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ON MULTIVARIATE OSTROWSKI TYPE INEQUALITIES AND THEIR APPLICATIONS

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Abstract. We prove sharp Ostorowski type inequality for multivariate Sobolev classes and apply it to the problem of optimal recovery of integrals.

1. Introduction

In 1938 Ostrowski [8] proved the following theorem.

THEOREM. *Let* $f\colon [-1,1]\to\mathbb{R}$ *be a differentiable function and let for all* $t\in(-1,1)$ $|f'(t)|\leqslant 1$. *Then for all* $x\in[-1,1]$ *the following inequality holds*

$$\left|\frac{1}{2}\int\limits_{-1}^{1} f(t)dt - f(x)\right| \leqslant \frac{1+x^2}{2}.$$

The inequality is sharp in the sense that for each fixed $x\in[-1,1]$ *the upper bound* $\frac{1+x^2}{2}$ *cannot be reduced.*

This result gave rise to a special branch in the Theory of Inequalities, namely inequalities that estimate the deviation of the value of a function from its mean value with the help of some characteristics of the function. Such inequalities are now called *Ostrowski type inequalities*. They have numerous applications in Analysis, Approximation Theory, Numerical Methods and other areas, in particular, applications in analysis of numerical integration errors.

There exist a lot of results in the area of Ostrowski type inequalities. We refer the reader to monographs [7, 3, 9] and references therein.

The main goal of this article is to obtain a new sharp Ostrowski type inequality on Sobolev classes of multivariate functions and to apply the obtained inequality to the problem of optimal recovery of integrals.

The article is organized as follows. Section 2 contains necessary definitions and several auxiliary results needed throughout the paper. Ostrowski type inequality for Sobolev classes is presented in Section 3. Section 4 is devoted to optimal integration formulae: in subsection 4.2 we present sharp estimates for errors of integral optimal recovery formulae on rather simple domains; subsection 4.3 contains asymptotically sharp estimates of the errors in the case when the domain is an arbitrary convex set.

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2. Notations and some auxiliary results

Let $Q\subset\mathbb{R}^d$, $d\in\mathbb{N}$, be a nonempty, bounded, open set. By $W^{1,p}(Q)$, $p\in[1,\infty]$, we denote the Sobolev space of functions $f\colon Q\to\mathbb{R}$, such that f and all their (distributional) first order partial derivatives belong to $L_p(Q)$. For $x=(x^1,\dots,x^d)\in\mathbb{R}^d$ and $q\in[1,\infty)$, we set $|x|_q:=\left(\sum\limits_{k=1}^{d}|x^k|^q\right)^{\frac{1}{q}}$, $|x|_\infty:=\max\limits_{k=1,\dots,d}|x^k|$. It is clear that for all $f\in W^{1,p}(Q)$ we have $\|\,|\nabla f|_1\,\|_{L_p(Q)}<\infty$. For $p\in[1,\infty]$ set $W_p^\nabla(Q):=\{f\in W^{1,p}(Q)\colon \|\,|\nabla f|_1\,\|_p\leqslant 1\}$.

Everywhere below we assume $d\geqslant 2$ and $p\in(d,\infty]$.

Definition 1. We call a nonempty bounded open set $Q\subset\mathbb{R}^d$ *admissible*, if there exists an embedding of the class $W^{1,p}(Q)$ into the space of bounded continuous on Q functions.

The family of admissible sets Q is rather large. For example, all sets Q, that satisfy the so-called cone condition (see Chapter 4 and Theorem 4.12 of [1]) are admissible (for all $p>d$). If Q is admissible, then the values $f(x)$, $x\in Q$, of the functions $f\in W^{1,p}(Q)$ are well defined.

For $x,y\in\mathbb{R}^d$ by (x,y) we denote the dot product of elements x and y. We need the following theorem, which follows from the results proved in Chapter 6.9 of [6].

Theorem 1. *Suppose $Q\subset\mathbb{R}^d$ is admissible. Let $f\in W^{1,p}(Q)$ and $x,y\in Q$ be such that the line segment connecting points x and y also belongs to Q. Then*

$$f(y)-f(x)=\int\limits_0^1(y-x,\nabla f[(1-t)x+ty])\,dt.$$

Everywhere below, for $h>0$ we set $\square_h^d:=\{x\in\mathbb{R}^d\colon |x|_\infty<h\}$; p' is such, that $\frac{1}{p}+\frac{1}{p'}=1$. It is easy to see that the set $\square_h^d$ is admissible.

We need the following analogue of the theorem about integration in spherical coordinate system.

Theorem 2. *Let $f:\square_h^d\to\mathbb{R}$ be an integrable function. Then*

$$\int\limits_{\square_h^d}f(x)dx=\int\limits_0^h\rho^{d-1}\int\limits_{\partial\square_1^d}f(\rho y)\,dy\,d\rho.$$

Proof. Set $D_j^\pm:=\{(x_1,\dots,x_d)\in\mathbb{R}^d\colon\ \pm x_j\in[0,1],\ 0\leqslant|x_k|\leqslant\pm x_j,\ k\neq j\}$, $j=1,\dots,d$. It is sufficient to prove that for each $j=1,\dots,d$

$$\int\limits_{D_j^\pm}f(x)dx=\int\limits_0^h\rho^{d-1}\int\limits_{z=(z_1,\dots,z_d)\in\square_1^d,\ z_j=\pm1}f(\rho z)\,dz\,d\rho$$

and then sum all these equalities. For the set D_1^+ the desired equality is proved as follows:

$$\int_{D_1^+} f(x)dx = \int_0^h \int_{[-\rho,\rho]^{d-1}} f(\rho,y)dyd\rho = \int_0^h \rho^{d-1} \int_{[-1,1]^{d-1}} f(\rho\cdot(1,z))dzd\rho.$$

For all other j the proof is similar. The theorem is proved. $\square$

3. Ostrowski type inequality

We need the following lemma.

