xi \in \mathbb{R}^n$ and the operator $A + \lambda + P_\varepsilon(\xi)$ is invertible in E. So, from (2.4) we obtain that the solution of equation (2.1) can be represented in the form

$$u(x) = F^{-1}[A + \lambda + P_\varepsilon(\xi)]^{-1} \hat{f}. \tag{2.5}$$

By definition of the positive operator A, the inverse of A^{-1} is bounded in E. Then the operator A is a closed linear operator (as an inverse of bounded linear operator A^{-1}). By the differential properties of the Fourier transform and by using (2.5) we have

$$\|Au\|_X = \left\| F^{-1} A [A + \lambda + P_\varepsilon(\xi)]^{-1} \hat{f} \right\|_X ,$$
$$\|D^\alpha u\|_X = \left\| F^{-1} \xi^\alpha [A + \lambda + P_\varepsilon(\xi)]^{-1} \hat{f} \right\|_X ,$$

where $X = L_p(\mathbb{R}^n; E)$. Hence, it suffices to show that operator-functions

$$\sigma(\varepsilon, \lambda, \xi) = A[A + \lambda + P_\varepsilon(\xi)]^{-1},$$
$$\sigma_\alpha(\varepsilon, \lambda, \xi) = \varepsilon(\alpha) |\lambda|^{1-\frac{|\alpha|}{m}} \xi^\alpha [A + \lambda + P_\varepsilon(\xi)]^{-1}$$

are collections of multipliers in X uniformly with respect to $\varepsilon_k \in (0, \varepsilon_0]$ and $\lambda \in S_{\varphi_2}$. By virtue of $[5, \text{Lemma } 2.3]$, for $\lambda \in S_{\varphi_1}$ and $\nu \in S_{\varphi_2}$ with $\varphi_1 + \varphi_2 < \pi$ there is a positive constant C such that

$$|\lambda + \nu| \geqslant C(|\lambda| + |\nu|). \tag{2.6}$$

By using the positivity properties of operator A, we get that

$$B(\lambda, \varepsilon) = [A + \lambda + P_\varepsilon(\xi)]^{-1}$$

is bounded for all $\xi \in \mathbb{R}^n$, $\lambda \in S_{\varphi_1}$, $\varepsilon_k \in (0\,, \varepsilon_0]$ and

$$\|B(\lambda, \varepsilon)\| \leqslant C(1 + |\lambda + P_\varepsilon(\xi)|)^{-1}.$$

By using Condition 2.1 and estimate (2.6), we obtain that

$$\|B(\lambda, \varepsilon)\| \leqslant C(1 + |\lambda| + |P_\varepsilon(\xi)|)^{-1} \leqslant C_2 \left[1 + |\lambda| + \sum_{k=1}^n \varepsilon_k |\xi_k|^m \right]^{-1}. \tag{2.7}$$

Then by resolvent properties of positive operators and uniform estimate (2.7) we obtain

$$\|\sigma(\varepsilon,\lambda,\xi)\| \leqslant \left\|I+(\lambda+P_\varepsilon(\xi))\left[A+\lambda+P_\varepsilon(\xi)\right]^{-1}\right\|$$
$$\leqslant 1+(|\lambda|+|P_\varepsilon(\xi)|)(1+|\lambda|+|P_\varepsilon(\xi)|)^{-1} \leqslant C_3,$$

where I is an identity operator in E. Moreover, by using the well known inequality

$$y_1^{\beta_1} y_2^{\beta_2},.,y_n^{\beta_n} \leqslant C\left(1+\sum_{k=1}^n y_k^m\right)$$

for $|\beta| \leqslant m$, $y_k > 0$ and $\beta = (\beta_1,\beta_2,...,\beta_n)$ for all $u \in E$, we have

$$\|\sigma_\alpha(\varepsilon,\lambda,\xi)u\|_E \leqslant \varepsilon(\alpha)|\lambda|^{1-\frac{|\alpha|}{m}}|\xi^\alpha|\,\|B(\lambda,t)u\|_E$$
$$\leqslant |\lambda|\prod_{k=1}^n\left(\varepsilon_k^{\frac{1}{m}}|\lambda|^{\frac{1}{m}}|\xi_k|\right)^{\alpha_k}\|B(\lambda,\varepsilon)u\|_E$$
$$\leqslant C_\alpha\left(|\lambda|+\sum_{k=1}^n \varepsilon_k|\xi_k|^m\right)\|B(\lambda,\varepsilon)u\|_E.$$

In view of estimate (2.7) and by Condition 2.1 we get from the above inequality

$$\|\sigma_\alpha(\varepsilon,\lambda,\xi)u\|_E \leqslant C_\alpha\|u\|_E.$$

So, we obtain that the operator functions $\sigma(\varepsilon,\lambda,\xi)$ and $\sigma_\alpha(\varepsilon,\lambda,\xi)$ are uniformly bounded, i.e.,

$$\|\sigma(\varepsilon,\lambda,\xi)\|_{B(E)} \leqslant C, \|\sigma_\alpha(\varepsilon,\lambda,\xi)\|_{B(E)} \leqslant C_\alpha. \tag{2.8}$$

