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ON THE ITERATED MEAN TRANSFORMS OF OPERATORS

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Abstract. Let $T = U|T|$ be the polar decomposition of an operator $T \in \mathscr{L}(\mathscr{H})$. For given $s,t \geqslant 0$, we say that $\widehat{T}_{s,t} := sU|T| + t|T|U$ is the weighted mean transform of T. In this paper, we study properties of the k-th iterated weighted mean transform $\widehat{T}_{s,t}^{(k)}$ of $T = U|T|$ when U is unitary. In particular, we give the polar decomposition of such $\widehat{T}_{s,t}^{(k)}$ and investigate its applications. Finally, we consider the iterated weighted mean transforms of a weighted shift.

1. Introduction

Let $\mathscr{H}$ be a separable complex Hilbert space and let $\mathscr{L}(\mathscr{H})$ denote the algebra of all bounded linear operators on $\mathscr{H}$. If $T \in \mathscr{L}(\mathscr{H})$, we write $\sigma(T)$, $\sigma_p(T)$, and $\sigma_{ap}(T)$ for the spectrum, the point spectrum, and the approximate point spectrum of T, respectively. For $0 < p < \infty$, we say that an operator $T \in \mathscr{L}(\mathscr{H})$ is p-hyponormal if $(T^*T)^p \geqslant (TT^*)^p$. In particular, 1-hyponormal (resp. $\frac{1}{2}$-hyponormal) operators are said to be *hyponormal* (resp. *semi-hyponormal*). By Löwner-Heinz inequality, p-hyponormality implies q-hyponormality for $0 < q < p < \infty$.

A closed subspace $\mathscr{M}$ of $\mathscr{H}$ is called an *invariant subspace* for an operator $T \in \mathscr{L}(\mathscr{H})$ if $T\mathscr{M} \subset \mathscr{M}$. The collection of all subspaces of $\mathscr{H}$ invariant under T is denoted by $\mathrm{Lat}(T)$. We say that $\mathscr{M} \subset \mathscr{H}$ is a *hyperinvariant subspace* for $T \in \mathscr{L}(\mathscr{H})$ if $\mathscr{M}$ is an invariant subspace for every $S \in \mathscr{L}(\mathscr{H})$ commuting with T (see [15] for more details).

For an operator $T \in \mathscr{L}(\mathscr{H})$, there exists a unique polar decomposition $T = U|T|$, where $|T| = (T^*T)^{\frac{1}{2}}$ and U is the partial isometry satisfying $\ker(U) = \ker(T)$. Under this polar decomposition, we define the operator $\widetilde{T}^A := |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$, so-called the *Aluthge transform* of T. Taking the Aluthge transform, we obtain the advantages to understand the structure of the original operator. For example, it is known that if

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$T \in \mathcal{L}(\mathcal{H})$ is p-hyponormal, then $\widetilde{T}^A$ is $(p+\frac{1}{2})$-hyponormal (see [1]). Furthermore, if $\widetilde{T}^A$ has a nontrivial invariant subspace, then so does T (see [6]). We refer to [1], [4],[5], [6], [7], [8], and [10] for the Aluthge transforms.

For an operator $T \in \mathcal{L}(\mathcal{H})$ with polar decomposition $T = U|T|$, we define the *weighted mean transform* of T as

$$\widehat{T}_{s,t} := sT + t\widetilde{T}^D = sU|T| + t|T|U,$$

where s and t are nonnegative real numbers and $\widetilde{T}^D$ denotes the *Duggal transform* of T given by $\widetilde{T}^D := |T|U$ (see [9], [13], etc.). In particular, if $s = t = \frac{1}{2}$,

$$\widehat{T}_{\frac{1}{2},\frac{1}{2}} := \frac{1}{2}(T + \widetilde{T}^D)$$

is called the *mean transform* of T.

The mean transform was introduced recently in [11]. According to [9], there are several connections between an operator and its mean transforms in terms of spectral and local spectral theory. Note that every operator $T \in \mathcal{L}(\mathcal{H})$ satisfies that $\|\widehat{T}_{s,t}\| \leqslant (s+t)\|T\|$ for $s,t \geqslant 0$.

Given $s,t \geqslant 0$, the *k-th iterated weighted mean transform* of an operator $T \in \mathcal{L}(\mathcal{H})$ is defined as $\widehat{T}^{(1)}_{s,t} = \widehat{T}_{s,t}$ and $\widehat{T}^{(k+1)}_{s,t} = \widehat{(\widehat{T}^{(k)}_{s,t})}_{s,t}$ for every positive integer k. We note that $\widehat{T}^{(k)}_{0,1}$ is the k-th iterated Duggal transform and $\widehat{T}^{(1)}_{0,1} = \widetilde{T}^D$. In [9], S. Jung, E. Ko and S. Park showed that if W is a weighted shift with weights $\{\beta_n\}_{n=0}^{\infty}$ of positive real numbers, then $\widehat{W}^{(k)}_{\frac{1}{2},\frac{1}{2}}$ is hyponormal if and only if

$$\sum_{n=0}^{k} \binom{k}{n} (\beta_{j+k} - \beta_{j+k+1}) \leqslant 0$$

for each nonnegative integer j. Thus, the hyponormality of a weighted shift is preserved under its iterated weighted mean transforms.

