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INCLUSION BETWEEN GENERALIZED STUMMEL CLASSES AND OTHER FUNCTION SPACES

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Abstract. We refine the definition of generalized Stummel classes and study inclusion properties of these classes. We also study the inclusion relation between Stummel classes and other function spaces such as generalized Morrey spaces, weak Morrey spaces, and Lorentz spaces. In addition, we show that these inclusions are proper. Our results extend some previous results in [2, 13].

1. Introduction and preliminaries

The definition of Stummel class was introduced in [4, 13]. For $0<\alpha<n$, the *Stummel class* $S_\alpha = S_\alpha(\mathbb{R}^n)$ is defined by

$$S_\alpha := \left\{f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \eta_\alpha f(r) \searrow 0 \quad \text{for} \quad r \searrow 0\right\},$$

where

$$\eta_\alpha f(r) := \sup_{x\in\mathbb{R}^n} \int_{|x-y|<r} \frac{|f(y)|}{|x-y|^{n-\alpha}}\,dy, \qquad r>0.$$

For $\alpha = 2$, S_2 is known as the *Stummel-Kato class*. Knowledge of Stummel classes is important when one is studying the regularity properties of the solutions of some partial differential equations (see [1, 2, 3, 5, 10]).

In the mean time, the study of Morrey spaces, which were introduced by C. B. Morrey in [11], has attracted many researchers, especially in the last two decades. For $1 \leqslant p < \infty$ and $0 \leqslant \lambda \leqslant n$, the *Morrey space* $L^{p,\lambda} = L^{p,\lambda}(\mathbb{R}^n)$ is defined to be the collection of all functions $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{L^{p,\lambda}} := \sup_{x\in\mathbb{R}^n, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L^p(B(x,r))} < \infty,$$

where $B(x,r) := \{y \in \mathbb{R}^n : |x-y| < r\}$ and

$$\|f\|_{L^p(B(x,r))} := \left(\int_{|x-y|<r} |f(y)|^p\,dy\right)^{\frac{1}{p}}.$$

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Note that $L^{p,0} = L^p$. As shown in [13], one may observe that $L^{1,\lambda} \subseteq S_\alpha$ provided that $n-\lambda < \alpha < n$. (For the case $\alpha = 2$, this fact was proved in [5].) Conversely, if $V \in S_\alpha$ for $0 < \alpha < n$ and $\eta_\alpha f(r) \sim r^\sigma$ for some $\sigma > 0$, then $V \in L^{1,n-\alpha+\sigma}$.

Eridani and Gunawan [6] developed the concept of generalized Stummel classes and studied the inclusion relation between these classes and generalized Morrey spaces. For $1 \leqslant p < \infty$ and a measurable function $\Psi : (0,\infty) \to (0,\infty)$, the *generalized Morrey space* $L^{p,\Psi} = L^{p,\Psi}(\mathbb{R}^n)$ is the collection of all functions $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{L^{p,\Psi}} := \sup_{x\in\mathbb{R}^n,\, r>0} \frac{|B(x,r)|^{-\frac{1}{p}}}{\Psi(r)} \|f\|_{L^p(B(x,r))} < \infty,$$

where $|B(x,r)|$ denotes the Lebesgue measure of $B(x,r)$. Observe that, for $\Psi(t) := t^{\frac{\lambda-n}{p}}$ $(0 \leqslant \lambda \leqslant n)$, we have $L^{p,\Psi} = L^{p,\lambda}$. Further works on the inclusion relation between generalized Stummel classes and Morrey spaces can be found in [8, 14].

The purpose of this paper is to refine the definition of generalized Stummel classes and study the inclusion relation between these classes. We also study the inclusion relation between Stummel classes and Morrey spaces using assumptions that are different from the assumptions used in [6, 8, 14]. We give an example of a function which belongs to the generalized Stummel class but not to the generalized Morrey space. Furthermore, we prove that the Stummel class contains weak Morrey spaces under certain conditions. For $1 \leqslant p < \infty$ and $0 \leqslant \lambda \leqslant n$, the *weak Morrey space* $wL^{p,\lambda} = wL^{p,\lambda}(\mathbb{R}^n)$ is the collection of all Lebesgue measurable functions f on $\mathbb{R}^n$ which satisfy

$$\|f\|_{wL^{p,\lambda}} := \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{wL^p(B(x,r))} < \infty,$$

where

$$\|f\|_{wL^p(B(x,r))} := \sup_{t>0} t\,|\{y \in B(x,r) : |f(y)| > t\}|^{\frac{1}{p}}.$$

Observe that, by taking $\lambda = 0$, we can recover the weak Lebesgue space wL^p. In this paper, we also study the relation between Stummel classes and Lorentz spaces.

Throughout this paper we assume that $\Psi : (0,\infty) \to (0,\infty)$ is a measurable function. Whenever required, we consider the following conditions on Ψ:

$$\int_0^1 \frac{\Psi(t)}{t} dt < \infty; \tag{1}$$

$$\frac{1}{A_1} \leqslant \frac{\Psi(s)}{\Psi(r)} \leqslant A_1 \quad \text{for} \quad 1 \leqslant \frac{s}{r} \leqslant 2; \tag{2}$$

$$\frac{\Psi(r)}{r^n} \leqslant A_2 \frac{\Psi(s)}{s^n} \quad \text{for} \quad s \leqslant r, \tag{3}$$

where $A_i > 0$, $i = 1,2$, are independent of $r,s > 0$. The condition (2) is known as the *doubling condition* on Ψ. In some cases, we can weaken the doubling condition by the *right doubling condition*:

$$\frac{\Psi(s)}{\Psi(r)} \leqslant A_3 \quad \text{for} \quad 1 \leqslant \frac{s}{r} \leqslant 2, \tag{4}$$

where A_3 is independent of $r,s>0$.

In this paper, the constant $c>0$ that appears in the proof of all theorems may vary from line to line, and the notation $c=c(\alpha,\beta,\ldots,\zeta)$ indicates that c depends on $\alpha,\beta,\ldots,\zeta$.

2. The generalized Stummel classes

We begin this section by defining the generalized Stummel class.