Due to R-positivity of A, by (2.8) and by Kahane's contraction principle $[6$, Lemma 3.5$]$, we obtain that the set $\{\sigma(\varepsilon,\lambda,\xi);\ \xi \in \mathbb{R}^n\setminus\{0\}\}$ is uniformly R-bounded, i.e.,

$$\sup_{\varepsilon,\lambda} R\{\sigma(\varepsilon,\lambda,\xi);\ \xi \in \mathbb{R}^n\setminus\{0\}\} \leqslant M_0.$$

In a similar way for $\beta = (\beta_1,\beta_2,...,\beta_n)$, $\beta_i \in \{0,1\}$, we obtain

$$R\left(\left\{|\xi|^{|\beta|}D_\xi^\beta\sigma(\varepsilon,\lambda,\xi) : \xi \in \mathbb{R}^n\setminus\{0\}\right\}\right) \leqslant M. \tag{2.9}$$

Consider the following sets

$$\sigma^\beta(\varepsilon,\lambda,\xi) = \left\{|\xi|^{|\beta|}D_\xi^\beta\sigma(\varepsilon,\lambda,\xi) : \xi \in \mathbb{R}^n\setminus\{0\}\right\},$$
$$\sigma_\alpha^\beta(\varepsilon,\lambda,\xi) = \left\{|\xi|^{|\beta|}D_\xi^\beta\sigma_\alpha(\varepsilon,\lambda,\xi) : \xi \in \mathbb{R}^n\setminus\{0\}\right\},$$
$$\beta = (\beta_1,\beta_2,...,\beta_n),\ \beta_i \in \{0,1\}.$$

In view of the R-positivity properties of operator A, due to Kahane's contraction, addition and product properties of the collection of R-bounded operators (see e.g. $[7, 23]$), by (2.9) for all $\left\{\xi^{(j)}\right\} \in \mathbb{R}^n$, $\left\{\sigma_\alpha^\beta\left(\varepsilon,\lambda,\xi^{(j)}\right)\right\}$, $j=1,2,...,\mu$ and $u_{1,}u_2,...,u_\mu \in E$ and independent symmetric $\{-1,1\}$-valued random variables $r_j(y)$, $\mu \in \mathbb{N}$ we obtain the following uniform estimate

$$\int\limits_\Omega \left\| \sum_{j=1}^{\mu} r_j(y)\,\sigma_\alpha^\beta\left(\varepsilon,\lambda,\xi^{(j)}\right) u_j \right\|_E dy$$
$$\leqslant C \int\limits_\Omega \left\| \sum_{j=1}^{\mu} \sigma^\beta\left(\varepsilon,\lambda,\xi^{(j)}\right) r_j(y)\, u_j \right\|_E dy \leqslant C \int\limits_\Omega \left\| \sum_{j=1}^{\mu} r_j(y)\, u_j \right\|_E dy,$$

i.e.,

$$R\left(\left\{\xi^\beta D_\xi^\beta \sigma_\alpha(\varepsilon,\lambda,\xi) : \xi \in \mathbb{R}^n \backslash \{0\}\right\}\right) \leqslant M_\beta .$$

Hence, we infer that the operator-valued functions $\sigma(\varepsilon,\lambda,\xi)$ and $\sigma_\alpha(\varepsilon,\lambda,\xi)$ are uniform R-bounded multipliers and it's R-bounds are independent of ε and λ. By virtue of Preposition A_0, the operator-valued functions $\sigma(\varepsilon,\lambda,\xi)$ and $\sigma_\alpha(\varepsilon,\lambda,\xi)$ are uniform collections of Fourier multipliers in X. So, we obtain that for all $f \in X$ there is a unique solution of equation (2.1) and estimate (2.3) holds.

Let O_ε denote the operator in X generated by problem (2.1) for $\lambda = 0$, i.e.,

$$D(O_\varepsilon) \subset H_p^m(\mathbb{R}^n; E(A), E),\ O_\varepsilon u = P_\varepsilon(D)\, u + Au.$$

Theorem 2.1 and the definition of the space $H_p^m(\mathbb{R}^n; E(A), E)$ imply the following result:

RESULT 2.1. Assume all conditions of Theorem 2.1 are satisfied. Then there are positive constants C_1 and C_2 so that

$$C_1 \|O_\varepsilon u\|_X \leqslant \|u\|_{H_{p,\varepsilon}^m(\mathbb{R}^n;E(A),E)} \leqslant C_2 \|O_\varepsilon u\|_X$$

for $u \in Y$. Indeed, if we put $\lambda = 1$ in (2.3), by Theorem 2.1 we get

$$\sum_{|\alpha| \leqslant m} \varepsilon(\alpha) \|D^\alpha u\|_X + \|Au\|_X \leqslant C \|O_\varepsilon u\|_X \tag{2.10}$$

for $u \in Y$. Due to the closedness of A and by the differential properties of the Fourier transform, we have

$$\|Au\|_X = \left\|F^{-1} A\hat{u}\right\|_X,\ \|D^\alpha u\|_X = \left\|F^{-1}\xi^\alpha \hat{u}\right\|_X .$$

So, in view of estimate (2.10) and by definition of Y, we obtain

$$\|u\|_{H_{p,\varepsilon}^m(\mathbb{R}^n;E(A),E)} \leqslant C_2 \|O_\varepsilon u\|_X .$$