In this paper, we study properties of the k-th iterated weighted mean transform $\widehat{T}^{(k)}_{s,t}$ of $T = U|T|$ when U is unitary. In particular, we give the polar decomposition of such $\widehat{T}^{(k)}_{s,t}$ and investigate its applications. Finally, we consider the iterated weighted mean transforms of a weighted shift.

2. Preliminaries

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have the *single-valued extension property* (or SVEP) if for every open set G in $\mathbb{C}$ and every analytic function $f : G \to \mathcal{H}$ with $(T - z)f(z) \equiv 0$ on G, we have $f(z) \equiv 0$ on G. For an operator $T \in \mathcal{L}(\mathcal{H})$ and a vector $x \in \mathcal{H}$, the set $\rho_T(x)$, called the *local resolvent* of T at x, consists of elements z_0 in $\mathbb{C}$ such that there exists an $\mathcal{H}$-valued analytic function $f(z)$ defined in a neighborhood of

z_0 which verifies $(T-z)f(z) \equiv x$. The *local spectrum* of T at x is given by $\sigma_T(x) := \mathbb{C} \setminus \rho_T(x)$. Moreover, we define the *local spectral subspace* of T as $\mathscr{H}_T(F) := \{x \in \mathscr{H} : \sigma_T(x) \subset F\}$, where F is a subset of $\mathbb{C}$. An operator $T \in \mathscr{L}(\mathscr{H})$ is said to have the *Dunford's property* (C) if $\mathscr{H}_T(F)$ is closed for each closed subset F of $\mathbb{C}$. We say that $T \in \mathscr{L}(\mathscr{H})$ has the *Bishop's property* (β) if for every open subset G of $\mathbb{C}$ and every sequence $f_n : G \to \mathscr{H}$ of $\mathscr{H}$-valued analytic functions such that $(T-z)f_n(z)$ converges uniformly to 0 in norm on compact subsets of G, then $f_n(z)$ converges uniformly to 0 in norm on compact subsets of G. The following implications are well known (see [3] and [12] for more details):

$$\text{Bishop's property } (\beta) \Rightarrow \text{Dunford's property } (C) \Rightarrow \text{SVEP}.$$

3. Main results

In this section, we study the iterated weighted mean transforms $\widehat{T}_{s,t}^{(k)}$ of an operator $T \in \mathscr{L}(\mathscr{H})$ and give various connections between T and $\widehat{T}_{s,t}^{(k)}$. If $t=0$, then $\widehat{T}_{s,t}^{(k)}$ becomes a scalar multiple of T, and hence we may assume that $t>0$. We first give the polar decomposition of the iterated weighted mean transforms of operators.

THEOREM 1. Let $T = U|T|$ be the polar decomposition of an operator $T \in \mathscr{L}(\mathscr{H})$ where U is unitary. Suppose that $s \geqslant 0$, $t > 0$, and k is a positive integer k. Then $\widehat{T}_{s,t}^{(k)}$ has the polar decomposition

$$\widehat{T}_{s,t}^{(k)} = U|\widehat{T}_{s,t}^{(k)}|$$

where

$$|\widehat{T}_{s,t}^{(k)}| = \sum_{j=0}^{k} \binom{k}{j} s^{k-j} t^j U^{*j} |T| U^j.$$

Moreover, if T is invertible, then $\widehat{(T^{-1})}_{s,t}^{(k)}$ has the polar decomposition

$$\widehat{(T^{-1})}_{s,t}^{(k)} = U^* |\widehat{(T^{-1})}_{s,t}^{(k)}|$$

where

$$|\widehat{(T^{-1})}_{s,t}^{(k)}| = \sum_{j=0}^{k} \binom{k}{j} s^{k-j} t^j U^j |T^*|^{-1} U^{*j} = \sum_{j=0}^{k} \binom{k}{j} s^{k-j} t^j U^{j+1} |T|^{-1} U^{*j+1}.$$

Proof. In order to find the polar decomposition of $\widehat{T}_{s,t}^{(k)}$, we use the induction on k. Since U is unitary, it is evident that $\widehat{T}_{s,t} = U(s|T| + tU^*|T|U)$. Moreover, we get that

$$(\widehat{T}_{s,t})^* \widehat{T}_{s,t} = (s|T|U^* + tU^*|T|)(sU|T| + t|T|U)$$

$$= s^2|T|^2 + t^2U^*|T|^2U + stU^*|T|U|T| + st|T|U^*|T|U$$
$$= (s|T| + tU^*|T|U)^2,$$

which gives $|\widehat{T}_{s,t}| = s|T| + tU^*|T|U$. It remains to prove $\ker(\widehat{T}_{s,t}) = \ker(U) = \{0\}$. If $x \in \ker(\widehat{T}_{s,t})$, then

$$0 = \langle|\widehat{T}_{s,t}|x,x\rangle = s\langle|T|x,x\rangle + t\langle U^*|T|Ux,x\rangle.$$

Since both $|T|$ and $U^*|T|U$ are positive operators and $t > 0$, we have $\langle U^*|T|Ux,x\rangle = 0$, i.e., $|T|^{\frac{1}{2}}Ux = 0$. Since $\ker(|T|^{\frac{1}{2}}) = \ker(U) = \{0\}$, we get that $x = 0$, namely $\ker(\widehat{T}_{s,t}) = \{0\}$.