DEFINITION 1. For $1\leqslant p<\infty$, we define the **generalized Stummel p-class** $S_{p,\Psi}=S_{p,\Psi}(\mathbb{R}^n)$ by

$$S_{p,\Psi}:=\left\{f\in L^p_{\mathrm{loc}}(\mathbb{R}^n):\eta_{p,\Psi}f(r)\searrow 0\quad\text{for}\quad r\searrow 0\right\},$$

where

$$\eta_{p,\Psi}f(r):=\sup_{x\in\mathbb{R}^n}\left(\int_{|x-y|<r}\frac{|f(y)|^p\Psi(|x-y|)}{|x-y|^n}\,dy\right)^{\frac{1}{p}},\quad r>0.$$

We call $\eta_{p,\Psi}f$ the *Stummel p-modulus* of f. Observe that the Stummel p-modulus is nondecreasing on $(0,\infty)$. For $p=1$, we have $S_{1,\Psi}:=S_\Psi$ — the generalized Stummel class introduced in [6]. For $\Psi(t):=t^\alpha$ $(0<\alpha<n)$, we write $S_{p,\alpha}$ instead of $S_{p,\Psi}$ and $\eta_{p,\alpha}$ instead of $\eta_{p,\Psi}$. Observe that $S_{1,\alpha}:=S_\alpha$ — the Stummel class introduced in [4, 13].

The following two theorems confirm that $\eta_{p,\Psi}f$ is continuous (hence measurable) and satisfies the doubling condition.

THEOREM 1. *If $f\in S_{p,\Psi}$, then $\eta_{p,\Psi}f$ is continuous on $(0,\infty)$.*

Proof. Let $\{r_k\}$ be a sequence in $(0,\infty)$ with $r_k\to r\in(0,\infty)$ and $x\in\mathbb{R}^n$. Choose $r_*>0$ such that $r,r_k\leqslant r_*$ for every $k\in\mathbb{N}$. Next, for every $k\in\mathbb{N}$, define

$$g_k(y):=\frac{|f(y)|^p\Psi(|y-x|)}{|y-x|^n}\chi_{B(x,r_k)}(y)\quad\text{and}\quad g(y):=\frac{|f(y)|^p\Psi(|y-x|)}{|y-x|^n}\chi_{B(x,r)}(y),$$

for $y\in B(x,r_*)$. We see that $\{g_k\}$ is a sequence of nonnegative measurable functions on $B(x,r_*)$, and $g_k\to g$ almost everywhere on $B(x,r_*)$. By the Dominated Convergence Theorem we obtain

$$\int_{|y-x|<r_*}g_k(y)dy\to\int_{|y-x|<r_*}g(y)dy.$$

Therefore

$$\left(\int_{|y-x|<r_k}\frac{|f(y)|^p\Psi(|y-x|)}{|y-x|^n}dy\right)^{\frac{1}{p}}\to\left(\int_{|y-x|<r}\frac{|f(y)|^p\Psi(|y-x|)}{|y-x|^n}dy\right)^{\frac{1}{p}}.\tag{5}$$

Let ε be any positive real number. By (5), there exists $k_0 \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ with $k \geqslant k_0$ we have

$$\begin{aligned}\left(\int_{|y-x|<r} \frac{|f(y)|^p \Psi(|y-x|)}{|y-x|^n} dy\right)^{\frac{1}{p}} - \varepsilon &< \left(\int_{|y-x|<r_k} \frac{|f(y)|^p \Psi(|y-x|)}{|y-x|^n} dy\right)^{\frac{1}{p}} \\ &< \left(\int_{|y-x|<r} \frac{|f(y)|^p \Psi(|y-x|)}{|y-x|^n} dy\right)^{\frac{1}{p}} + \varepsilon.\end{aligned}$$

Since $x \in \mathbb{R}^n$ is arbitrary, we conclude that

$$\eta_{p,\Psi} f(r) - \varepsilon \leqslant \eta_{p,\Psi} f(r_k) \leqslant \eta_{p,\Psi} f(r) + \varepsilon.$$

Thus, we have proved that $\eta_{p,\Psi} f(r_k) \to \eta_{p,\Psi} f(r)$ for any sequence $\{r_k\}$ in $(0,\infty)$ with $r_k \to r \in (0,\infty)$. This means that $\eta_{p,\Psi} f$ is continuous on $(0,\infty)$. $\square$

THEOREM 2. *Let Ψ satisfy the condition (3). If $f \in S_{p,\Psi}$, then $\eta_{p,\Psi} f$ satisfies the doubling condition.*

Proof. Let $x \in \mathbb{R}^n$ and $r > 0$. Choose $m = m(n) \in \mathbb{N}$ and $x_1, \ldots, x_m \in B(x,r)$ such that

$$B(x,r) \subseteq \bigcup_{i=1}^{m} B\left(x_i, \frac{r}{2}\right).$$

Note that

$$\begin{aligned}\left(\int_{|y-x|<r} \frac{|f(y)|^p \Psi(|y-x|)}{|y-x|^n} dy\right)^{\frac{1}{p}} &\leqslant \sum_{i=1}^{m} \left(\int_{|y-x_i|<\frac{r}{2}} \frac{|f(y)|^p \Psi(|y-x|)}{|y-x|^n} dy\right)^{\frac{1}{p}} \\ &= \sum_{i=1}^{m} I_i. \end{aligned} \tag{6}$$

For $i = 1, \ldots, m$, we have

$$\begin{aligned} I_i &= \left(\int_{|y-x_i|<\frac{r}{2}} \frac{|f(y)|^p \Psi(|y-x|)}{|y-x|^n} dy\right)^{\frac{1}{p}} \\ &\leqslant \left(\int_{|y-x|>|y-x_i|, |y-x_i|<\frac{r}{2}} \frac{|f(y)|^p \Psi(|y-x|)}{|y-x|^n} dy\right)^{\frac{1}{p}} \\ &\quad + \left(\int_{|y-x|\leqslant|y-x_i|<\frac{r}{2}} \frac{|f(y)|^p \Psi(|y-x|)}{|y-x|^n} dy\right)^{\frac{1}{p}} \\ &= A_i + B_i. \end{aligned} \tag{7}$$

By the condition (3) on Ψ, we obtain

$$\begin{aligned}
A_i &= \left(\int_{|y-x|>|y-x_i|,|y-x_i|<\frac{r}{2}} \frac{|f(y)|^p\Psi(|y-x|)}{|y-x|^n}dy\right)^{\frac{1}{p}} \\
&\leqslant c(p)\left(\int_{|y-x|>|y-x_i|,|y-x_i|<\frac{r}{2}} \frac{|f(y)|^p\Psi(|y-x_i|)}{|y-x_i|^n}dy\right)^{\frac{1}{p}} \\
&\leqslant c(p)\left(\int_{|y-x_i|<\frac{r}{2}} \frac{|f(y)|^p\Psi(|y-x_i|)}{|y-x_i|^n}dy\right)^{\frac{1}{p}} \leqslant c(p)\,\eta_{p,\Psi}f\left(\frac{r}{2}\right).
\end{aligned}$$