The first inequality is equivalent to the following estimate

$$\left\|F^{-1} A\hat{u}\right\|_X + \left\|F^{-1} P_\varepsilon(\xi)\,\hat{u}\right\|_X$$
$$\leqslant C\left\{ \left\|F^{-1} A\hat{u}\right\|_X + \left\| F^{-1}\left[1 + \left(\sum_{k=1}^{n} \varepsilon_k^{\frac{2}{m}} \xi_k^2\right)^{\frac{1}{2}}\right]^m \hat{u} \right\|_X \right\}.$$

So, it suffices to show that the operator functions

$$A\left\{A+\left[1+\left(\sum_{k=1}^{n}\varepsilon_k^{\frac{2}{m}}\xi_k^2\right)^{\frac{1}{2}}\right]^m\right\}^{-1},\ \varepsilon(\alpha)\,\xi^\alpha\left[1+\left(\sum_{k=1}^{n}\varepsilon_k^{\frac{2}{m}}\xi_k^2\right)^{\frac{1}{2}}\right]^{-m}$$

are uniform Fourier multipliers in X. This fact is proved in a similar way as in the proof of Theorem 2.1.

From Theorem 2.1, we have:

RESULT 2.2. Assume all conditions of Theorem 2.1 hold. Then, for all $\lambda\in S_\varphi$ the resolvent of operator O_ε exists and the following sharp uniform estimate holds

$$\sum_{|\alpha|\leqslant m}\varepsilon(\alpha)\left\|D^\alpha(O_\varepsilon+\lambda)^{-1}\right\|_{B(X)}+\left\|A(O_\varepsilon+\lambda)^{-1}\right\|_{B(X)}\leqslant C. \tag{2.11}$$

Indeed, we infer from Theorem 2.1 that the operator $O_\varepsilon+\lambda$ has a bounded inverse from X to Y. So, the solution u of the equation (2.1) can be expressed as $u(x)=(O_\varepsilon+\lambda)^{-1}f$ for all $f\in X$. Then estimate (2.3) implies the estimate (2.11).

RESULT 2.3. Theorem 2.1 particularly implies that the operator O_ε is positive in X. Then the operators O_ε^σ are generators of analytic semigroups in X for $\sigma\leqslant\frac{1}{2}$ (see e.g. [20, §1.14.5]).

Now consider the problem (1.1). By using Theorem 2.1 and the perturbation theory of linear operators, we have:

THEOREM 2.2. *Assume all conditions of Theorem 2.1 are satisfied. Suppose* $A_\alpha(x)A^{-\left(1-\frac{|\alpha|}{m}-\mu\right)}\in L_\infty(\mathbb{R}^n;B(E))$ *for* $\mu\in\left(0,1-\frac{|\alpha|}{m}\right)$. *Then for* $f\in X$, $\lambda\in S_{\varphi_2}$, $0\leqslant\varphi_2<\pi-\varphi_1$, $\varphi_1+\varphi_2\leqslant\varphi$ *and for sufficiently large* $|\lambda|$ *there is a unique solution* u *of the equation* (1.1) *belonging to* Y *and the following coercive uniform estimate holds*

$$\sum_{|\alpha|\leqslant m}\varepsilon(\alpha)\,|\lambda|^{1-\frac{|\alpha|}{m}}\,\|D^\alpha u\|_X+\|Au\|_X\leqslant C\,\|f\|_X. \tag{2.12}$$

Proof. It is clear that $Q_\varepsilon=O_\varepsilon+L_\varepsilon$, where O_ε is the operator in X generated by problem (2.1) for $\lambda=0$ and

$$L_\varepsilon u=\sum_{|\alpha|<m}\varepsilon(\alpha)A_\alpha(x)D^\alpha u \text{ for } u\in Y.$$

In view of the condition on $A_\alpha(x)$ and by Theorem A_1 for $u\in Y$ we have

$$C_\mu\sum_{|\alpha|<m}\varepsilon(\alpha)\left\|A^{1-\frac{|\alpha|}{m}-\mu}D^\alpha u\right\|_X\leqslant C_\mu\left[h^\mu\|u\|_{H^m_{p,\varepsilon}(R^n;E(A),E)}+h^{-(1-\mu)}\|u\|_X\right]. \tag{2.13}$$

$$\|L_\varepsilon u\|_X\leqslant\sum_{|\alpha|<m}\varepsilon(\alpha)\,\|A_\alpha(x)D^\alpha u\|_X.$$

Then from estimates (2.12), (2.13) and for $u \in Y$ we obtain

$$\|L_\varepsilon u\|_X \leqslant C\left[h^\mu \|O_\varepsilon u\|_X + h^{-(1-\mu)}\|u\|_X\right]. \tag{2.14}$$

Since $\|u\|_X = \frac{1}{\lambda}\|(O_\varepsilon+\lambda)u + L_\varepsilon u\|_X$ for $\lambda \in S_{\varphi_2}$, we get

$$\begin{aligned}\|u\|_X &\leqslant \frac{1}{|\lambda|}\left[\|(O_\varepsilon+\lambda)u\|_X + \|O_\varepsilon u\|_X\right] \\ &\leqslant \frac{1}{|\lambda|}\|(O_\varepsilon+\lambda)u\|_X + \frac{1}{|\lambda|}\left[\sum_{|\alpha|<m}\varepsilon(\alpha)\|D^\alpha u\|_X + \|Au\|_X\right].\end{aligned} \tag{2.15}$$