We now assume that the result is true for $k = n$. Then

$$\begin{aligned}
\widehat{T}_{s,t}^{(n+1)} &= sU|\widehat{T}_{s,t}^{(n)}| + t|\widehat{T}_{s,t}^{(n)}|U \\
&= U\Big(\sum_{j=0}^{n}\binom{n}{j}s^{n+1-j}t^jU^{*j}|T|U^j\Big) + UU^*\Big(\sum_{j=0}^{n}\binom{n}{j}s^{n-j}t^{j+1}U^{*j}|T|U^j\Big)U \\
&= U\Big(\sum_{j=0}^{n}\binom{n}{j}s^{n+1-j}t^jU^{*j}|T|U^j + \sum_{j=1}^{n+1}\binom{n}{j-1}s^{n+1-j}t^jU^{*j}|T|U^j\Big) \\
&= U\Big(\sum_{j=0}^{n+1}\binom{n+1}{j}s^{n+1-j}t^jU^{*j}|T|U^j\Big). \qquad (1)
\end{aligned}$$

Since $U^{*j}|T|U^j \geqslant 0$ for each nonnegative integer j, it is not difficult to show that

$$|\widehat{T}_{s,t}^{(n+1)}| = \sum_{j=0}^{n+1}\binom{n+1}{j}s^{n+1-j}t^jU^{*j}|T|U^j$$

and $\ker(|\widehat{T}_{s,t}^{(n+1)}|) = \ker(U) = \{0\}$. Hence, (1) is the polar decomposition of $\widehat{T}_{s,t}^{(n+1)}$.

If T is invertible, then U is unitary and

$$T^{-1} = |T|^{-1}U^* = (U^*|T^*|U)^{-1}U^* = U^*|T^*|^{-1}.$$

Since $(T^{-1})^*T^{-1} = (TT^*)^{-1} = (|T^*|^{-1})^2$, we have $|T^{-1}| = |T^*|^{-1}$. Moreover, since $\ker(T^{-1}) = \ker(U^*) = \{0\}$, the factorization $T^{-1} = U^*|T^*|^{-1}$ is the polar decomposition of T^{-1}. Using the polar decomposition of $\widehat{T}_{s,t}^{(k)}$, we obtain that $\widehat{(T^{-1})}_{s,t}^{(k)} = U^*|\widehat{(T^{-1})}_{s,t}^{(k)}|$ is the polar decomposition of $\widehat{(T^{-1})}_{s,t}^{(k)}$ with

$$|\widehat{(T^{-1})}_{s,t}^{(k)}| = \sum_{j=0}^{k}\binom{k}{j}s^{k-j}t^jU^j|T^*|^{-1}U^{*j}.$$

Since $|T^*|^{-1} = U|T|^{-1}U^*$, the latter representation also holds. □

An operator $T \in \mathscr{L}(\mathscr{H})$ is a *quasiaffinity* if it has trivial kernel and dense range. Remark that the partial isometric part U of a quasiaffinity $T = U|T|$ must be unitary.

COROLLARY 1. Let $s \geqslant 0$ and $t > 0$. If $T \in \mathscr{L}(\mathscr{H})$ is a semi-hyponormal operator with dense range, then $\widehat{T}_{s,t}^{(k)}$ is semi-hyponormal for every positive integer k.

Proof. Assume that $T = U|T|$ is the polar decomposition and k is any positive integer. If T is semi-hyponormal and has dense range, then $\ker(T) \subset \ker(T^*) = \{0\}$ by [1], which ensures that T is a quasiaffinity and U is unitary. From Theorem 1, we obtain that

$$\begin{aligned}
|\widehat{T}_{s,t}^{(k)}| - |(\widehat{T}_{s,t}^{(k)})^*| &= |\widehat{T}_{s,t}^{(k)}| - U|\widehat{T}_{s,t}^{(k)}|U^* \\
&= \sum_{j=0}^{k} \binom{k}{j} s^{k-j} t^j U^{*j}(|T| - U|T|U^*)U^j \\
&= \sum_{j=0}^{k} \binom{k}{j} s^{k-j} t^j U^{*j}(|T| - |T^*|)U^j \\
&\geqslant 0.
\end{aligned}$$

Hence $\widehat{T}_{s,t}^{(k)}$ is semi-hyponormal. □

COROLLARY 2. Assume $T = U|T|$ is the polar decomposition of T in $\mathscr{L}(\mathscr{H})$ where U is unitary. If k is a positive integer, then $\widehat{T}_{0,1}^{(k)}$ is hyponormal if and only if T is hyponormal. In particular, $\widehat{T}_{0,1}^{(k)}$ is hyponormal for some positive integer k, then T has the Bishop's property (β), the Dunford's property (C), and the single-valued extension property.

Proof. If $\widehat{T}_{0,1}^{(k)}$ is hyponormal for some positive integer k, then Theorem 1 implies that

$$\begin{aligned}
0 &\leqslant (\widehat{T}_{0,1}^{(k)})^*(\widehat{T}_{0,1}^{(k)}) - (\widehat{T}_{0,1}^{(k)})(\widehat{T}_{0,1}^{(k)})^* \\
&= (U^{*k}|T|U^{k-1})(U^{*k-1}|T|U^k) - (U^{*k-1}|T|U^k)(U^{*k}|T|U^{k-1}) \\
&= U^{*k}|T|^2U^k - U^{*k-1}|T|^2U^{k-1}.
\end{aligned}$$

Hence $U^{*k}|T|^2U^k \geqslant U^{*k-1}|T|^2U^{k-1}$, i.e., $|T|^2 \geqslant U|T|^2U^*$. Therefore T is hyponormal.