It is clear that

$$B_i \leqslant \left(\int_{|y-x|<\frac{r}{2}} \frac{|f(y)|^p\Psi(|y-x|)}{|y-x|^n}dy\right)^{\frac{1}{p}} \leqslant \eta_{p,\Psi}f\left(\frac{r}{2}\right).$$

From (6) and (7), we get

$$\begin{aligned}
\left(\int_{|y-x|<r} \frac{|f(y)|^p\Psi(|y-x|)}{|y-x|^n}dy\right)^{\frac{1}{p}} &\leqslant m(n)(c(p)+1)\,\eta_{p,\Psi}f\left(\frac{r}{2}\right) \\
&= c(n,p)\,\eta_{p,\Psi}f\left(\frac{r}{2}\right). \qquad (8)
\end{aligned}$$

Since the inequality (8) holds for all $x\in\mathbb{R}^n$, we obtain

$$\eta_{p,\Psi}f(r) \leqslant c(n,p)\,\eta_{p,\Psi}f\left(\frac{r}{2}\right).$$

According to the fact that $\eta_{p,\Psi}f$ is nondecreasing, we conclude that $\eta_{p,\Psi}f$ satisfies the doubling condition. $\square$

As for the classical Stummel class [13], we also have more information about functions in $S_{p,\psi}$, as presented in the following theorem and its corollary.

THEOREM 3. *Let $1\leqslant p<\infty$, Ψ satisfy the condition (3), and $\Phi:(0,\infty)\to(0,\infty)$ be continuous and nondecreasing. If $f\in S_{p,\Psi}$ and*

$$\int_0^1 \left[\eta_{p,\Psi}f(t)\right]^p[\Phi(t)]^{-1}t^{-1}dt<\infty,$$

then there exists a positive constant $c(n,p)$ independent of f such that

$$\int_{|x-y|<r}\frac{|f(y)|^p\Psi(|x-y|)}{|x-y|^n\Phi(|x-y|)}dy \leqslant c(n,p)\int_0^{\frac{r}{2}}\left[\eta_{p,\Psi}f(t)\right]^p[\Phi(t)]^{-1}t^{-1}dt.$$

holds for all $x\in\mathbb{R}^n$ and $r>0$.

Proof. Let $f \in S_{p,\Psi}$. By the hypotheses, we observe that for each $x \in \mathbb{R}^n$ and $r > 0$, we have

$$\begin{aligned}
\int_{|x-y|<r} \frac{|f(y)|^p \Psi(|x-y|)}{|x-y|^n \Phi(|x-y|)}\, dy &= \sum_{k=0}^{\infty} \int_{\frac{r}{2^{k+1}} \leqslant |x-y| < \frac{r}{2^k}} \frac{|f(y)|^p \Psi(|x-y|)}{|x-y|^n \Phi(|x-y|)}\, dy \\
&\leqslant \sum_{k=0}^{\infty} \left[\Phi\left(\frac{r}{2^{k+1}}\right)\right]^{-1} \int_{|x-y|<\frac{r}{2^k}} \frac{|f(y)|^p \Psi(|x-y|)}{|x-y|^n} dy \\
&\leqslant \sum_{k=0}^{\infty} \left[\Phi\left(\frac{r}{2^{k+1}}\right)\right]^{-1} \left[\eta_{p,\Psi} f\left(\frac{r}{2^k}\right)\right]^p. \qquad (9)
\end{aligned}$$

Combining the inequality (9) and Theorem 2, we get

$$\begin{aligned}
\int_{|x-y|<r} \frac{|f(y)|^p \Psi(|x-y|)}{|x-y|^n \Phi(|x-y|)}\, dy &\leqslant c(n,p) \sum_{k=0}^{\infty} \left[\Phi\left(\frac{r}{2^{k+2}}\right)\right]^{-1} \left[\eta_{p,\Psi} f\left(\frac{r}{2^{k+2}}\right)\right]^p \\
&\leqslant c(n,p) \sum_{k=0}^{\infty} \int_{\frac{r}{2^{k+2}}}^{\frac{r}{2^{k+1}}} \left[\eta_{p,\Psi} f(t)\right]^p [\Phi(t)]^{-1} t^{-1} dt \\
&= c(n,p) \int_0^{\frac{r}{2}} \left[\eta_{p,\Psi} f(t)\right]^p [\Phi(t)]^{-1} t^{-1} dt,
\end{aligned}$$

which is the desired inequality. □

REMARK 1. If we put

$$\Theta(r) := \int_0^{\frac{r}{2}} \left[\eta_{p,\Psi} f(t)\right]^p [\Phi(t)]^{-1} t^{-1} dt, \quad r > 0,$$

then we see that $\Theta(r)$ is nondecreasing and that $\lim_{r \to 0^+} \Theta(r) = 0$.

For $f \in S_{p,\Psi}$, the function $\eta_{p,\Psi} f$ is continuous according to Theorem 1. Moreover, it is nondecreasing and $\lim_{t \to 0^+} \eta_{p,\Psi} f(t) = 0$. Accordingly, we have the next corollary which generalizes the result in [13, p. 58].

COROLLARY 1. *Let $1 \leqslant p < \infty$. If $f \in S_{p,\Psi}$ and*

$$\int_0^1 \left[\eta_{p,\Psi} f(t)\right]^{p-\vartheta} t^{-1} dt < \infty,$$

for some $\vartheta \in (0,1)$, then there exists a positive constant $c(n,p)$ independent of f such that for each $x \in \mathbb{R}^n$ and $r > 0$, we have

$$\int_{|x-y|<r} \frac{|f(y)|^p \Psi(|x-y|)}{|x-y|^n \left[\eta_{p,\Psi} f(|x-y|)\right]^{\vartheta}}\, dy \leqslant c(n,p) \Theta(r),$$

where $\Theta(r) := \int_0^{\frac{r}{2}} \left[\eta_{p,\Psi} f(t)\right]^{p-\vartheta} t^{-1} dt$.