LEMMA 1. *Let $d\in\mathbb{N}$ and $p\in(d,\infty]$. Then*

$$\frac{1}{|\cdot|_\infty^{d-1}} \in L_{p'}(\square_1^d).$$

Proof. By assumption on p and definition of p', we obtain that $\alpha := p'\cdot(d-1) < d$. Using Theorem 2 and taking into account that $|\rho z|_\infty = \rho$ for all $z\in\partial\square_1^d$, we obtain

$$\int_{\square_1^d} \frac{dy}{|y|_\infty^\alpha} = \int_0^1 \rho^{d-1} \int_{\partial\square_1^d} \frac{dzd\rho}{|\rho z|_\infty^\alpha} = |\partial\square_1^d| \int_0^1 \rho^{d-1-\alpha}d\rho < \infty,$$

since $d-1-\alpha > -1$. The lemma is proved. $\square$

The following theorem gives an Ostrowski type inequality for functions from Sobolev class $W^{1,p}(\square_h^d)$.

THEOREM 3. *Let $p\in(d,\infty]$ and $f\in W^{1,p}(\square_h^d)$. Then*

$$\left|\int_{\square_h^d} f(y)dy - (2h)^d f(0)\right| \leqslant c(d,p)\cdot h^{1+\frac{d}{p'}} \|\,|\nabla f|_1\,\|_{L_p(\square_h^d)}, \tag{1}$$

where $c(d,p) := \frac{1}{d}\left\|\frac{1}{|\cdot|_\infty^{d-1}} - |\cdot|_\infty\right\|_{L_{p'}(\square_1^d)}$. The inequality is sharp. Equality occurs for the function

$$f_e(y) = f_{e,h}(y) = \int_0^{|y|_\infty} \left|\frac{h^{d-1}}{u^{d-1}} - \frac{u}{h}\right|^{p'-1} du. \tag{2}$$

REMARK 1. The constant $c(d,p)$ in (1) is finite due to Lemma 1.

Proof. We prove the theorem in three steps.
Step 1. First, we prove that inequality (1) is true for all $f\in W^{1,p}(\square_h^d)$. Indeed,

$$\left|\int\limits_{\square_h^d} f(y)dy-(2h)^d f(0)\right| = \left|\int\limits_{\square_h^d}[f(y)-f(0)]dy\right|$$

$$\text{(by Theorem 1)} = \left|\int\limits_{\square_h^d}\int\limits_0^1 (y,\nabla f(ty))dtdy\right| \leqslant \int\limits_{\square_h^d}\int\limits_0^1 |y|_\infty |\nabla f(ty)|_1 dtdy \tag{3}$$

$$\text{(by Theorem 2)} = \int\limits_{\partial\square_1^d}\int\limits_0^h\int\limits_0^1 \rho^{d-1}|\rho z|_\infty|\nabla f(\rho t z)|_1 dtd\rho dz$$

(substitution $s=\rho t$; note that $|\rho z|_\infty=\rho$ for all $z\in\partial\square_1^d$)

$$= \int\limits_{\partial\square_1^d}\int\limits_0^h\int\limits_0^\rho \rho^{d-1}|\nabla f(sz)|_1 dsd\rho dz$$

(change the order of integration in two internal integrals)

$$= \int\limits_{\partial\square_1^d}\int\limits_0^h |\nabla f(sz)|_1\int\limits_s^h \rho^{d-1}d\rho dsdz = \frac{1}{d}\int\limits_{\partial\square_1^d}\int\limits_0^h |\nabla f(sz)|_1(h^d-s^d)dsdz$$

$$= \frac{h}{d}\int\limits_0^h s^{d-1}\int\limits_{\partial\square_1^d}|\nabla f(sz)|_1\left(\frac{h^{d-1}}{|sz|_\infty^{d-1}}-\frac{|sz|_\infty}{h}\right)dzds$$

$$\text{(by Theorem 2)} = \frac{h}{d}\int\limits_{\square_h^d}|\nabla f(y)|_1\left(\frac{h^{d-1}}{|y|_\infty^{d-1}}-\frac{|y|_\infty}{h}\right)dy$$

$$\text{(by Holder's inequality)} \leqslant \frac{h}{d}\|\,|\nabla f|_1\,\|_{L_p(\square_h^d)}\left\|\frac{h^{d-1}}{|\cdot|_\infty^{d-1}}-\frac{|\cdot|_\infty}{h}\right\|_{L_{p'}(\square_h^d)}. \tag{4}$$

Taking into account that

$$\left\|\frac{h^{d-1}}{|\cdot|_\infty^{d-1}}-\frac{|\cdot|_\infty}{h}\right\|_{L_{p'}(\square_h^d)}^{p'} = \int\limits_{\square_h^d}\left[\frac{h^{d-1}}{|y|_\infty^{d-1}}-\frac{|y|_\infty}{h}\right]^{p'}dy$$

$$= h^d\int\limits_{\square_1^d}\left[\frac{1}{|x|_\infty^{d-1}}-|x|_\infty\right]^{p'}dx = \left[h^{\frac{d}{p'}}\left\|\frac{1}{|\cdot|_\infty^{d-1}}-|\cdot|_\infty\right\|_{L_{p'}(\square_1^d)}\right]^{p'}$$

we obtain inequality (1).

Step 2. Next, we prove that the function f_e, defined by (2), belongs to the class $W^{1,p}(\square_h^d)$.

First of all, note that since $p > d$ we have $(p'-1)(d-1) = \frac{d-1}{p-1} < 1$ and hence the function $g(u) := \left(\frac{h^{d-1}}{u^{d-1}} - \frac{u}{h}\right)^{p'-1}$ is integrable on all intervals $[0,t]$, $0 < t \leqslant h$.