From estimates $(2.13)-(2.15)$ for $u \in Y$, we obtain

$$\|L_\varepsilon u\|_X \leqslant Ch^\mu \|(O_\varepsilon+\lambda)u\|_X + C_1|\lambda|^{-1}h^{-(1-\mu)}\|(O_\varepsilon+\lambda)u\|_X. \tag{2.16}$$

Then choosing h and λ such that $Ch^\mu < 1$, $C_1|\lambda|^{-1}h^{-(1-\mu)} < 1$ from (2.16), we obtain that

$$\left\|L_\varepsilon(O_\varepsilon+\lambda)^{-1}\right\|_{B(X)} < 1. \tag{2.17}$$

From Theorem 2.1 and (2.17) we get that the operator $(Q_\varepsilon+\lambda)$ has a bounded inverse in X. Moreover, it is clear that

$$(Q_\varepsilon+\lambda)^{-1} = \left[I + L_\varepsilon(O_\varepsilon+\lambda)^{-1}\right](O_\varepsilon+\lambda), \tag{2.18}$$

where I is an identity operator in X. Using relation (2.18), estimates (2.3), (2.17) and perturbation theory of linear operators, we obtain that the operator $Q_\varepsilon+\lambda$ has a bounded inverse from X into Y and the estimate (2.12) holds.

3. The Cauchy problem for parabolic FDOE with parameter

In this section, we shall consider the following Cauchy problem for the parabolic FDOE

$$\frac{\partial u}{\partial t} + P_\varepsilon(D)u + Au = f(t,x),\ u(0,x) = 0, \tag{3.1}$$

where $P_\varepsilon(D)$ is the fractional differential operator defined by (2.2) and A is a linear operator in E, $\varepsilon = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_n)$, ε_k are positive parameters.

By applying Theorem 2.1 we establish the maximal regularity of the problem (3.1) in E-valued mixed $L_{\mathbf{p}}$ spaces, where $\mathbf{p} = (p_1, p)$. Let O_ε denote the operator generated by problem (2.1). For this aim we need the following result:

THEOREM 3.1. *Suppose Condition 2.1 holds, E is an UMD space and operator A is R-positive in E with $0 \leqslant \varphi < \pi - \varphi_1$. Then operator O_ε is uniformly R-positive in $L_p(\mathbb{R}^n; E)$.*

Proof. From Result 2.3 we obtain that the operator O_ε is positive in $X = L_p(\mathbb{R}^n; E)$. We have to prove the R-boundedness of the set

$$\sigma(\varepsilon,\lambda,\xi) = \left\{\lambda(O_\varepsilon+\lambda)^{-1} : \lambda \in S_\varphi\right\}.$$

From the proof of Theorem 2.1, we have

$$\lambda \left(O_{\varepsilon}+\lambda\right)^{-1} f = F^{-1}\Phi\left(\varepsilon,\xi,\lambda\right)\hat{f},\ f\in X,$$

where

$$\Phi\left(\varepsilon,\xi,\lambda\right) = \lambda \left(A+P_{\varepsilon}\left(\xi\right)+\lambda\right)^{-1}.$$

By reasoning as in the proof of Theorem 2.1, we obtain the following uniform estimate

$$\left\|\Phi\left(\varepsilon,\xi,\lambda\right)\right\|_{B(E)} \leqslant \left|\lambda\right| \left\|\left(A+P_{\varepsilon}\left(\xi\right)+\lambda\right)^{-1}\right\|_{B(E)} \leqslant C.$$

By definition of R-boundedness, it suffices to show that the operator function $\Phi\left(\varepsilon,\xi,\lambda\right)$ (which depends on variable λ and parameters $\xi,\ \varepsilon$) is a multiplier in X uniformly with respect to ξ and ε. Indeed, by reasoning as in Theorem 2.1 we can easily show that $\Phi\left(\varepsilon,\xi,\lambda\right)$ is a uniform multiplier in $L_p\left(\mathbb{R};E\right)$. Then, by the definition of a R-bounded set we have

$$\int_0^1 \left\|\sum_{j=1}^m r_j\left(y\right)\lambda_j\left(O_{\varepsilon}+\lambda_j\right)^{-1} f_j\right\|_X dy = \int_0^1 \left\|\sum_{j=1}^m r_j\left(y\right)F^{-1}\Phi\left(\varepsilon,\xi,\lambda_j\right)\hat{f}_j\right\|_X dy$$

$$= \int_0^1 \left\|F^{-1}\sum_{j=1}^m r_j\left(y\right)\Phi\left(\varepsilon,\xi,\lambda_j\right)\hat{f}_j\right\|_X dy \leqslant C\int_0^1 \left\|\sum_{j=1}^m r_j\left(y\right)f_j\right\|_X dy$$

for all $\xi\in\mathbb{R}^n,\ \lambda_1,\lambda_2,\ldots,\lambda_m\in S_{\varphi},\ f_1,f_2,\ldots,f_m\in X,\ m\in N$, where $\left\{r_j\right\}$ is a sequence of independent symmetric $\{-1,1\}$-valued random variables on $[0,1]$. Hence, the set $\Phi\left(\varepsilon,\xi,\lambda\right)$ is uniformly R-bounded.