3. Inclusion between generalized Stummel classes

In this section, we are going to investigate the inclusion between two Stummel classes. The first theorem discusses the relationship between Stummel classes with different parameters Ψ. (Unless otherwise stated, we always assume that $1 \leqslant p < \infty$.)

THEOREM 4. *Suppose that Ψ_2 satisfies the condition (3) and that there exist $c > 0$ and $\delta > 0$ such that $\Psi_2(t) \leqslant c\Psi_1(t)$ for every $t \in (0,\delta)$. Then $S_{p,\Psi_1} \subseteq S_{p,\Psi_2}$.*

Proof. Let $f \in S_{p,\Psi_1}$, $x \in \mathbb{R}^n$, and $r > 0$. For $r \leqslant \delta$, we have

$$\left(\int_{|y-x|<r} \frac{|f(y)|^p \Psi_2(|y-x|)}{|y-x|^n} dy\right)^{\frac{1}{p}} \leqslant c^{\frac{1}{p}} \left(\int_{|y-x|<r} \frac{|f(y)|^p \Psi_1(|y-x|)}{|y-x|^n} dy\right)^{\frac{1}{p}},$$

whence $\eta_{p,\Psi_2} f(r) \leqslant c^{\frac{1}{p}} \eta_{p,\Psi_1} f(r) \searrow 0$ for $r \searrow 0$. Hence $f \in S_{p,\Psi_2}$. □

As an immediate consequence of Theorem 4, we have the following corollary.

COROLLARY 2. *If $0 < \alpha \leqslant \beta < n$, then $S_{p,\alpha} \subseteq S_{p,\beta}$.*

REMARK 2. For $0 < \alpha < \beta < n$, the above inclusion is proper. Indeed, for $0 < \beta < n$, define $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ by the formula

$$f(y) := \left(\frac{\chi_B(y)}{|y|^\beta \, |\ln|y||^2}\right)^{\frac{1}{p}}, \quad y \in \mathbb{R}^n,$$

where $B := B(0, e^{-\frac{2}{\beta}})$. Then $f \in S_{p,\beta} \backslash S_{p,\alpha}$ whenever $0 < \alpha < \beta$. The fact that $f \in S_{p,\beta}$ is proved in general in Example 1. We will show here that $f \notin S_{p,\alpha}$. Let $0 < r < e^{-\frac{2}{\beta}}$. Using polar coordinates and the fact that $1/(t^\beta |\ln(t)|^2)$ is decreasing on $(0, e^{-\frac{2}{\beta}})$, we have

$$\begin{aligned}
(\eta_{p,\beta} f(r))^p &\geqslant \int_{|y|<r} \frac{1}{|y|^\beta \, |\ln|y||^2 |y|^{n-\alpha}} dy \\
&\geqslant \frac{1}{r^\beta |\ln(r)|^2} \int_{|y|<r} \frac{1}{|y|^{n-\alpha}} dy \\
&= c(n,\alpha) \frac{1}{r^{\beta-\alpha} |\ln(r)|^2}.
\end{aligned}$$

Therefore

$$\eta_{p,\beta} f(r) \geqslant c(n,\alpha,p) \left(\frac{1}{r^{\beta-\alpha} |\ln(r)|^2}\right)^{\frac{1}{p}} \to \infty,$$

for $r \searrow 0$. This verifies that $f \notin S_{p,\alpha}$.

The next theorem shows the relationship between two Stummel classes with different parameters p.

THEOREM 5. *If* $1 \leqslant p_2 \leqslant p_1 < \infty$ *and* Ψ *satisfies (1), then* $S_{p_1,\Psi} \subseteq S_{p_2,\Psi}$.

Proof. Let $f \in S_{p_1,\Psi}$, $x \in \mathbb{R}^n$, and $0 < r \leqslant 1$. Then by Hölder's inequality we have

$$\begin{aligned}\int_{|y-x|<r} \frac{|f(y)|^{p_2}\Psi(|y-x|)}{|y-x|^n}\,dy &\leqslant \left(\int_{|y-x|<r} \frac{|f(y)|^{p_1}\Psi(|y-x|)}{|y-x|^n}\,dy\right)^{\frac{p_2}{p_1}} \\ &\quad \times \left(\int_{|y-x|<r} \frac{\Psi(|y-x|)}{|y-x|^n}\,dy\right)^{1-\frac{p_2}{p_1}} \\ &= \left(\int_{|y-x|<r} \frac{|f(y)|^{p_1}\Psi(|y-x|)}{|y-x|^n}\,dy\right)^{\frac{p_2}{p_1}} \\ &\quad \times \left(c(n)\int_0^r \frac{\Psi(t)}{t}\,dt\right)^{1-\frac{p_2}{p_1}}.\end{aligned}$$

Therefore

$$\eta_{p_2,\Psi} f(r) \leqslant c(n,p_1,p_2)\,\eta_{p_1,\Psi} f(r) \left(\int_0^r \frac{\Psi(t)}{t}\,dt\right)^{\frac{1}{p_2}-\frac{1}{p_1}} \searrow 0 \quad \text{for} \quad r \searrow 0,$$

which tells us that $f \in S_{p_2,\Psi}$. We conclude that $S_{p_1,\Psi} \subseteq S_{p_2,\Psi}$. $\square$

As a consequence of Theorem 5, we have the following corollary.

COROLLARY 3. *If* $1 \leqslant p_2 \leqslant p_1 < \infty$, *then* $S_{p_1,\alpha} \subseteq S_{p_2,\alpha}$.

REMARK 3. For $1 \leqslant p_2 < p_1 < \infty$, the above inclusion is proper. Indeed, for $\frac{\alpha}{p_1} < \gamma < \min\{\frac{\alpha}{p_2}, \frac{n}{p_1}\}$, we have $f(y) := |y|^{-\gamma} \in S_{p_2,\alpha} \backslash S_{p_1,\alpha}$.

From Theorem 4 and Theorem 5, we get the following corollary.

COROLLARY 4. *Suppose that* $1 \leqslant p_2 \leqslant p_1 < \infty$, Ψ_2 *satisfies the conditions (1) and (3), and there exist* $c > 0$ *and* $\delta > 0$ *such that* $\Psi_2(t) \leqslant c\Psi_1(t)$ *for every* $t \in (0,\delta)$. *Then* $S_{p_1,\Psi_1} \subseteq S_{p_2,\Psi_2}$.