The function $f_e(y)$ is constant on each of subsets $|y|_\infty = const$. Moreover, if $y = (y_1,\ldots,y_d)$, then for all $k = 1,\ldots,d$ inside the regions

$$|y_k| > \max\{|y_1|,\ldots,|y_{k-1}|,|y_{k+1}|,\ldots,|y_d|\}, \tag{5}$$

we have

$$\frac{\partial f_e}{\partial y_s} = 0, \quad s \neq k, \qquad \text{and} \qquad \frac{\partial f_e}{\partial y_k} = \operatorname{sgn} y_k \cdot g(|y_k|). \tag{6}$$

Hence, we obtain that almost everywhere on $\square_h^d$

$$|\nabla f_e(y)|_1 = g(|y|_\infty). \tag{7}$$

Since $(p'-1)p = p'$, from Lemma 1 it follows that the integral $\int\limits_{\square_h^d} g^p(|y|_\infty)dy$ is finite and, hence, $f \in W^{1,p}(\square_h^d)$.

Step 3. Finally, we prove that inequality (1) becomes equality for the functions f_e defined by (2). In order to do so, we take a closer look at all the places with inequalities in the proof of (1) and show that in each place equalities hold for the function f_e.

Inside regions (5) we have (6), so we obtain that $(y, \nabla f_e(ty)) = |y_k| \cdot g(t|y_k|) = |y|_\infty \cdot |\nabla f_e(ty)|_1 \geqslant 0$ for all $t \in [0,1]$, hence, inequality (3) becomes equality for f_e. Furthermore, (4) also turns into equality, since from (7) we obtain

$$|\nabla f_e(y)|_1^p = g^p(|y|_\infty) = \left(\frac{h^{d-1}}{|y|_\infty^{d-1}} - \frac{|y|_\infty}{h}\right)^{(p'-1)p} = \left(\frac{h^{d-1}}{|y|_\infty^{d-1}} - \frac{|y|_\infty}{h}\right)^{p'}.$$

The theorem is proved. $\square$

4. On optimal quadrature

4.1. Statement of the problem and extremal functions

In what follows X denotes a class of continuous functions defined on a bounded measurable subset Q of $\mathbb{R}^n$. By a *method of recovery* we shall mean any function $\Phi\colon \mathbb{R}^n \to \mathbb{R}$. For given points $x_1,\ldots,x_n \in Q$, the *error of recovery* of the integral using information $f(x_1),...,f(x_n)$ by the method Φ is defined by the following equality

$$e(X,\Phi,x_1,\ldots,x_n) := \sup_{f\in X} \left| \int\limits_Q f(x)dx - \Phi(f(x_1),\ldots,f(x_n)) \right|.$$

The problem of the optimal recovery of the integral is to find the best error of the recovery

$$E_n(X) := \inf_{x_1,\dots,x_n\in Q} \inf_{\Phi\colon \mathbb{R}^n\to\mathbb{R}} e(X,\Phi,x_1,\dots,x_n), \tag{8}$$

the best method of recovery, and the best position of the *information points* $x_1,\dots,x_n$ (i. e. such method $\Phi\colon \mathbb{R}^n \to \mathbb{R}$ and points $x_1,\dots,x_n \in Q$, for which the infima in (8) are attained).

We consider the case $X = W_p^{\nabla}(Q)$, where $p > d$ and the set Q is admissible. Note that in this case it is sufficient to consider only linear methods of recovery in (8). The existence of an optimal linear method of recovery is well known in many situations. See for example [10]. We will not prove it here.

Let $x_1,\dots,x_n \in Q$ and $h > 0$. Recall, that the function $f_{e,h}$, defined by (2), is well defined on the boundary $\partial\Box_h^d$ and is constant there; hence we can continuously extend the function $f_{e,h}$ to all of $\mathbb{R}^d$ by setting $f_{e,h}(y)$ equal to the value of $f_{e,h}$ on the boundary of $\Box_h^d$ for all $y \notin \Box_h^d$. For all $y \in \mathbb{R}^d$, we set

$$f_h(x_1,\dots,x_n;y) := \min_{k=1,\dots,n} f_{e,h}(y - x_k). \tag{9}$$

It is easy to see that $f_h(x_1,\dots,x_n;y) \in W^{1,p}(Q)$ for all $p \in (d,\infty]$.

4.2. Domain composed of cubes

DEFINITION 2. We say that a domain $Q \subset \mathbb{R}^d$ *is composed of* $n \in \mathbb{N}$ *cubes* if there exist $h > 0$ and points $x_1,\dots,x_n$, such that the cubes

$$C_k := \{x \in \mathbb{R}^d \colon |x - x_k|_\infty < h\},$$

$k = 1,\dots,n$, are pairwise disjoint and *mes* $\left[Q \setminus \bigcup_{k=1}^{n} C_k\right] = 0$

LEMMA 2. *Let a domain $R \subset \mathbb{R}^d$ be composed of n cubes $\Box_{h,k}^d$, with the length of edges equal to $h > 0$ and centers $\bar{x}_k$, $k = 1,\dots,n$. Let also $R \subset Q$ and $x_1,\dots,x_n \in Q$. Then*

$$\int_Q f_h(x_1,\dots,x_n;y)dy \geqslant \int_Q f_h(\bar{x}_1,\dots,\bar{x}_n;y)dy \geqslant 0 \tag{10}$$

and for all $p \in (d,\infty]$

$$\|\,|\nabla f_h(x_1,\dots,x_n)|_1\,\|_{L_p(Q)} \leqslant \|\,|\nabla f_h(\bar{x}_1,\dots,\bar{x}_n)|_1\,\|_{L_p(Q)}. \tag{11}$$