Let E be a Banach space. For $\mathbf{p}=\left(p,p_1\right),\ Z=L_{\mathbf{p}}\left(\mathbb{R}_+^{n+1};E\right)$ will denote the space of all $\mathbf{p}$-summable E-valued functions with mixed norm, i.e., the space of all measurable E-valued functions f defined on $\mathbb{R}_+^{n+1}$ for which

$$\left\|f\right\|_{L_{\mathbf{p}}\left(\mathbb{R}_+^{n+1};E\right)} = \left(\int_{\mathbb{R}^n}\left(\int_0^{\infty}\left\|f\left(t,x\right)\right\|_E^p dx\right)^{\frac{p_1}{p}} dt\right)^{\frac{1}{p_1}} < \infty.$$

Let E be a Banach space and A be a positive operator in E. Suppose, l is a positive integer. $W_p^l\left(0,\infty;E\left(A\right),E\right)$ denotes the space of all functions $u\in L_p\left(0,\infty;E\left(A\right)\right)$ possessing the generalized derivatives $u^{(l)}\in L_p\left(0,\infty;E\right)$ with the norm

$$\left\|u\right\|_{W_p^l\left(0,\infty;E(A),E\right)} = \left\|Au\right\|_{L_p\left(0,\infty;E\right)} + \left\|u^{(l)}\right\|_{L_p\left(0,\infty;E\right)}.$$

Let m be a positive number. $W_{\mathbf{p}}^{1,m}\left(\mathbb{R}_+^{n+1};E\left(A\right),E\right)$ denotes the space of all functions $u\in L_{\mathbf{p}}\left(\mathbb{R}_+^{n+1};E\left(A\right)\right)$ possessing the generalized derivative $D_t u=\frac{\partial u}{\partial t}\in Z$ with respect to y and fractional derivatives $D_x^{\alpha}u\in Z$ with respect to x for $\left|\alpha\right|\leqslant m$ with the norm

$$\left\|u\right\|_{W_{\mathbf{p}}^{1,m}\left(\mathbb{R}_+^{n+1};E(A),E\right)} = \left\|Au\right\|_Z + \left\|\frac{du}{dt}\right\|_Z + \sum_{|\alpha|\leqslant m}\left\|D_x^{\alpha}u\right\|_Z,$$

where $u = u(t,x)$.

Now, we are ready to state the main result of this section.

THEOREM 3.2. *Assume the conditions of Theorem 2.1 hold for* $\varphi \in \left(\frac{\pi}{2}, \pi\right)$. *Then for* $f \in Z$ *problem* (3.1) *has a unique solution*

$$u \in W_{\mathbf{p}}^{1,m}\left(\mathbb{R}_+^{n+1}; E(A), E\right)$$

satisfying the following unform coercive estimate

$$\left\| \frac{du}{dt} \right\|_Z + \|P_\varepsilon(D)u\|_Z + \|Au\|_Z \leqslant C\|f\|_Z.$$

Proof. By definition of $X = L_p(\mathbb{R}^n; E)$ and mixed space $L_{\mathbf{p}}\left(\mathbb{R}_+^{n+1}; E\right)$, $\mathbf{p} = (p, p_1)$, we have

$$\|u\|_{L_{p_1}(0,\infty;X)} = \left(\int_0^\infty \|u(y)\|_X^{p_1} dt \right)^{\frac{1}{p_1}} = \left(\int_0^\infty \|u(y)\|_{L_p(\mathbb{R}^n;E)}^{p_1} dt \right)^{\frac{1}{p_1}}$$

$$= \left(\int_{\mathbb{R}^n} \left(\int_0^\infty \|u(y,x)\|_E^p dt \right)^{\frac{p_1}{p}} dx \right)^{\frac{1}{p_1}} = \|u\|_Z.$$

Moreover, by definition of the space $W_p^m(0,\infty; E(A), E)$ and by Result 2.1, we obtain

$$\begin{aligned} \|u\|_{W_{p_1}^1(0,\infty;D(O_\varepsilon),X)} &= \|O_\varepsilon u\|_{L_p(0,\infty;X)} + \left\|u'\right\|_{L_p(0,\infty;X)} \\ &= \|Au\|_Z + \|D_t u\|_Z + \sum_{|\alpha| \leqslant m} \|D_x^\alpha u\|_Z \\ &= \|u\|_{W_{\mathbf{p}}^{1,m}\left(\mathbb{R}_+^{n+1}; E(A), E\right)}. \end{aligned} \tag{3.2}$$

Hence, we deduced from the above equalities that,

$$\begin{aligned} Z = L_{\mathbf{p}}\left(\mathbb{R}_+^{n+1}; E\right) = L_{p_1}(0,\infty;X),\ & W_{\mathbf{p}}^{1,m}\left(\mathbb{R}_+^{n+1}; E(A), E\right) \\ &= W_{p_1}^1(0,\infty; D(O_\varepsilon), X). \end{aligned}$$

Therefore, the problem (3.1) can be expressed as the following Cauchy problem for the abstract parabolic equation

$$\frac{du}{dt} + O_\varepsilon u(t) = f(t),\ u(0) = 0,\ t \in (0,\infty). \tag{3.3}$$