4. Inclusion between Stummel classes and Morrey spaces

Our next theorem gives an inclusion relation between generalized Morrey spaces and generalized Stummel classes. We also give an example of a function that belongs to the generalized Stummel class but not to the generalized Morrey space.

THEOREM 6. *Let* $1 \leqslant p_2 \leqslant p_1 < \infty$. *Assume that* Ψ_1 *satisfies* (2) *and that* Ψ_2 *satisfies the right-doubling condition* (4). *If*

$$\int_0^1 \frac{\Psi_1(t)^{p_2}\Psi_2(t)}{t}\,dt < \infty, \tag{10}$$

then $L^{p_1,\Psi_1} \subseteq S_{p_2,\Psi_2}$.

REMARK 4. Let $p_1 = p_2 = 1$, $\Psi_1(t) := t^{\lambda-n}$ where $0 \leqslant \lambda \leqslant n$, and $\Psi_2(t) := t^{\alpha}$ where $n-\lambda < \alpha < n$. Then the above theorem reduces to the result in [13, p. 56].

Proof of Theorem 6. Let $f \in L^{p_1,\Psi_1}$, $x \in \mathbb{R}^n$, and $r > 0$. Since Ψ_2 satisfies (4), we have

$$\int_{|x-y|<r} \frac{|f(y)|^{p_2}\Psi_2(|x-y|)}{|x-y|^n}\,dy = \sum_{k=-\infty}^{-1} \int_{2^k r \leqslant |x-y| < 2^{k+1} r} \frac{|f(y)|^{p_2}\Psi_2(|x-y|)}{|x-y|^n}\,dy$$
$$\leqslant c \sum_{k=-\infty}^{-1} \frac{\Psi_2(2^k r)}{|B(x,2^{k+1}r)|} \int_{B(x,2^{k+1}r)} |f(y)|^{p_2}\,dy.$$

Combining the last inequality and Hölder's inequality, we get

$$\int_{|x-y|<r} \frac{|f(y)|^{p_2}\Psi_2(|x-y|)}{|x-y|^n}\,dy \leqslant c \sum_{k=-\infty}^{-1} \frac{\Psi_2(2^k r)}{|B(x,2^{k+1}r)|^{p_2/p_1}} \|f\|^{p_2}_{L^{p_1}(B(x,2^{k+1}r))}$$
$$\leqslant c\,\|f\|^{p_2}_{L^{p_1,\Psi_1}} \sum_{k=-\infty}^{-1} \Psi_1(2^{k+1}r)^{p_2}\Psi_2(2^k r). \tag{11}$$

Using (4) and the monotonicity of Ψ_1, we get

$$\sum_{k=-\infty}^{-1} \Psi_1(2^{k+1}r)^{p_2}\Psi_2(2^k r) \leqslant c \sum_{k=-\infty}^{-1} \int_{2^{k-1}r}^{2^k r} \frac{\Psi_1(t)^{p_2}\Psi_2(t)}{t}\,dt$$
$$= c \int_0^{r/2} \frac{\Psi_1(t)^{p_2}\Psi_2(t)}{t}\,dt. \tag{12}$$

We combine (11) and (12) to obtain

$$\eta_{p_2,\Psi_2} f(r) \leqslant c \left(\int_0^{r/2} \frac{\Psi_1(t)^{p_2}\Psi_2(t)}{t}\,dt \right)^{\frac{1}{p_2}} \|f\|_{L^{p_1,\Psi_1}}. \tag{13}$$

Since $\displaystyle\int_0^1 \frac{\Psi_1(t)^{p_2}\Psi_2(t)}{t}\,dt < \infty$, we see that $\displaystyle\lim_{r\to 0^+} \int_0^{r/2} \frac{\Psi_1(t)^{p_2}\Psi_2(t)}{t}\,dt = 0$. This fact and (13) imply $\displaystyle\lim_{r\to 0^+} \eta_{p_2,\Psi_2} f(r) = 0$. Hence, $f \in S_{p_2,\Psi_2}$. This shows that $L^{p_1,\Psi_1} \subseteq S_{p_2,\Psi_2}$. $\square$

The following example shows that the inclusion in Theorem 6 is proper.

EXAMPLE 1. Let $1 \leqslant p_2 \leqslant p_1 < \infty$, Ψ_2 satisfy the condition (4), $\Psi_2(t)\,|\ln(t)|^2$ be nondecreasing on $(0,\delta)$ for some $\delta > 0$, and $\Psi_1(r)^{p_2}\Psi_2(r)\,|\ln(r)|^2 \searrow 0$ as $r \searrow 0$. Define $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ by the formula

$$f(y) := \left(\frac{\chi_B(y)}{\Psi_2(|y|)\,|\ln|y||^2} \right)^{\frac{1}{p_2}}, \quad y \in \mathbb{R}^n,$$

where $B := B(0,\delta)$. Then $f \in S_{p_2,\Psi_2} \backslash L^{p_1,\Psi_1}$.

First we show that $f \in S_{p_2,\Psi_2}$. Let $0 < r < \min\{\delta, 1\}$. Since the function f is radial and nonincreasing, the supremum in the Stummel modulus is attained at the origin, so that

$$\eta_{p_2,\Psi_2} f(r) = \left(\int_{|y|<r} \frac{|f(y)|^{p_2} \Psi_2(|y|)}{|y|^n} \, dy \right)^{\frac{1}{p_2}} = \left(\int_{|y|<r} \frac{1}{|\ln|y||^2 |y|^n} \, dy \right)^{\frac{1}{p_2}}.$$

Converting to polar coordinates, we get

$$\int_{|y|<r} \frac{1}{|\ln|y||^2 |y|^n} \, dy = c \int_0^r \frac{1}{s(\ln s)^2} \, ds = -\frac{c}{\ln r}.$$

Therefore,

$$\eta_{p_2,\Psi_2} f(r) = c \left(-\frac{1}{\ln r} \right)^{\frac{1}{p_2}}.$$

Since $\lim_{r \to 0^+} \frac{1}{\ln r} = 0$, we conclude that $\eta_{p_2,\Psi_2} f(r) \searrow 0$ for $r \searrow 0$. This proves that $f \in S_{p_2,\Psi_2}$.