Conversely, if T is hyponormal and k is any positive integer, then $|T|^2 \geqslant U|T|^2U^*$. Since U is unitary, we get that $U^*|T|^2U \geqslant |T|^2$. Hence

$$(\widehat{T}_{0,1}^{(k)})^*(\widehat{T}_{0,1}^{(k)}) - (\widehat{T}_{0,1}^{(k)})(\widehat{T}_{0,1}^{(k)})^* = U^{*k-1}(U^*|T|^2U - |T|^2)U^{k-1} \geqslant 0.$$

Hence $\widehat{T}_{0,1}^{(k)}$ is hyponormal.

If $\widehat{T}_{0,1}^{(k)}$ is hyponormal for some positive integer k, then T is hyponormal. Every hyponormal operator has the Bishop's property (β) (see [14]). So, we complete the proof by [3] or [12]. □

Recall that an operator $T \in \mathscr{L}(\mathscr{H})$ is called *quasinormal* if $T(T^*T) = (T^*T)T$. We say that $T \in \mathscr{L}(\mathscr{H})$ is *binormal* if $(T^*T)(TT^*) = (TT^*)(T^*T)$. It is known that quasinormal operators are hyponormal and binormal. For an operator $T \in \mathcal{L}(\mathcal{H})$ with polar decomposition $T = U|T|$ and $s,t > 0$, it is easy to see that the equation $\widehat{T}_{s,t} = (s+t)T$ is equivalent to $|T|U = U|T|$, that is, T is quasinormal.

COROLLARY 3. Let $T \in \mathcal{L}(\mathcal{H})$ have the polar decomposition $T = U|T|$ where U is unitary. Suppose that $s,t > 0$ and k is a positive integer. If $U^2|T| = |T|U^2$, then the following statements hold:
(i) $\widehat{T}_{s,t}^{(k)}$ is quasinormal if and only if $s = t$ or T is quasinormal.
(ii) $\widehat{T}_{s,t}^{(k)}$ is binormal if and only if $s = t$ or T is binormal.
In particular, $\widehat{T}_{0,1}^{(k)}$ is quasinormal (resp. binormal) if and only if T is quasinormal (resp. binormal).

Proof. (i) From Theorem 1, we know that $\widehat{T}_{s,t}^{(k)}$ is quasinormal if and only if U and $|\widehat{T}_{s,t}^{(k)}|$ commute where $|\widehat{T}_{s,t}^{(k)}| = \sum_{j=0}^{k} \binom{k}{j} s^{k-j}t^j U^{*j}|T|U^j$. Note that

$$\begin{aligned} U|\widehat{T}_{s,t}^{(k)}| &= \sum_{j=0}^{k} \binom{k}{j} s^{k-j}t^j UU^{*j}|T|U^j \\ &= s^k U|T| + \sum_{j=1}^{k} \binom{k}{j} s^{k-j}t^j U^{*j-1}|T|U^j. \end{aligned}$$

Since $U^2|T| = |T|U^2$, one can compute that

$$U^{*j-1}|T|U^j = \begin{cases} U^{*j-1}U^j|T| = U|T| & \text{if } j \text{ is even} \\ |T|U^{*j-1}U^j = |T|U & \text{if } j \text{ is odd.} \end{cases}$$

Thus, it holds that

$$\begin{aligned} U|\widehat{T}_{s,t}^{(k)}| &= \sum_{\substack{0 \leqslant j \leqslant k \\ j\,:\,\text{even}}} \binom{k}{j} s^{k-j}t^j U|T| + \sum_{\substack{0 \leqslant j \leqslant k \\ j\,:\,\text{odd}}} \binom{k}{j} s^{k-j}t^j |T|U \\ &= a_k U|T| + b_k|T|U \end{aligned}$$

where $a_k = \frac{(s+t)^k + (s-t)^k}{2}$ and $b_k = \frac{(s+t)^k - (s-t)^k}{2}$. Similarly, we have

$$\begin{aligned} |\widehat{T}_{s,t}^{(k)}|U &= \sum_{j=0}^{k} \binom{k}{j} s^{k-j}t^j U^{*j}|T|U^{j+1} \\ &= \sum_{\substack{0 \leqslant j \leqslant k \\ j\,:\,\text{even}}} \binom{k}{j} s^{k-j}t^j |T|U + \sum_{\substack{0 \leqslant j \leqslant k \\ j\,:\,\text{odd}}} \binom{k}{j} s^{k-j}t^j U|T| \\ &= a_k|T|U + b_k U|T|. \end{aligned}$$

Since

$$U|\widehat{T}_{s,t}^{(k)}| - |\widehat{T}_{s,t}^{(k)}|U = (a_k - b_k)(U|T| - |T|U),$$

it follows that $\widehat{T}_{s,t}^{(k)}$ is quasinormal if and only if $a_k = b_k$ or $U|T| = |T|U$. Since $a_k = b_k$ is equivalent to $s = t$, we obtain the quasinormality of $\widehat{T}_{s,t}^{(k)}$ exactly when $s = t$ or T is quasinormal.