By virtue of $[1$, Theorem 4.5.2$]$, the condition $E \in UMD$ implies $X \in UMD$ for $p \in (1,\infty)$. Then due to the R-positivity of O_ε and by virtue of [23, Theorem 4.2], we obtain that for $f \in L_{p_1}(0,\infty;X)$ equation (3.3) has a unique solution $u \in W_{p_1}^1(0,\infty; D(O_\varepsilon), X)$ satisfying the following uniform estimate

$$\left\| \frac{du}{dt} \right\|_{L_{p_1}(0,\infty;X)} + \|O_\varepsilon u\|_{L_{p_1}(0,\infty;X)} \leqslant C\|f\|_{L_{p_1}(0,\infty;X)}.$$

From the Theorem 2.1, relation (3.2) and from the above estimate we get the assertion.

4. BVP for anisotropic FDE

In this section, the maximal regularity properties of the anisotropic FDE is studied.

Let $\tilde{\Omega}=\Omega\times\mathbb{R}^{n}$, where $\Omega\subset\mathbb{R}^{\mu}$ is an open connected set with compact C^{2l}-boundary $\partial\Omega$. Consider the BVP for the FDE

$$P_{\varepsilon}(D)u+\sum_{|\alpha|\leqslant 2l}\left(b_{\alpha}D_{y}^{\alpha}+\lambda\right)u=f(x,y),\ y\in\Omega, \tag{4.1}$$

$$B_{j}u=\sum_{|\beta|\leqslant l_{j}}b_{j\beta}(y)D_{y}^{\beta}u(x,y)=0,\ x\in\mathbb{R}^{n}, \tag{4.2}$$

$$y\in\partial\Omega,\ j=1,2,...,l,$$

where $u=u(x,y)$, $P_{\varepsilon}(D)$ is the fractional differential operator defined by (2.1),

$$D_{j}=-i\frac{\partial}{\partial y_{j}},\ y=\left(y_{1},...,y_{\mu}\right),\ b_{\alpha}=b_{\alpha}(y),$$

$\alpha=\left(\alpha_{1},\alpha_{2},\ldots,\alpha_{\mu}\right)$, $\beta=\left(\beta_{1},\beta_{2},\ldots,\beta_{\mu}\right)$ are nonnegative integer numbers, and $\varepsilon=(\varepsilon_{1},\varepsilon_{2},\ldots,\varepsilon_{n})$ and ε_{k} are positive parameters. For $\tilde{\Omega}=\mathbb{R}^{n}\times\Omega$, $\mathbf{p}=(p_{1},p)$ here, $L_{\mathbf{p}}(\tilde{\Omega})$ will denote the space of all $\mathbf{p}$-summable scalar-valued functions with mixed norm i.e., the space of all measurable functions f defined on $\tilde{\Omega}$, for which

$$\|f\|_{L_{\mathbf{p}}(\tilde{\Omega})}=\left(\int_{\mathbb{R}^{n}}\left(\int_{\Omega}|f(x,y)|^{p_{1}}dx\right)^{\frac{p}{p_{1}}}dy\right)^{\frac{1}{p}}<\infty.$$

Analogously, $W_{\mathbf{p}}^{m,2l}(\tilde{\Omega})$ denotes the anisotropic fractional Sobolev space with corresponding mixed norm, i.e., $W_{\mathbf{p}}^{m,2l}(\tilde{\Omega})$ denotes the space of all functions $u\in L_{\mathbf{p}}(\tilde{\Omega})$ possessing the fractional derivatives $D_{x}^{\alpha}u\in L_{\mathbf{p}}(\tilde{\Omega})$ with respect to x for $|\alpha|\leqslant m$ and generalized derivative $\frac{\partial^{2l}u}{\partial y_{k}^{2l}}\in L_{\mathbf{p}}(\tilde{\Omega})$ with respect to y with the norm

$$\|u\|_{W_{\mathbf{p}}^{m,2l}(\tilde{\Omega})}=\sum_{|\alpha|\leqslant m}\|D_{x}^{\alpha}u\|_{L_{\mathbf{p}}(\tilde{\Omega})}+\sum_{k=1}^{\mu}\left\|\frac{\partial^{2l}u}{\partial y_{k}^{2l}}\right\|_{L_{\mathbf{p}}(\tilde{\Omega})}.$$

Let Q denote the operator generated by problem $(4.1)-(4.2)$, i.e.,

$$D(Q)=W_{\mathbf{p}}^{m,2l}\left(\tilde{\Omega},B_{j}\right)=\left\{u:u\in W_{\mathbf{p}}^{m,2l}(\tilde{\Omega}),\ B_{j}u=0,\ j=1,2,...l\right\},$$

$$Qu=P_{\varepsilon}(D)u+\sum_{|\alpha|\leqslant 2l}b_{\alpha}D_{y}^{\alpha}u.$$

Let $\xi' = (\xi_1, \xi_2, ..., \xi_{\mu-1}) \in \mathbb{R}^{\mu-1}$, $\alpha' = (\alpha_1, \alpha_2, ..., \alpha_{\mu-1}) \in Z^{\mu}$ and

$$A(y_0, \xi', D_y) = \sum_{|\alpha'|+j \leqslant 2l} a_{\alpha'}(y_0) \xi_1^{\alpha_1} \xi_2^{\alpha_2}, .., \xi_{\mu-1}^{\alpha_{\mu-1}} D_\mu^j \text{ for } y_0 \in \bar{G}$$

$$B_j(y_0, \xi', D_y) = \sum_{|\beta'|+j \leqslant l_j} b_{j\beta'}(y_0) \xi_1^{\beta_1} \xi_2^{\beta_2}, .., \xi_{\mu-1}^{\beta_{\mu-1}} D_\mu^j \text{ for } y_0 \in \partial G.$$