Now, we will show that $f \notin L^{p_1,\Psi_1}$. Let $0 < r < \delta$. Since $\Psi_2(t)\,|\ln(t)|^2$ is nondecreasing on $(0,r) \subseteq (0,\delta)$, we have

$$\begin{aligned}
\frac{1}{\Psi_1(r)^{p_1}} \frac{1}{|B(0,r)|} \int_{B(0,r)} |f(y)|^{p_1} \, dy &= \frac{1}{\Psi_1(r)^{p_1}} \frac{1}{|B(0,r)|} \int_{B(0,r)} \left(\frac{1}{\Psi_2(|y|)\,|\ln|y||^2} \right)^{\frac{p_1}{p_2}} dy \\
&\geqslant \frac{1}{\Psi_1(r)^{p_1} |B(0,r)|} \left(\frac{1}{\Psi_2(r)\,|\ln r|^2} \right)^{\frac{p_1}{p_2}} \int_{B(0,r)} dy \\
&= \left(\frac{1}{\Psi_1(r)^{p_2} \Psi_2(r)\,|\ln r|^2} \right)^{\frac{p_1}{p_2}}.
\end{aligned}$$

Note that $\Psi_1(r)^{p_2} \Psi_2(r)\,|\ln r|^2 \searrow 0$ as $r \searrow 0$. Then

$$\left(\frac{1}{\Psi_1(r)^{p_2} \Psi_2(r)\,|\ln r|^2} \right)^{\frac{p_1}{p_2}} \to \infty \quad \text{for} \quad r \searrow 0.$$

We conclude that $f \notin L^{p_1,\Psi_1}$.

REMARK 5. Let $1 \leqslant p_2 \leqslant p_1 < \infty$, $\Psi_1(t) := t^{\frac{\lambda-n}{p_1}}$ where $0 \leqslant \lambda \leqslant n$, and $\Psi_2(t) := t^\alpha$ where $\frac{(n-\lambda)p_2}{p_1} < \alpha < n$. It can be shown that Ψ_1 and Ψ_2 satisfy all conditions in Theorem 6 and Example 1.

As a counterpart of Theorem 6, we have the following result.

THEOREM 7. *Let $1 \leqslant p_2 \leqslant p_1 < \infty$ and assume that Ψ_1 satisfies (3). If $f \in S_{p_1,\Psi_1}$ and*

$$\eta_{p_1,\Psi_1} f(r) \leqslant c \Psi_1(r)^{\frac{1}{p_1}} \Psi_2(r) \tag{14}$$

for some Ψ_2 and for every $r > 0$, then $f \in L^{p_2,\Psi_2}$.

Proof. Let $a \in \mathbb{R}^n$ and $r > 0$. Then, by Hölder's inequality, we have

$$\int_{B(a,r)} |f(x)|^{p_2}\, dx \leqslant c\, r^{n\left(1-\frac{p_2}{p_1}\right)} \left(\int_{B(a,r)} |f(x)|^{p_1}\, dx\right)^{\frac{p_2}{p_1}}$$
$$= \frac{c\, r^n}{\Psi_1(r)^{\frac{p_2}{p_1}}} \left(\int_{B(a,r)} \frac{|f(x)|^{p_1}\Psi_1(r)}{r^n}\, dx\right)^{\frac{p_2}{p_1}}.$$

We combine (3), (14), and Definition 1 to obtain

$$\int_{B(a,r)} |f(x)|^{p_2}\, dx \leqslant \frac{c\, r^n}{\Psi_1(r)^{\frac{p_2}{p_1}}} \left(\int_{B(a,r)} \frac{|f(x)|^{p_1}\Psi_1(|x-a|)}{|x-a|^n}\, dx\right)^{\frac{p_2}{p_1}}$$
$$\leqslant \frac{c\, r^n}{\Psi_1(r)^{\frac{p_2}{p_1}}} [\eta_{p_1,\Psi_1} f(r)]^{p_2} \leqslant c\, r^n \Psi_2(r)^{p_2}.$$

Consequently,

$$\frac{1}{|B(a,r)|\Psi_2(r)} \left(\int_{B(a,r)} |f(x)|^{p_2}\, dx\right)^{\frac{1}{p_2}} \leqslant c.$$

Since a and r are arbitrary, we conclude that $f \in L^{p_2,\Psi_2}$. □

Taking $\Psi_1(t) := t^{\alpha}$ and $\Psi_2(t) := t^{\frac{\sigma}{p_2}-\frac{\alpha}{p_1}}$ where $0 < \alpha < n$, $1 \leqslant p_2 \leqslant p_1 < \infty$, and $0 < \sigma < \frac{\alpha p_2}{p_1}$, we get the following corollary.

COROLLARY 5. *Let $1 \leqslant p_2 \leqslant p_1 < \infty$ and $0 < \alpha < n$. If $f \in S_{p_1,\alpha}$ and $\eta_{p_1,\alpha} f(r) \leqslant c r^{\frac{\sigma}{p_2}}$ for some $0 < \sigma < \frac{\alpha p_2}{p_1}$ and for every $r > 0$, then $f \in L^{p_2, n+\sigma-\frac{\alpha p_2}{p_1}}$.*

Next, we are going to investigate the relation between generalized Stummel classes and generalized weak Morrey spaces. The generalized weak Morrey spaces are defined as follows.

DEFINITION 2. Let $1 \leqslant p < \infty$ and $\Psi : (0,\infty) \to (0,\infty)$. The **generalized weak Morrey space** $wL^{p,\Psi} = wL^{p,\Psi}(\mathbb{R}^n)$ is defined to be the set of all measurable functions f for which

$$\|f\|_{wL^{p,\Psi}} := \sup_{a\in\mathbb{R}^n,\, r>0,\, t>0} \frac{t|\{x \in B(a,r) : |f(x)| > t\}|^{1/p}}{\Psi(r)|B(a,r)|^{1/p}} < \infty$$

The inclusion between generalized Stummel classes and generalized weak Morrey spaces is given in the following theorems.

THEOREM 8. *Let $1 \leqslant p_2 < p_1 < \infty$. Assume that Ψ_1 satisfies (2) and that Ψ_2 satisfies (4). If*

$$\int_0^1 \frac{\Psi_1(t)^{p_2}\Psi_2(t)}{t}\, dt < \infty,$$

then $wL^{p_1,\Psi_1} \subseteq S_{p_2,\Psi_2}$.