(ii) Note that $\widehat{T}_{s,t}^{(k)}$ is binormal if and only if

$$|\widehat{T}_{s,t}^{(k)}||(\widehat{T}_{s,t}^{(k)})^*| = |(\widehat{T}_{s,t}^{(k)})^*||\widehat{T}_{s,t}^{(k)}|. \tag{2}$$

Claim. If k is any positive integer, then

$$\begin{cases} |\widehat{T}_{s,t}^{(k)}| = a_k|T| + b_k|T^*| \\ |(\widehat{T}_{s,t}^{(k)})^*| = b_k|T| + a_k|T^*| \end{cases}$$

where $a_k = \frac{(s+t)^k+(s-t)^k}{2}$ and $b_k = \frac{(s+t)^k-(s-t)^k}{2}$.

Since $U^2|T| = |T|U^2$, we have $|T^*| = U|T|U^* = U^*|T|U$. This implies that

$$\begin{cases} |\widehat{T}_{s,t}| = s|T| + t|T^*| = a_1|T| + b_1|T^*| \\ |(\widehat{T}_{s,t})^*| = U|\widehat{T}_{s,t}|U^* = b_1|T| + a_1|T^*|. \end{cases}$$

Hence, the claim is true for $k = 1$. If the claim holds for $k = n$, then

$$\begin{aligned} |\widehat{T}_{s,t}^{(n+1)}| &= |\widehat{(\widehat{T}_{s,t}^{(n)})}_{s,t}| \\ &= a_1|\widehat{T}_{s,t}^{(n)}| + b_1|(\widehat{T}_{s,t}^{(n)})^*| \\ &= a_1(a_n|T| + b_n|T^*|) + b_1(b_n|T| + a_n|T^*|) \\ &= (a_1a_n + b_1b_n)|T| + (a_1b_n + b_1a_n)|T^*| \\ &= a_{n+1}|T| + b_{n+1}|T^*| \end{aligned}$$

and

$$|(\widehat{T}_{s,t}^{(n+1)})^*| = U|\widehat{T}_{s,t}^{(n+1)}|U^* = b_{n+1}|T| + a_{n+1}|T^*|.$$

Therefore, our claim is satisfied for all positive integers k.

Applying the claim above, we see that

$$\begin{aligned} &|\widehat{T}_{s,t}^{(k)}||(\widehat{T}_{s,t}^{(k)})^*| - |(\widehat{T}_{s,t}^{(k)})^*||\widehat{T}_{s,t}^{(k)}| \\ &= (a_k|T| + b_k|T^*|)(b_k|T| + a_k|T^*|) - (a_k|T^*| + b_k|T|)(b_k|T^*| + a_k|T|) \\ &= (a_k^2 - b_k^2)(|T||T^*| - |T^*||T|). \end{aligned}$$

According to (2), we conclude that $\widehat{T}_{s,t}$ is binormal if and only if $a_k = b_k$ or T is binormal. So, we complete the proof. □

REMARK 1. In [11], the authors showed that if $T \in \mathscr{L}(\mathscr{H})$ has the polar decomposition $T = U|T|$ where $U^2|T| = |T|U^2$ and U is unitary, then $\widehat{T}_{\frac{1}{2},\frac{1}{2}}$ is quasinormal.

We give some properties for the case when $s = t$ in Corollary 3, as follows:

COROLLARY 4. Let $T = U|T|$ be the polar decomposition of $T \in \mathcal{L}(\mathcal{H})$, where U is unitary, and let $s > 0$. If $U^2|T| = |T|U^2$, then $\widehat{T}_{s,s}^{(k)}$ is quasinormal and $\widehat{T}_{s,s}^{(k)} = (2s)^{k-1}\widehat{T}_{s,s}$ for each positive integer k.

Proof. We know from Corollary 3 that $\widehat{T}_{s,s}^{(k)}$ is quasinormal for each positive integer k. In particular, $\widehat{T}_{s,s}$ is quasinormal and then $\widehat{T}_{s,s}^{(2)} = 2s\widehat{T}_{s,s}$. Since $\widehat{T}_{s,s}^{(2)}$ is quasinormal, it follows that $\widehat{T}_{s,s}^{(3)} = 2s\widehat{T}_{s,s}^{(2)} = (2s)^2\widehat{T}_{s,s}$ and $\widehat{T}_{s,s}^{(3)}$ is also quasinormal. Repeating this method, we derive that $\widehat{T}_{s,s}^{(k)}$ is quasinormal and $\widehat{T}_{s,s}^{(k)} = (2s)^{k-1}\widehat{T}_{s,s}$ for all positive integers k. □

We next provide some examples for Theorem 1 and Corollary 3.

EXAMPLE 1. Let $T = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix} \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ where A is a quasiaffinity that is not an isometry. Then the polar decomposition $T = U|T|$ is given by $U = \begin{pmatrix} 0 & I \\ U_A & 0 \end{pmatrix}$ and $|T| = \begin{pmatrix} |A| & 0 \\ 0 & I \end{pmatrix}$ where U_A is the partial isometric part of A. Note that U_A is unitary since A is a quasiaffinity. Fix $s \geqslant 0$ and $t > 0$. A simple calculation shows that $\widehat{T}_{s,t} = \begin{pmatrix} 0 & s + t|A| \\ sA + tU_A & 0 \end{pmatrix}$ and $\widehat{T}_{s,t}$ has the polar decomposition $\widehat{T}_{s,t} = U|\widehat{T}_{s,t}|$ with

$$U = \begin{pmatrix} 0 & I \\ U_A & 0 \end{pmatrix} \text{ and } |\widehat{T}_{s,t}| = \begin{pmatrix} s|A| + t & 0 \\ 0 & s + t|A| \end{pmatrix}$$

due to Theorem 1. Observe that $\widehat{T}_{s,t}$ is not necessarily binormal, although T is binormal. But, if A is quasinormal, then $U^2|T| = |T|U^2$. Hence, $\widehat{T}_{s,t}$ is binormal by Corollary 3. We also indicate that $\widehat{T}_{s,t}$ is not quasinormal whenever $s \neq t$.