CONDITION 4.1. Let the following conditions be satisfied:

(1) $b_\alpha \in C(\bar{\Omega})$ for each $|\alpha| = 2l$ and $b_\alpha \in L_\infty(\Omega) + L_{r_k}(\Omega)$ for each $|\alpha| = k < 2l$ with $r_k \geqslant p_1$, $p_1 \in (1, \infty)$ and $2l - k > \frac{l}{r_k}$;

(2) $b_{j\beta} \in C^{2l-l_j}(\partial\Omega)$ for each j, β, $l_j < 2l$, $p \in (1, \infty)$, $\lambda \in S_\varphi$, $\varphi \in [0, \pi)$;

(3) for $y \in \bar{\Omega}$, $\xi \in R^\mu$, $\sigma \in S_{\varphi_0}$, $\varphi_0 \in (0, \frac{\pi}{2})$, $|\xi| + |\sigma| \neq 0$ let $\sigma + \sum_{|\alpha|=2l} b_\alpha(y) \xi^\alpha \neq 0$;

(4) for each $y_0 \in \partial\Omega$ local BVP in local coordinates corresponding to y_0,

$$\lambda + A(y_0, \xi', D_y) \vartheta(y) = 0,$$
$$B_j(y_0, \xi', D_y) \vartheta(0) = h_j,\ j = 1, 2, ..., l$$

has a unique solution $\vartheta \in C_0(0, \infty)$ for all $h = (h_1, h_2, ..., h_l) \in \mathbb{C}^l$ and for $\xi' \in \mathbb{R}^{n-1}$.

Suppose $\nu = (\nu_1, \nu_2, ..., \nu_n)$ are nonnegative real numbers. In this section, we present the following result:

THEOREM 4.1. *Assume Condition 2.1 and Condition 4.1 are satisfied. Then for* $f \in L_{\mathbf{p}}(\tilde{\Omega})$, $\lambda \in S_\varphi$, $\varphi \in (0, \pi]$, *problem* $(4.1) - (4.2)$ *has a unique solution* $u \in W_p^{m,2l}(\tilde{\Omega})$ *and the following coercive uniform estimate holds*

$$\sum_{|\nu| \leqslant m} \prod_{k=1}^{n} \varepsilon_k^{\frac{\nu_k}{m}} |\lambda|^{1-\frac{|\nu|}{m}} \|D_x^\nu u\|_{L_{\mathbf{p}}(\tilde{\Omega})} + \sum_{|\alpha| \leqslant 2l} \left\|D_y^\alpha u\right\|_{L_{\mathbf{p}}(\tilde{\Omega})} \leqslant C \|f\|_{L_{\mathbf{p}}(\tilde{\Omega})}.$$

Proof. Let $E = L_{p_1}(\Omega)$. It is known [4] that $L_{p_1}(\Omega)$ is an *UMD* space for $p_1 \in (1, \infty)$. Consider the operator A defined by

$$D(A) = W_{p_1}^{2l}(\Omega; B_j u = 0),\ Au = \sum_{|\alpha| \leqslant 2l} b_\alpha(x) D^\alpha u(y).$$

Therefore, the problem $(4.1) - (4.2)$ can be rewritten in the form of (2.1), where $u(x) = u(x, .)$, $f(x) = f(x, .)$ are functions with values in $E = L_{p_1}(\Omega)$. From [6, Theorem 8.2] we get that the following problem

$$\eta u(y) + \sum_{|\alpha| \leqslant 2l} b_\alpha(y) D^\alpha u(y) = f(y), \tag{4.3}$$

$$B_j u = \sum_{|\beta| \leqslant l_j} b_{j\beta}(y) D^\beta u(y) = 0,\ j = 1, 2, ..., l$$

has a unique solution for $f \in L_{p_1}(\Omega)$ and arg $\eta \in S(\varphi_1)$, $|\eta| \to \infty$. Moreover, the operator A generated by (4.3) is R-positive in L_{p_1}. Then from Theorem 2.1 we obtain the assertion.