Proof. First, we prove inequality (10). For each $\lambda \geqslant 0$ and arbitrary $x_1,\dots,x_n \in Q$, we consider the set

$$S(x_1,\dots,x_n;\lambda) := \{y \in Q\colon f_h(x_1,\dots,x_n;y) \leqslant \lambda\}.$$

From the definition of the function $f_h(x_1,\dots,x_n;y)$, it follows that $S(x_1,\dots,x_n;\,\lambda)$ is the intersection of Q with the union of n cubes with the centers in the points x_k,

$k = 1, \dots, n$, and equal length of the edges (the length of the edges grows with λ). Moreover, if $\lambda_0 := \inf\{\lambda \geqslant 0\colon mes\, S(\bar{x}_1, \dots, \bar{x}_n; \lambda) = mes\, R\}$, then the cubes with the centers in the points $\bar{x}_1, \dots, \bar{x}_n$ that define the set $S(\bar{x}_1, \dots, \bar{x}_n; \lambda)$ are pairwise disjoint for all $\lambda < \lambda_0$ and $S(\bar{x}_1, \dots, \bar{x}_n; \lambda) = Q$ for $\lambda \geqslant \lambda_0$. This implies that for all $\lambda \geqslant 0$ $mes\, S(\bar{x}_1, \dots, \bar{x}_n; \lambda) \geqslant mes\, S(x_1, \dots, x_n; \lambda)$ and, hence,

$$\begin{aligned} mes\{y \in Q\colon f_h(x_1, \dots, x_n; y) > \lambda\} &= mes Q - mes\, S(x_1, \dots, x_n; \lambda) \\ \geqslant mes Q - mes\, S(\bar{x}_1, \dots, \bar{x}_n; \lambda) &= mes\{y \in Q\colon f_h(\bar{x}_1, \dots, \bar{x}_n; y) > \lambda\}. \end{aligned}$$

The latter inequality implies the first inequality in (10) (see §1 in Chapter 1 of [11]). The second inequality in (10) follows from the definition of the functions $f_h(x_1, \dots, x_n)$.

Next, we prove inequality (11). For $k = 1, \dots, n$, we set

$$A_k := \{x \in Q\colon |x - x_k|_\infty < |x - x_s|_\infty, \ \forall s \neq k\}.$$

From the definition of the function $f_h(x_1, \dots, x_n)$, it follows that

$$f_h(x_1, \dots, x_n; x) = f_{e,h}(x - x_k)$$

on $B_k := \{x \in A_k\colon |x - x_k|_\infty < h\}$, $k = 1, \dots, n$, and

$$|\nabla f_h(x_1, \dots, x_n; x)|_1 = 0 \tag{12}$$

almost everywhere on the set $Q \setminus \left(\bigcup_{k=1}^{n} B_k \right)$. For all $k = 1, \dots, n$,

$$\begin{aligned} \| |\nabla f_h(x_1, \dots, x_n; \cdot)|_1 \|_{L_p(B_k)} &= \| |\nabla f_{e,h}(\cdot - x_k)|_1 \|_{L_p(B_k)} \leqslant \| |\nabla f_{e,h}(\cdot)|_1 \|_{L_p(\square_h^d)} \\ &= \| |\nabla f_{e,h}(\cdot - \bar{x}_k)|_1 \|_{L_p\left(\square_{h,k}^d\right)} = \| |\nabla f_h(\bar{x}_1, \dots, \bar{x}_n; \cdot)|_1 \|_{L_p\left(\square_{h,k}^d\right)}. \end{aligned}$$

The latter together with (12) implies inequality (11). The lemma is proved. □

LEMMA 3. *Let $d \in \mathbb{N}$, a domain $Q \subset \mathbb{R}^d$ be composed of n cubes $\square_k$ with centers $\bar{x}_k$, $k = 1, \dots, n$, and $p \in (d; \infty]$. Then for all $f \in W^{1,p}(Q)$*

$$\left| \int_Q f(x)dx - \frac{mes Q}{n} \sum_{k=1}^{n} f(\bar{x}_k) \right| \leqslant \frac{c(d,p)}{n^{\frac{1}{d}}} \left[\frac{mes Q}{2^d} \right]^{\frac{1}{d} + \frac{1}{p'}} \| |\nabla f|_1 \|_{L_p(Q)},$$

where the constant $c(d,p)$ is defined in Theorem 3. The inequality is sharp. Equality holds for the functions $a \cdot f_h(\bar{x}_1, \dots, \bar{x}_n)$, where $a \in \mathbb{R}$, $h = \frac{1}{2} \left(\frac{mes Q}{n} \right)^{\frac{1}{d}}$ and the function $f_h(\bar{x}_1, , \dots, , \bar{x}_n)$ is defined in (9).

Proof. Applying Theorem 3 for each of the cubes $\square_k$, we obtain

$$\left|\int_Q f(x)dx - \frac{mesQ}{n}\sum_{k=1}^n f(\bar{x}_k)\right| \leqslant c(d,p)\left[\frac{1}{2}\left(\frac{mesQ}{n}\right)^{\frac{1}{d}}\right]^{\frac{p'+d}{p'}} \sum_{k=1}^n \|\,|\nabla f|_1\|_{L_p(\square_k)}$$

$$\text{(by Holder's inequality)} \leqslant \frac{c(d,p)}{n^{\frac{1}{d}+\frac{1}{p'}}}\left[\frac{mesQ}{2^d}\right]^{\frac{1}{d}+\frac{1}{p'}}\left(\sum_{k=1}^n \|\,|\nabla f|_1\|^p_{L_p(\square_k)}\right)^{\frac{1}{p}} \cdot n^{\frac{1}{p'}}$$
$$= \frac{c(d,p)}{n^{\frac{1}{d}}}\left[\frac{mesQ}{2^d}\right]^{\frac{1}{d}+\frac{1}{p'}} \|\,|\nabla f|_1\|_{L_p(Q)}.$$