For another example, we consider some finite matrices.

EXAMPLE 2. Consider the matrix $T = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \\ 2 & -1 & 0 \end{pmatrix}$ on $\mathbb{C}^3$. Then it is straightforward to see that $|T| = \begin{pmatrix} \sqrt{5} & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $U = T|T|^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \end{pmatrix}$. Using Theorem 1, we know that

$$|\widehat{T}_{s,t}| = s|T| + tU^*|T|U$$

$$= s\begin{pmatrix}\sqrt{5} & 0 & 0\\ 0 & \sqrt{5} & 0\\ 0 & 0 & 1\end{pmatrix} + t\begin{pmatrix}\frac{4+\sqrt{5}}{5} & \frac{-2+2\sqrt{5}}{5} & 0\\ \frac{-2+2\sqrt{5}}{5} & \frac{1+4\sqrt{5}}{5} & 0\\ 0 & 0 & \sqrt{5}\end{pmatrix}$$
$$= \begin{pmatrix}\sqrt{5}s+\frac{4+\sqrt{5}}{5}t & \frac{-2+2\sqrt{5}}{5}t & 0\\ \frac{-2+2\sqrt{5}}{5}t & \sqrt{5}s+\frac{1+4\sqrt{5}}{5}t & 0\\ 0 & 0 & s+\sqrt{5}t\end{pmatrix}$$

for $s \geqslant 0$ and $t > 0$. Therefore, the weighted mean transform $\widehat{T}_{s,t}$ has the polar decomposition

$$\widehat{T}_{s,t} = U|\widehat{T}_{s,t}| = \begin{pmatrix}0 & 0 & 1\\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0\\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0\end{pmatrix}\begin{pmatrix}\sqrt{5}s+\frac{4+\sqrt{5}}{5}t & \frac{-2+2\sqrt{5}}{5}t & 0\\ \frac{-2+2\sqrt{5}}{5}t & \sqrt{5}s+\frac{1+4\sqrt{5}}{5}t & 0\\ 0 & 0 & s+\sqrt{5}t\end{pmatrix}$$

for $s \geqslant 0$ and $t > 0$. We note that T is binormal, but $\widehat{T}_{s,t}$ is not for any $s \geqslant 0$ and $t > 0$; indeed,

$$|(\widehat{T}_{s,t})^*| = U|\widehat{T}_{s,t}|U^* = \begin{pmatrix}s+\sqrt{5}t & 0 & 0\\ 0 & \sqrt{5}(s+t) & 0\\ 0 & 0 & \sqrt{5}s+t\end{pmatrix}$$

does not commute with $|\widehat{T}_{s,t}|$. Hence $U^2|T| \neq |T|U^2$ by Corollary 3. When $s = t = \frac{1}{2}$, we also compute that none of $\widehat{T}^{(2)}_{\frac{1}{2},\frac{1}{2}}$, $\widehat{T}^{(10)}_{\frac{1}{2},\frac{1}{2}}$, and $\widehat{T}^{(20)}_{\frac{1}{2},\frac{1}{2}}$ are binormal using the Maple program.

Recall that an operator T is *normal* if $T^*T - TT^* = 0$ and an operator T is *essentially normal* if $T^*T - TT^*$ is compact. Let $\pi : \mathscr{L}(\mathscr{H}) \to \mathscr{L}(\mathscr{H})/\mathscr{K}(\mathscr{H})$ be the Calkin map for the ideal $\mathscr{K}(\mathscr{H})$ of compact operators on $\mathscr{H}$.

THEOREM 2. If $T = U|T|$ is essentially normal, then $\pi(T) = \pi(\widehat{T}^{(k)}_{0,1})$, so $\widehat{T}^{(k)}_{0,1}$ is essentially normal. Conversely, if $\widehat{T}^{(k)}_{0,1}$ is essentially normal, and U is unitary, then T is essentially normal.

Proof. If T is essentially normal, then $T^*T - TT^*$ is compact, and we obtain that $|T|^2 - U|T|^2U^*$ and $|T|^2U - U|T|^2$ are also compact. That is, $\pi(U)$ commutes with $\pi(|T|^2)$. Hence $\pi(U)$ commutes with the positive square root $\pi(|T|)$ of $\pi(|T|^2)$. Therefore,

$$\pi(\widehat{T}_{0,1}) = \pi(|T|U) = \pi(|T|)\pi(U) = \pi(U)\pi(|T|) = \pi(T),$$

i.e., Thus $\widehat{T}_{0,1}$ is essentially normal. Assume that $\pi(T) = \pi(\widehat{T}^{(n)}_{0,1})$ for some positive integer n. Then we have that

$$\pi(\widehat{T}^{(n+1)}_{0,1}) = \pi(|\widehat{T}^{(n)}_{0,1}|U) = \pi(|\widehat{T}^{(n)}_{0,1}|)\pi(U) = \pi(U)\pi(|\widehat{T}^{(n)}_{0,1}|) = \pi(\widehat{T}^{(n)}_{0,1}) = \pi(T),$$

implying that $\pi(T)=\pi(\widehat{T}_{0,1}^{(k)})$ and $\widehat{T}_{0,1}^{(k)}$ is essentially normal for each positive integer k by induction.