Proof. Since $p_2 < p_1$, by virtue of [9, Theorem 5.1], we have $wL^{p_1,\Psi_1} \subseteq L^{p_2,\Psi_1}$. By Theorem 6, we have $L^{p_2,\Psi_1} \subseteq S_{p_2,\Psi_2}$. It thus follows that $wL^{p_1,\Psi_1} \subseteq S_{p_2,\Psi_2}$. □

THEOREM 9. *Let $1 \leqslant p_1 \leqslant p_2 < \infty$ and assume that Ψ_1 satisfies (3). If $f \in S_{p_1,\Psi_1}$ and the inequality (14) holds for some Ψ_2 and for every $r > 0$, then $f \in wL^{p_2,\Psi_2}$.*

Proof. The assertion follows from Theorem 7 and the inclusion $L^{p_2,\Psi_2} \subseteq wL^{p_2,\Psi_2}$. □

For the classical weak Morrey spaces and Stummel classes, we have the following result.

THEOREM 10. *For $1 \leqslant p_2 < p_1 < \infty$, if $0 \leqslant \lambda < n$ and $\frac{(n-\lambda)p_2}{p_1} < \alpha < n$, then $wL^{p_1,\lambda} \subseteq S_{p_2,\alpha}$. Conversely, for $1 \leqslant p < \infty$, if $f \in S_{p,\alpha}$ for $0 < \alpha < n$ and $\eta_{p,\alpha} f(r) \leqslant cr^{\frac{\sigma}{p}}$ for some $\sigma > 0$, then $f \in wL^{p,n-\alpha+\sigma}$.*

Proof. The first assertion follows from Theorem 8 by taking $\Psi_1(t) := t^{\frac{\lambda-n}{p_1}}$, and $\Psi_2(t) := t^{\alpha}$ where $0 \leqslant \lambda < n$ and $\frac{(n-\lambda)p_2}{p_1} < \alpha < n$. The second part is a consequence of Corollary 5 when $p_1 = p_2 = p$ and the inclusion $L^{p,n-\alpha+\sigma} \subseteq wL^{p,n-\alpha+\sigma}$. □

REMARK 6. The second part of Theorem 10 generalizes the result in [13, p. 57]. For the case $p = 1$, the first part of Theorem 10 does not generally hold. To see this, consider the function $f(y) := |y|^{-n}$, $y \in \mathbb{R}^n$. Then $f \in wL^{1,\lambda}$ for $0 \leqslant \lambda < n$, but $f \notin S_{\alpha}$ for $n - \lambda < \alpha < n$.

5. Further results

In this section, we study the relation between bounded Stummel modulus classes $\tilde{S}_{p,\alpha}$ and Stummel classes. We also study the inclusion between $\tilde{S}_{p,\alpha}$ and Lorentz spaces. For $0 < \alpha < n$ and $1 \leqslant p < \infty$, recall the definition of the Stummel modulus

$$\eta_{p,\alpha} f(r) := \sup_{x \in \mathbb{R}^n} \left(\int_{|x-y|<r} \frac{|f(y)|^p}{|x-y|^{n-\alpha}}\, dy \right)^{\frac{1}{p}}, \quad r > 0.$$

DEFINITION 3. For $0 < \alpha < n$ and $1 \leqslant p < \infty$, we define **the bounded Stummel modulus class** $\tilde{S}_{p,\alpha} = \tilde{S}_{p,\alpha}(\mathbb{R}^n)$ by

$$\tilde{S}_{p,\alpha} := \left\{ f \in L^p_{\mathrm{loc}}(\mathbb{R}^n) : \eta_{p,\alpha} f(r) < \infty \quad \text{for all} \quad r > 0 \right\}.$$

Note that the inclusions similar to Corollary 2 and Corollary 3 also hold for $\tilde{S}_{p,\alpha}$. Moreover, we have $S_{p,\alpha} \subseteq \tilde{S}_{p,\alpha}$. This inclusion is proper due to the following example which we adapt from [1, p. 250–251].

EXAMPLE 2. Let $0 < \alpha < n$ and $1 \leqslant p < \infty$. For every $k \in \mathbb{N}$ with $k \geqslant 3$, let $x_k := (2^{-k}, 0, \ldots, 0) \in \mathbb{R}^n$ and

$$V_k(y) := \begin{cases} 8^{\alpha k}, & y \in B(x_k, 8^{-k}), \\ 0, & y \notin B(x_k, 8^{-k}). \end{cases}$$

Define $V : \mathbb{R}^n \to \mathbb{R}$ by the formula

$$V(y) := \left(\sum_{k=3}^{\infty} V_k(y) \right)^{\frac{1}{p}}.$$

Since

$$\int_{B(x,r)} |V(y)|^p \, dy = \sum_{k=3}^{\infty} \int_{B(x,r)} |V_k(y)| \, dy \leqslant c(n) \sum_{k=3}^{\infty} 8^{(\alpha-n)k} < \infty$$

for every $x \in \mathbb{R}^n$ and $r > 0$ where $c(n) := |B(0,1)|$, we obtain $V \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$.

We will show that $V \in \tilde{S}_{p,\alpha}$. Let

$$\rho_k(x) := \int_{\mathbb{R}^n} \frac{|V_k(y)|}{|x-y|^{n-\alpha}} \, dy = 8^{\alpha k} \int_{|y-x_k|<8^{-k}} \frac{1}{|x-y|^{n-\alpha}} \, dy, \quad x \in \mathbb{R}^n.$$

There are two cases: (i) $|x - x_k| \geqslant 2^{-2k+1}$, or, (ii) $|x - x_k| < 2^{-2k+1}$.

Suppose that the case (i) holds, that is, $|x - x_k| \geqslant 2^{-2k+1}$. We have,

$$\rho_k(x) \leqslant c(n) 2^{(\alpha-n)k}. \tag{15}$$

For the case (ii) $|x - x_k| < 2^{-2k+1}$, we have

$$\rho_k(x) \leqslant c(n,\alpha) \tag{16}$$

where $c(n,\alpha) := \max\{c(n), \frac{3^\alpha}{\alpha} c(n)\}$.