The inequality is proved. Moreover, from the definitions of the function $f_h(\bar{x}_1,\dots,\bar{x}_n)$ and of the extremal function in Theorem 3, we obtain that this inequality becomes equality on the function $f_h(\bar{x}_1,\dots,\bar{x}_n)$ (and, hence, on all functions $a\cdot f_h(\bar{x}_1,\dots,\bar{x}_n)$, $a\in\mathbb{R}$). $\square$

The solution of the problem on optimal recovery of the integral on the class $W_p^\nabla(Q)$, where Q is composed of cubes, is given by the following theorem.

THEOREM 4. *Let $d\in\mathbb{N}$, a domain $Q\subset\mathbb{R}^d$ be composed of n cubes with centers $\bar{x}_k$, $k=1,\dots,n$, and $p\in(d;\infty]$. Then*

$$E_n\left(W_p^\nabla(Q)\right) = \frac{c(d,p)}{n^{\frac{1}{d}}}\left[\frac{mesQ}{2^d}\right]^{\frac{1}{d}+\frac{1}{p'}},$$

where the constant $c(d,p)$ is defined in Theorem 3. The optimal information set is $\{\bar{x}_k\}_{k=1}^n$ and the best recovery method is

$$\tilde{\Phi}(f(x_1),\dots,f(x_n)) = \frac{mesQ}{n}\sum_{k=1}^n f(x_k).$$

Using Lemma 3, we obtain

$$\inf_{x_1,\dots,x_n\in Q}\ \inf_{\Phi:\,\mathbb{R}^n\to\mathbb{R}}\ \sup_{f\in W_p^\nabla(Q)}\left|\int_Q f(x)dx - \Phi(f(x_1),\dots,f(x_n))\right|$$
$$\leqslant \sup_{f\in W_p^\nabla(Q)}\left|\int_Q f(x)dx - \frac{mesQ}{n}\sum_{k=1}^n f(\bar{x}_k)\right|$$
$$\leqslant \sup_{f\in W_p^\nabla(Q)} \frac{c(d,p)}{n^{\frac{1}{d}}}\left[\frac{mesQ}{2^d}\right]^{\frac{1}{d}+\frac{1}{p'}} \|\,|\nabla f|_1\|_{L_p(Q)} = \frac{c(d,p)}{n^{\frac{1}{d}}}\left[\frac{mesQ}{2^d}\right]^{\frac{1}{d}+\frac{1}{p'}}.$$

On the other hand, taking into account Lemma 2 (with $R=Q$ and $h=\frac{1}{2}\left(\frac{mesQ}{n}\right)^{\frac{1}{d}}$), we obtain that for all $x_1,\dots,x_n\in Q$ and all $c_1,\dots,c_n\in\mathbb{R}$

$$\begin{aligned}
\sup_{f\in W_p^{\nabla}(Q)}\left|\int_Q f(x)dx-\sum_{k=1}^n c_k f(x_k)\right| &\geqslant \sup_{\substack{f\in W_p^{\nabla}(Q),\\ f(x_k)=0,\,k=1,\dots,n}}\left|\int_Q f(x)dx\right| \\
&\geqslant \frac{1}{\||\nabla f_h(x_1,\dots,x_n)|_1\|_{L_p(Q)}}\int_Q f_h(x_1,\dots,x_n;x)dx \\
&\geqslant \frac{1}{\||\nabla f_h(\bar{x}_1,\dots,\bar{x}_n)|_1\|_{L_p(Q)}}\int_Q f_h(\bar{x}_1,\dots,\bar{x}_n;x)dx \\
\text{(by Lemma 3)} &= \frac{c(d,p)}{n^{\frac{1}{d}}}\left[\frac{mesQ}{2^d}\right]^{\frac{1}{d}+\frac{1}{p'}}.
\end{aligned}$$

Finally, taking into account the existence of the optimal linear method of recovery and arbitrariness of $x_1,\dots,x_n\in Q$ and $c_1,\dots,c_n\in\mathbb{R}$, we obtain the estimate for $E_n\left(W_p^{\nabla}(Q)\right)$ from below, which completes the proof of the theorem. $\square$

4.3. Case of a convex set Q

Below we consider the case when domain Q is a bounded, open, convex set. According to Theorem 5 in Chapter 1 of [4], every bounded open convex set is Jordan measurable. Moreover, it is easy to see that convex sets satisfy the cone condition and, hence, are admissible (see Definition 1).

4.3.1. Asymptotically optimal information sets and weights

We use the construction of the information set similar to the one used in [2].

Everywhere below we use the following notation. For a finite set A, by $|A|$ we denote the number of its elements.