Conversely, if $\widehat{T}_{0,1}^{(k)}$ is essentially normal, and U is unitary, then we have that $\widehat{T}_{0,1}^{(k)*}\widehat{T}_{0,1}^{(k)}-\widehat{T}_{0,1}^{(k)}\widehat{T}_{0,1}^{(k)*}$ is compact. Therefore, we ensure that $|\widehat{T}_{0,1}^{(k)}|^2-U|\widehat{T}_{0,1}^{(k)}|^2U^*$ and $|\widehat{T}_{0,1}^{(k)}|^2U-U|\widehat{T}_{0,1}^{(k)}|^2$ are also compact. It follows that $\pi(U)$ commutes with the positive square root $\pi(|\widehat{T}_{0,1}^{(k)}|)$ of $\pi(|\widehat{T}_{0,1}^{(k)}|^2)$. Thus

$$\pi(\widehat{T}_{0,1}^{(k)})=\pi(U)\pi(|\widehat{T}_{0,1}^{(k)}|)=\pi(|\widehat{T}_{0,1}^{(k)}|)\pi(U)=\pi(\widehat{T}_{0,1}^{(k-1)}),$$

i.e., $\widehat{T}_{0,1}^{(k-1)}$ is essentially normal. By the induction hypothesis, T is essentially normal. □

EXAMPLE 3. Let $S=\begin{pmatrix}0 & Q^2\\ I & 0\end{pmatrix}$ where Q is a positive semidefinite operator in $\mathscr{L}(\mathscr{H})$ with trivial kernel. Then $S^*S=\begin{pmatrix}I & 0\\ 0 & Q^4\end{pmatrix}$ and $SS^*=\begin{pmatrix}Q^4 & 0\\ 0 & I\end{pmatrix}$. Hence $|S|=\begin{pmatrix}I & 0\\ 0 & Q^2\end{pmatrix}$ and $S=\begin{pmatrix}0 & Q^2\\ I & 0\end{pmatrix}=U\begin{pmatrix}I & 0\\ 0 & Q^2\end{pmatrix}$ where $U=\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}$. Thus $\widehat{S}_{0,1}^{(1)}=|S|U=\begin{pmatrix}0 & I\\ Q^2 & 0\end{pmatrix}$ and U is unitary. Hence $S^*S-SS^*=\begin{pmatrix}I-Q^4 & 0\\ 0 & Q^4-I\end{pmatrix}$ and $(\widehat{S}_{0,1}^{(1)})^*\widehat{S}_{0,1}^{(1)}-\widehat{S}_{0,1}^{(1)}(\widehat{S}_{0,1}^{(1)})^*=\begin{pmatrix}Q^4-I & 0\\ 0 & I-Q^4\end{pmatrix}$. If $I-Q^4$ is compact, then S and $\widehat{S}_{0,1}^{(1)}$ are essenially normal.

Let us recall Berberian's technique in [2]. Denote by $\mathfrak{M}$ a linear space of all sequences $\{x_n\}\subset\mathscr{H}$ such that $\sup_n\|x_n\|<\infty$. Consider the quotient space $\mathfrak{M}/\mathfrak{N}$ where $\mathfrak{N}:=\{\{x_n\}\in\mathfrak{M}: glim\{\|x_n\|\}=0\}$ and $glim$ is Banach generalized limit (see [2] or [16] for more details). We will represent an equivalence class of $\mathfrak{M}/\mathfrak{N}$ containing a sequence $\{x_n\}$ as $[\{x_n\}]$. It is easy to show that

$$\langle x^\circ,y^\circ\rangle=glim\{\langle x_n,y_n\rangle\},\ x^\circ=[\{x_n\}],y^\circ=[\{y_n\}]\in\mathfrak{M}/\mathfrak{N}$$

is an inner product in $\mathfrak{M}/\mathfrak{N}$. Moreover, $\mathfrak{M}/\mathfrak{N}$ can be completed to a Hilbert space $\mathscr{H}^\circ$ and the Hilbert space $\mathscr{H}^\circ$ is an extension of $\mathscr{H}$ by identifying a vector $x\in\mathscr{H}$ with $[\{x,x,x,\cdots\}]\in\mathscr{H}^\circ$. Let T° be the operator on $\mathscr{H}^\circ$ determined by the relation $T^\circ x^\circ=[\{Tx_n\}]$ for $x^\circ=[\{x_n\}]\in\mathscr{H}^\circ$. Under the same notations as above, the Hilbert space $\mathscr{H}^\circ$ and the mapping $\circ:\mathscr{L}(\mathscr{H})\to\mathscr{L}(\mathscr{H}^\circ)$ satisfy the following proposition.