5. The system of FDE of infinite many order

Consider the following system of FDEs

$$P_{\varepsilon}(D)u_i+\sum_{j=1}^{N}(a_{ij}+\lambda)u_j(x)=f_i(x),\, x\in\mathbb{R}^n, \tag{5.1}$$
$$i=1,2,...,N,\ N\in[1,\infty],$$

where $P_{\varepsilon}(D)$ is the fractional differential operator defined by (2.2), $\varepsilon=(\varepsilon_1,\varepsilon_2,...,\varepsilon_n)$ and ε_k are positive parameters. Let a_{ij} be real numbers and

$$l_q(A)=\left\{u\in l_q, \|u\|_{l_q(A)}=\|Au\|_{l_q}\right.$$
$$\left.=\left(\sum_{i=1}^{N}\left|(Au)_i\right|^q\right)^{\frac{1}{q}}=\left(\sum_{i=1}^{N}\left|\sum_{j=1}^{N}a_{ij}u_j\right|^q\right)^{\frac{1}{q}}<\infty\right\},$$
$$u=\{u_j\},\ Au=\left\{\sum_{j=1}^{N}a_{ij}u_j\right\},\ i,\ j=1,2,...,N.$$

CONDITION 5.1. Let

$$a_{ij}=a_{ji},\ \sum_{i,j=1}^{N}a_{ij}\xi_i\xi_j\geqslant C_0|\xi|^2 \text{ for } \xi\neq 0.$$

Let

$$f(x)=\{f_i(x)\}_1^N,\, u=\{u_i(x)\}_1^N.$$

THEOREM 5.1.. *Assume that the Condition 2.1 and Condition 5.1 are satisfied. Then, for* $f(x)\in L_p(\mathbb{R}^n;l_q)$, $|\arg\lambda|\leqslant\varphi$, $\varphi\in(0\,,\pi]$ *and for sufficiently large* $|\lambda|$, *problem* (5.1) *has a unique solution* $u\in H_p^m(\mathbb{R}^n,l_q(A),l_q)$ *and the following uniform coercive estimate holds*

$$\sum_{|\alpha|\leqslant m}\varepsilon(\alpha)|\lambda|^{1-\frac{|\alpha|}{m}}\left[\int_{\mathbb{R}^n}\left(\sum_{j=1}^{N}\left|D^{\alpha}u_j(x)\right|^q\right)^{\frac{p}{q}}dx\right]^{\frac{1}{p}}$$
$$+\left[\int_{\mathbb{R}^n}\left(\sum_{i=1}^{N}\left|\sum_{j=1}^{N}a_{ij}u_j\right|^q\right)^{\frac{p}{q}}dx\right]^{\frac{1}{p}}\leqslant C\left[\int_{\mathbb{R}^n}\left(\sum_{i=1}^{N}|f_i(x)|^q\right)^{\frac{p}{q}}dx\right]^{\frac{1}{p}}.$$

Proof. Let $E=l_q$ and A be a matrix such that $A=[a_{ij}]$, $i,\ j=1,2,...,N$. It is easy to see that

$$B(\lambda)=\lambda(A+\lambda)^{-1}=\frac{\lambda}{D(\lambda)}[A_{ji}(\lambda)],\, i,\ j=1,2,...N,$$

where $D(\lambda) = \det(A - \lambda I)$, $A_{ji}(\lambda)$ are entries of the corresponding adjoint matrix of $A - \lambda I$. Since the matrix A is symmetric and positive definite, it generates a positive operator in l_q for $q \in (1,\infty)$. For all $u_1, u_2, ..., u_\mu \in l_q$, $\lambda_1, \lambda_2, ..., \lambda_\mu \in \mathbb{C}$ and independent symmetric $\{-1,1\}$-valued random variables $r_k(y)$, $k = 1,2,...,\mu$, $\mu \in \mathbb{N}$ we have

$$
\begin{aligned}
&\int\limits_\Omega \left\| \sum_{k=1}^{\mu} r_k(y) B(\lambda_k) u_k \right\|_{l_q}^q dy \\
&\qquad \leqslant C \left\{ \int\limits_\Omega \sum_{j=1}^{N} \left| \sum_{k=1}^{\mu} \sum_{j=1}^{N} \frac{\lambda_k}{D(\lambda_k)} A_{ji}(\lambda_k) r_k(y) u_{ki} \right|^q dy \right. \\
&\qquad \leqslant \sup_{k,i} \sum_{j=1}^{N} \left| \frac{\lambda_k}{D(\lambda_k)} A_{ji}(\lambda_k) \right|^q \int\limits_\Omega \left| \sum_{k=1}^{\mu} r_k(y) u_{kj} \right|^q dy.
\end{aligned} \tag{5.2}
$$

Since A is symmetric and positive definite, we have

$$
\sup_{k,i} \sum_{j=1}^{N} \left| \frac{\lambda_k}{D(\lambda_k)} A_{ji}(\lambda_k) \right|^q \leqslant C.
$$

From (5.2) and (5.3) we get

$$
\int\limits_\Omega \left\| \sum_{k=1}^{\mu} r_k(y) B(\lambda_k) u_k \right\|_{l_q}^q dy \leqslant C \int\limits_\Omega \left\| \sum_{k=1}^{\mu} r_k(y) u_k \right\|_{l_q}^q dy,
$$

i.e., the operator A is R-positive in l_q. From Theorem 2.1, we obtain that problem (5.1) has a unique solution $u \in H_p^m(\mathbb{R}^n; l_q(A), l_q)$ for $f \in L_p(\mathbb{R}^n; l_q)$ and the following estimate holds

$$
\sum_{|\alpha| \leqslant m} \varepsilon(\alpha) |\lambda|^{1 - \frac{|\alpha|}{m}} \|D^\alpha u\|_{L_p(\mathbb{R}^n; l_q)} + \|Au\|_{L_p(\mathbb{R}^n; l_q)} \leqslant M \|f\|_{L_p(\mathbb{R}^n; l_q)}.
$$

From the above estimate we obtain the assertion.

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