For $h>0$, we consider the lattice Λ in $\mathbb{R}^d$ generated by the vectors $(2h,0,0,\dots,0)$, $(0,2h,0,0,\dots,0),\dots,$ $(0,\dots,0,2h)\in\mathbb{R}^d$. By $P_k(h)$ we denote the cubes

$$2hk+[0,2h]^d,\qquad k\in\mathbb{Z}^d;$$

their volumes are equal to $(2h)^d$. By $A(h)$, we denote the set of all cubes $P_k(h)$ that are contained in Q. Let $a(h)$ be the set of the centers of the cubes from $A(h)$. By $B(h)$ we denote the set of all cubes $P_k(h)$ that have non-empty set of common with Q interior points. Let $b(h)$ be the set of the centers of the cubes from $B(h)$. Since Q is Jordan measurable, we have $\lim\limits_{h\to 0}|A(h)|\cdot(2h)^d=\lim\limits_{h\to 0}|B(h)|\cdot(2h)^d=mesQ$ or, equivalently,

$$|A(h)|=\frac{mesQ}{(2h)^d}+o\left(\frac{1}{h^d}\right)\text{ and }|B(h)|=\frac{mesQ}{(2h)^d}+o\left(\frac{1}{h^d}\right)\text{ as }h\to 0.\tag{13}$$

Let $n\in\mathbb{N}$ be fixed. We set

$$h_n:=\frac{1}{2}\left(\frac{mesQ}{n}\right)^{\frac{1}{d}}.\tag{14}$$

Then due to (13)

$$|A(h_n)| = n + o(n) \text{ and } |B(h_n)| = n + o(n) \qquad \text{as} \qquad n \to \infty. \tag{15}$$

For each cube P from the set $B(3h_n)$, we choose a point z according to the following rule: z is the center of the cube P if it belongs to $a(h_n)$; otherwise, z is an arbitrary point from $P \bigcap a(h_n)$ if the intersection is not empty, and z is an arbitrary internal point of $Q \bigcap P$ if the intersection is empty.

By $S_1(n)$ we denote the set of such points z. From (13) it follows that the number $|S_1(n)|$ of points in $S_1(n)$ satisfies

$$|S_1(n)| = \frac{n}{3^d} + o(n), \text{ as } n \to \infty. \tag{16}$$

Since for all $n \in \mathbb{N}$ $a(3h_n) \subset a(h_n)$, we obtain that

$$|S_1(n) \setminus a(h_n)| \leqslant |B(3h_n)| - |A(3h_n)| = o(n), \text{ as } n \to \infty. \tag{17}$$

Denote by $S_2(n)$ arbitrary subset of the set $a(h_n) \setminus S_1(n)$, that contains $n - |S_1(n)|$ points (for large enough n this number is positive; if $|a(h_n) \setminus S_1(n)| \leqslant n - |S_1(n)|$, then we set $S_2(n) := a(h_n) \setminus S_1(n)$). Set $S(n) := S_1(n) \cup S_2(n)$.

Let $S(n) = \{x_1^*, \ldots, x_{|S(n)|}^*\}$. For each $k = 1, \ldots, |S(n)|$, we define the set

$$V_k = V_k(n) := \{x \in Q \cap P(3h_n; x_k^*) \colon |x - x_k^*|_\infty < |x - x_s^*|_\infty, \ s \neq k\}, \tag{18}$$

where $P(3h_n; x_k^*)$ is the cube from $B(3h_n)$ that contains x_k^*. Then sets V_k are pairwise disjoint. Moreover, $\bigcup\limits_{k=1}^{|S(n)|} V_k \subset Q$ and $mes\left(Q \setminus \bigcup\limits_{k=1}^{|S(n)|} V_k\right) = 0$.

We set

$$c_k^* := mes V_k, k = 1, \ldots, |S(n)|. \tag{19}$$

The following lemma states some of the properties of the sets $S(n)$ and V_k defined above.

LEMMA 4. *Let $Q \subset \mathbb{R}^d$ be a Jordan measurable set, $n \in \mathbb{N}$ be sufficiently large, and h_n be defined by* (14)*. Then the following properties hold:*

1. *$S(n) \subset Q$ and $|S(n)| \leqslant n$.*
2. *If $x \in V_k$ then $|x - x_k^*|_\infty \leqslant 6h_n$.*
3. *By R_n we denote the union of cubes $P \in A(h_n)$ with centers that belong to $S(n)$. Then $mes R_n = mes Q + o(1)$ as $n \to \infty$.*
4. *For each cube $P \in B(h_n)$, $|P \cap S(n)| \leqslant 1$.*
5. *If $x \in S(n) \cap V_k$ is the center of the cube $P \in A(h_n)$, then $P = V_k \cap R_n$.*
6. *We denote by $U_k := V_k \setminus R_n$. Then $mes \bigcup\limits_{k=1}^{|S(n)|} U_k = o(1)$, $n \to \infty$.*

Proof. Let n be sufficiently large, so that $|S_1(n)| \leqslant n$ (see (16)). Properties 1 and 4 follow from the definition of the set $S(n)$, Property 2 follows from the definition of the sets V_k, since $V_k \subset P(3h_n; x_k^*)$.

Property 5 holds, since by the construction of the set $S(n)$, for every cube P from the set $B(3h_n)$ either $P \bigcap S(n) \subset a(h_n)$ (and $R_n \bigcap P$ is composed of $|P \bigcap S(n)|$ cubes, with centers that generate their own V_k), or $|P \bigcap S(n)| = 1$, $P \bigcap a(h_n) = \emptyset$, and, hence, $P \bigcap R_n = \emptyset$.

From (15) and (17) it follows, that $|a(h_n) \bigcap S(n)| = n + o(n)$, $n \to \infty$. Hence $mesR_n = n(1+o(1))(2h_n)^d = mesQ + o(1)$ as $n \to \infty$, and Property 3 is proved. Property 6 follows from Property 3. The lemma is proved. □

4.3.2. Optimal recovery formulae

An asymptotically optimal solution of the integral optimal recovery problem in the case of bounded convex domain is given by the following theorem.

THEOREM 5. *Let $d \in \mathbb{N}$, $p \in (d, \infty]$, and a bounded, open, convex set Q be given. Then*

$$E_n\left(W_p^{\nabla}(Q)\right) = c(d,p)\left(\frac{mesQ}{2^d}\right)^{\frac{1}{d}+\frac{1}{p'}} \cdot \frac{1+o(1)}{n^{\frac{1}{d}}}, \qquad n \to \infty,$$

where the constant $c(d,p)$ is defined in Theorem 3. The asymptotically optimal information set is $S(n)$ that was defined in Chapter 4.3.1. The optimal recovery method is

$$\tilde{\Phi}_n(f(x_1), \ldots, f(x_{|S(n)|})) = \sum_{k=1}^{|S(n)|} c_k^* f(x_k),$$

where the weights c_k^* are defined by (19).