Given $x \in \mathbb{R}^n$, we have $x \notin B(x_k, 2^{-2k+1})$ for all $k \geqslant 3$, or $x \in B(x_j, 2^{-2j+1})$ for some $j \geqslant 3$. Assume that $x \notin B(x_k, 2^{-2k+1})$ for all $k \geqslant 3$. Hence, from (15), we have

$$\int_{\mathbb{R}^n} \frac{|V(y)|^p}{|x-y|^{n-\alpha}} \, dy \leqslant \sum_{k=3}^{\infty} \rho_k(x) \leqslant c(n) \sum_{k=3}^{\infty} 2^{(\alpha-n)k} < \infty. \tag{17}$$

Now assume that $x \in B(x_j, 2^{-2j+1})$ for some $j \geqslant 3$. Since $\left\{B(x_k, 2^{-2k+1})\right\}_{k \geqslant 3}$ is a disjoint collection, we find that there is only one $j \in \mathbb{N}$, $j \geqslant 3$, such that $x \in B(x_j, 2^{-2j+1})$. Using (15) and (16), we get

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{|V(y)|^p}{|x-y|^{n-\alpha}} \, dy &\leqslant c(n,\alpha) + \sum_{\substack{k=3 \\ k \neq j}}^{\infty} \rho_k(x) \qquad (18) \\ &\leqslant c(n,\alpha) + c(n) \sum_{\substack{k=3 \\ k \neq j}}^{\infty} 2^{(\alpha-n)k} < \infty. \end{aligned}$$

According to (17) and (18), for every $r>0$, we have

$$\int_{|x-y|<r} \frac{|V(y)|^p}{|x-y|^{n-\alpha}}\,dy \leqslant \int_{\mathbb{R}^n} \frac{|V(y)|^p}{|x-y|^{n-\alpha}}\,dy < \infty.$$

Therefore $\eta_{p,\alpha}V(r)<\infty$, and we conclude that $V \in \tilde{S}_{p,\alpha}$.

Now, we will show that $V \notin S_{p,\alpha}$. Let $r>0$. By Archimedean property, there is $k \geqslant 3$ such that $8^{-k}<r$. Note that

$$\begin{aligned}
(\eta_{p,\alpha}V(r))^p &\geqslant \int_{|y-x_k|<r} \frac{|V(y)|^p}{|y-x_k|^{n-\alpha}}\,dy \\
&\geqslant \int_{|y-x_k|<r} \frac{|V_k(y)|}{|y-x_k|^{n-\alpha}}\,dy \\
&\geqslant 8^{\alpha k} \int_{|y-x_k|<8^{-k}} \frac{1}{|y-x_k|^{n-\alpha}}\,dy = \frac{c(n)}{\alpha}.
\end{aligned}$$

This shows that $\eta_{p,\alpha}V$ stays away from zero. Thus $V \notin S_{p,\alpha}$.

Given a measurable function $f:\mathbb{R}^n \to \mathbb{R}$, consider the *distribution function* D_f of f which is given by

$$D_f(\sigma) := |\{x \in \mathbb{R}^n : |f(x)| > \sigma\}|, \quad \sigma > 0.$$

The *decreasing rearrangement* of f is the function f^* defined on $[0,\infty)$ by

$$f^*(t) := \inf\{\sigma : D_f(\sigma) \leqslant t\}, \quad t \geqslant 0.$$

DEFINITION 4. Let $0<\kappa, p \leqslant \infty$. The **Lorentz space** $L^p_\kappa = L^p_\kappa(\mathbb{R}^n)$ is the collection of all measurable functions $f:\mathbb{R}^n \to \mathbb{R}$ satisfying $\|f\|_{L^p_\kappa} < \infty$, where

$$\|f\|_{L^p_\kappa} := \begin{cases} \left(\int_0^\infty \left(t^{\frac{1}{\kappa}} f^*(t) \right)^p \frac{dt}{t} \right)^{\frac{1}{p}}, & \text{if } p<\infty, \\ \sup\limits_{t>0} t^{\frac{1}{\kappa}} f^*(t), & \text{if } p=\infty. \end{cases}$$

Note that $L^\infty_\kappa = wL^\kappa$ for $\kappa \geqslant 1$. The following lemma is a well-known inclusion relation between Lorentz spaces (see [7, p. 49] or [12, p. 305] for its proof).

LEMMA 1. *If* $0<\kappa\leqslant\infty$ *and* $0<p_2\leqslant p_1\leqslant\infty$, *then* $L^{p_2}_\kappa \subseteq L^{p_1}_\kappa$.

Moreover, we have the following relation between Lorentz spaces and bounded Stummel modulus classes.

LEMMA 2. [2, Lemma 2.7] *Let* $0<\alpha<n$. *Then* $L^1_{\frac{n}{\alpha}} \subseteq \tilde{S}_{1,\alpha}$.

Our theorem below is an extension of Lemma 2.

THEOREM 11. *Let* $1 \leqslant p < \infty$ *and* $0 < \alpha < n$. *If* $\frac{np}{\alpha} \leqslant \kappa < \infty$, *then*

$$L^p_\kappa \subseteq \tilde{S}_{p,\alpha}.$$

Proof. We first prove the case where $\kappa = \frac{np}{\alpha}$. Let $f \in L^p_{\frac{np}{\alpha}}$. Then $|f|^p \in L^1_{\frac{n}{\alpha}}$. By virtue of Lemma 2, we have $|f|^p \in \tilde{S}_{1,\alpha}$. According to Definition 3, we see that $f \in \tilde{S}_{p,\alpha}$. Thus, we obtain $L^p_{\frac{np}{\alpha}} \subseteq \tilde{S}_{p,\alpha}$.

Let us now consider the case where $\kappa > \frac{np}{\alpha}$. Since $0 < \alpha < n$, we have $\kappa > p$. Hence by Theorem 10 (for $\lambda = 0$), we obtain $wL^\kappa \subseteq S_{p,\alpha}$. Now, combining this with Lemma 1 and the remark after Definition 3, we see that

$$L^p_\kappa \subseteq wL^\kappa \subseteq S_{p,\alpha} \subseteq \tilde{S}_{p,\alpha}.$$

This completes the proof. $\square$

REMARK 7. For $\frac{n}{\alpha} < \kappa < \infty$, we observe that $L^1_\kappa \not\subseteq L^1_{\frac{n}{\alpha}}$. To see this, one may check that $f(x) := |x|^{-\alpha}\chi_{\{x:|x|>1\}} \in L^1_\kappa \setminus L^1_{\frac{n}{\alpha}}$.

REMARK 8. It follows from Theorem 11 that, for $1 \leqslant p_2 \leqslant p_1 < \infty$ and $\frac{np_1}{\alpha} \leqslant \kappa < \infty$, the inclusion $L^{p_1}_\kappa \subseteq \tilde{S}_{p_2,\alpha}$ holds.

REMARK 9. By using the same trick as in the proof of the first part of Theorem 11, one can extend [8, Theorem 3.1] to the corresponding function spaces with parameter p instead of 1.

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