The estimate from below (for arbitrary admissible domain Q) is proved in the following lemma.

LEMMA 5. *Let $d \in \mathbb{N}$, $p \in (d, \infty]$ and an admissible set Q be given. Then*

$$E_n\left(W_p^{\nabla}(Q)\right) \geqslant c(d,p)\left(\frac{mesQ}{2^d}\right)^{\frac{1}{d}+\frac{1}{p'}} \cdot \frac{1+o(1)}{n^{\frac{1}{d}}}, \qquad n \to \infty.$$

Let R_n be as defined in Property 3 of Lemma 4. Using arguments similar to the ones used to obtain the estimate from below in the proof of Theorem 4 (Lemma 2 needs to be applies to the domain $R = R_n$), we obtain the inequality

$$E_n\left(W_p^{\nabla}(Q)\right) \geqslant \frac{c(d,p)}{n^{\frac{1}{d}}}\left[\frac{mesR_n}{2^d}\right]^{\frac{1}{d}+\frac{1}{p'}} = \frac{c(d,p)}{n^{\frac{1}{d}}}\left[\frac{mesQ+o(1)}{2^d}\right]^{\frac{1}{d}+\frac{1}{p'}}$$

$$= c(d,p)\left(\frac{mesQ}{2^d}\right)^{\frac{1}{d}+\frac{1}{p'}} \cdot \frac{1+o(1)}{n^{\frac{1}{d}}}, \qquad n \to \infty. \tag{20}$$

The lemma is proved.

The next lemma gives an estimate from above for arbitrary admissible domain Q.

LEMMA 6. *Let $d \in \mathbb{N}$, $p \in (d,\infty]$, and an admissible set Q be given. Then*

$$E_n\left(W_p^{\nabla}(Q)\right) \leqslant \sup_{f \in W_p^{\nabla}(Q)} \left| \sum_{k=1}^{|S(n)|} \int_{U_k} [f(x) - f(x_k^*)]dx \right| + c(d,p)\left(\frac{mesQ}{2^d}\right)^{\frac{1}{d}+\frac{1}{p'}} \cdot \frac{1+o(1)}{n^{\frac{1}{d}}},\ n \to \infty,$$

where sets U_k are defined in Property 6 of Lemma 4.

Proof. Indeed,

$$\begin{aligned}
E_n\left(W_p^{\nabla}(Q)\right) &= \inf_{x_1,\dots,x_n \in Q} \inf_{\Phi\colon \mathbb{R}^n \to \mathbb{R}} \sup_{f \in W_p^{\nabla}(Q)} \left| \int_Q f(x)dx - \Phi(f(x_1),\dots,f(x_n)) \right| \\
&\leqslant \sup_{f \in W_p^{\nabla}(Q)} \left| \int_Q f(x)dx - c_k^* \sum_{k=1}^{|S(n)|} f(x_k^*) \right| = \sup_{f \in W_p^{\nabla}(Q)} \left| \sum_{k=1}^{|S(n)|} \int_{V_k} [f(x) - f(x_k^*)]dx \right| \\
&\leqslant \sup_{f \in W_p^{\nabla}(Q)} \left| \sum_{k=1}^{|S(n)|} \int_{V_k \bigcap R_n} [f(x) - f(x_k^*)]dx \right| + \sup_{f \in W_p^{\nabla}(Q)} \left| \sum_{k=1}^{|S(n)|} \int_{U_k} [f(x) - f(x_k^*)]dx \right|.
\end{aligned}$$

Lemma 3, Property 5 of Lemma 4, and (17) imply that

$$\sup_{f \in W_p^{\nabla}(Q)} \left| \sum_{k=1}^{|S(n)|} \int_{V_k \bigcap R_n} [f(x) - f(x_k^*)]dx \right| \leqslant \frac{c(d,p)}{n^{\frac{1}{d}} \cdot (1+o(1))} \left[\frac{mesR_n}{2^d}\right]^{\frac{1}{d}+\frac{1}{p'}}$$

$$= c(d,p)\left(\frac{mesQ}{2^d}\right)^{\frac{1}{d}+\frac{1}{p'}} \cdot \frac{1+o(1)}{n^{\frac{1}{d}}}, \qquad n \to \infty.$$

The lemma is proved. □

In order to finish the proof of the theorem, it is now sufficient to prove the following lemma.

LEMMA 7. *Let $d \in \mathbb{N}$, $p \in (d,\infty]$, and a bounded, open, convex set Q be given. Then*

$$\sup_{f \in W_p^{\nabla}(Q)} \left| \sum_{k=1}^{|S(n)|} \int_{U_k} (f(x) - f(x_k^*))dx \right| \leqslant o(n^{-\frac{1}{d}}), \qquad n \to \infty. \tag{21}$$

Proof. Using Theorem 1 and the fact that Q is a convex domain, we obtain the following estimate for each function $f \in W_p^{\nabla}(Q)$:

$$\left| \sum_{k=1}^{|S(n)|} \int_{U_k} [f(x)-f(x_k^*)]dx \right| = \left| \sum_{k=1}^{|S(n)|} \int_{U_k} \int_0^1 (x-x_k^*, \nabla f(x_k^*+t(x-x_k^*)) \right| dtdx$$

$$\leqslant \sum_{k=1}^{|S(n)|} \int_0^1 \int_{U_k} |x-x_k^*|_\infty \cdot |\nabla f(x_k^*+t(x-x_k^*))|_1 dxdt$$

$$\leqslant \sum_{k=1}^{|S(n)|} \int_0^1 \left(\int_{U_k} |x-x_k^*|_\infty^{p'} dx \right)^{\frac{1}{p'}} \cdot \left(\int_{U_k} |\nabla f(x_k^*+t(x-x_k^*))|_1^p dx \right)^{\frac{1}{p}} dt$$

$$\leqslant 6h_n \sum_{k=1}^{|S(n)|} (mes\, U_k)^{\frac{1}{p'}} \cdot \int_0^1 \left(\int_{U_k} |\nabla f(x_k^*+t(x-x_k^*))|_1^p dx \right)^{\frac{1}{p}} dt. \qquad (22)$$

For each $k = 1, \dots, |S(n)|$, set $\psi_{k,t}(x) := tx + (1-t)x_k^*$. Substituting $y = \psi_{k,t}(x)$, we obtain

$$\int_0^1 \left(\int_{U_k} |\nabla f(x_k^*+t(x-x_k^*))|_1^p dx \right)^{\frac{1}{p}} dt = \int_0^1 \left(\int_{\psi_{k,t}(U_k)} |\nabla f(y)|_1^p t^{-d} dy \right)^{\frac{1}{p}} dt$$

$$= \int_0^1 t^{-\frac{d}{p}} \left(\int_{\psi_{k,t}(U_k)} |\nabla f(y)|_1^p dy \right)^{\frac{1}{p}} dt.$$

Each of the sets V_k is convex, since by the definition V_k, $k = 1, \dots, |S(n)|$, is an intersection of a convex set Q and several half-spaces. Hence, for each $t \in [0,1]$ and $k = 1, \dots, |S(n)|$ $\psi_{k,t}(U_k) \subset V_k$. Since $p > d$

$$\int_0^1 t^{-\frac{d}{p}} \left(\int_{\psi_{k,t}(U_k)} |\nabla f(y)|_1^p dx \right)^{\frac{1}{p}} dt \leqslant \int_0^1 t^{-\frac{d}{p}} dt \left(\int_{V_k} |\nabla f(y)|_1^p dx \right)^{\frac{1}{p}}$$

$$= \frac{p}{p-d} \left(\int_{V_k} |\nabla f(y)|_1^p dx \right)^{\frac{1}{p}}.$$

Using the latter inequality and (22), we obtain

$$\begin{aligned}
\left|\sum_{k=1}^{|S(n)|}\int\limits_{U_k}[f(x)-f(x_k^*)]dx\right| &\leqslant \tfrac{6ph_n}{p-d}\sum_{k=1}^{|S(n)|}(mes\,U_k)^{\frac{1}{p'}}\cdot\left(\int\limits_{V_k}|\nabla f(y)|_1^p dx\right)^{\frac{1}{p}}\\
&\leqslant \tfrac{6ph_n}{p-d}\left(\sum_{k=1}^{|S(n)|}mes\,U_k\right)^{\frac{1}{p'}}\cdot\left(\sum_{k=1}^{|S(n)|}\int\limits_{V_k}|\nabla f(y)|_1^p dx\right)^{\frac{1}{p}}\\
&= \tfrac{6ph_n}{p-d}(mes\,Q\setminus R_n)^{\frac{1}{p'}}\cdot\left(\int\limits_{Q}|\nabla f(y)|_1^p dx\right)^{\frac{1}{p}}\\
&\leqslant \tfrac{6ph_n}{p-d}(mes\,Q\setminus R_n)^{\frac{1}{p'}}.
\end{aligned}$$

Equality (14) and Property 3 in Lemma 4 imply that

$$\frac{6ph_n}{p-d}(mes\,Q\setminus R_n)^{\frac{1}{p'}} = o(n^{-\frac{1}{d}}), \qquad n\to\infty.$$

The lemma is proved. □

Combining Lemmas 5– 7, we obtain the proof of Theorem 5.

REFERENCES

[1] R. A. ADAMS, J. J. F. FOURNIER, *Sobolev Spaces*, Academic Press, 2003, 320.

[2] V. F. BABENKO, *Asymptotically sharp bounds for the remainder for the best quadrature formulas for several classes of functions*, Mathematical Notes, 19, 3, 1976, 187–193.

[3] S. S. DRAGOMIR AND T. M.RASSIAS (EDS), *Ostrowski type inequalities and applications in numerical integration*, Kluwer Academic Publishers, Dordrecht, 2002.

[4] P. M. GRUBER AND C. G. LEKKERKERKER, *Geometry of Numbers*, North-Holland Mathematical Library, V. 37, 1987, 732.

[5] N. KORNEICHUK, *Exact Constants in Approximation Theory*, Encyclopedia of Mathematics and its Applications **38**, Cambridge University Press, 1991, 452.

[6] E. H. LIEB AND M. LOSS, *Analysis*, American Mathematical Soc., 2001, 346.

[7] D. S. MITRINOVIC, J. E. PEČARIĆ AND A. M. FINK, *Inequalities for functions and their integrals and derivatives*, Kluwer Academic Publishers, Dordrecht, 1994.

[8] A. OSTROWSKI, *Über die Absolut Abweichung einer differentienbaren Funktionen von ihren Integralmittelwert*, Comment. Math. Hel, 10, 1938, 226–227.

[9] B. G. PACHPATTE, *Analytic Inequalities*, Atlantis Press, 2012.

[10] S. A. SMOLYAK, *On optimal restoration of functions and functionals of them (in Russian)*, Candidate's Dissertation, Physical-Mathematical Sciences, Moscow State University, 1965.

[11] E. STEIN, *Singular integrals and differentiability properties of functions*, Princeton University Press, 1970, 287.

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