PROPOSITION 1. [2] Let $\mathscr{H}$ be a complex Hilbert space. Then there exist a Hilbert space $\mathscr{H}^\circ\supset\mathscr{H}$ and a unital linear map $\circ:\mathscr{L}(\mathscr{H})\to\mathscr{L}(\mathscr{H}^\circ)$ such that
(i) $(ST)^\circ=S^\circ T^\circ$, $(T^\circ)^*=(T^*)^\circ$, $\|T^\circ\|=\|T\|$,
(ii) $S^\circ\leqslant T^\circ$ whenever $S\leqslant T$,
(iii) $\sigma(T)=\sigma(T^\circ)$, $\sigma_{ap}(T)=\sigma_{ap}(T^\circ)=\sigma_p(T^\circ)$.

LEMMA 1. If $T = U|T|$ is the polar decomposition of T in $\mathscr{L}(\mathscr{H})$, then

$$T^\circ = U^\circ |T|^\circ$$

is the polar decomposition of T°.

Proof. Since $(T^\circ)^* T^\circ = (T^* T)^\circ = (|T|^2)^\circ = (|T|^\circ)^2$ and $|T|^\circ \geqslant 0$, we have $|T^\circ| = |T|^\circ$. Since $T^\circ = U^\circ |T|^\circ$, it is enough to show that U° is partial isometric and $\ker(U^\circ) = \ker(T^\circ)$. Using $UU^*U = U$, we see that $U^\circ (U^\circ)^* U^\circ = U^\circ$ and so U° is a partial isometry.

To obtain $\ker(U^\circ) = \ker(T^\circ)$, let $x^\circ \in \ker(T^\circ)$ be given. Write $x^\circ = y^\circ + z^\circ$ where $y^\circ = [\{y_n\}]$ and $z^\circ = [\{z_n\}]$ for some $\{y_n\} \subset \ker(T)^\perp$ and $\{z_n\} \subset \ker(T)$. Then $z^\circ \in \ker(T^\circ)$ clearly. Since $y_n \in \overline{\operatorname{ran}(|T|)}$, choose $\{w_n\} \subset \mathscr{H}$ such that $\|y_n - |T|w_n\| < \frac{1}{n}$. Since

$$\liminf_{n\to\infty} \|y_n - |T|w_n\| \leqslant glim\{\|y_n - |T|w_n\|\} \leqslant \limsup_{n\to\infty} \|y_n - |T|w_n\|,$$

it holds that

$$y^\circ = |T|^\circ w^\circ \in \overline{\operatorname{ran}(|T|^\circ)} = \overline{\operatorname{ran}(|T^\circ|)} = \ker(T^\circ)^\perp$$

where $w^\circ = [\{w_n\}]$. Since $\mathscr{H}^\circ = \ker(T^\circ)^\perp \oplus \ker(T^\circ)$, we have $x^\circ = z^\circ$. Thus we see that

$$\ker(T^\circ) = \{x^\circ = [\{x_n\}] \in \mathscr{H}^\circ : \{x_n\} \subset \ker(T)\}.$$

Since this is true for any $T \in \mathscr{L}(\mathscr{H})$, we get that

$$\begin{aligned}\ker(T^\circ) &= \{x^\circ = [\{x_n\}] \in \mathscr{H}^\circ : \{x_n\} \subset \ker(T)\} \\ &= \{x^\circ = [\{x_n\}] \in \mathscr{H}^\circ : \{x_n\} \subset \ker(U)\} \\ &= \ker(U^\circ).\end{aligned}$$

Therefore, $T^\circ = U^\circ |T|^\circ$ is the polar decomposition of T°. □

THEOREM 3. Assume that $T = U|T|$ is the polar decomposition of an operator $T \in \mathscr{L}(\mathscr{H})$ where U is unitary. Let $s \geqslant 0$ and $t > 0$. For each positive integer k,

$$\widehat{(T^\circ)}_{s,t}^{(k)} = (\widehat{T}_{s,t}^{(k)})^\circ$$

and $\widehat{(T^\circ)}_{s,t}^{(k)} = U^\circ |\widehat{(T^\circ)}_{s,t}^{(k)}|$ is the polar decomposition where

$$|\widehat{(T^\circ)}_{s,t}^{(k)}| = \sum_{j=0}^{k} \binom{k}{j} s^{k-j} t^j (U^\circ)^{*j} |T|^\circ (U^\circ)^j.$$

Proof. Since T° has the polar decomposition $T^\circ = U^\circ|T|^\circ$ from Lemma 1, we obtain that

$$\widehat{(T^\circ)}_{s,t} = sU^\circ|T|^\circ + t|T|^\circ U^\circ = (sU|T| + t|T|U)^\circ = (\widehat{T}_{s,t})^\circ.$$

Since

$$\widehat{(T^\circ)}_{s,t}^{(k+1)} = \left(\widehat{\widehat{(T^\circ)}_{s,t}}\right)_{s,t}^{(k)} = \widehat{(\widehat{T}_{s,t})^\circ}_{s,t}^{(k)},$$

one can show that $\widehat{(T^\circ)}_{s,t}^{(k)} = (\widehat{T}_{s,t}^{(k)})^\circ$ for each positive integer k using the induction on k, and the expression of $|\widehat{(T^\circ)}_{s,t}^{(k)}|$ follows by Theorem 1 and Lemma 1. $\square$

COROLLARY 5. Assume $T = U|T|$ is the polar decomposition of T in $\mathscr{L}(\mathscr{H})$ where U is unitary. If $s \geqslant 0$, $t > 0$, and k is a positive integer, then $\sigma(\widehat{T}_{s,t}^{(k)}) = \sigma(\widehat{(T^\circ)}_{s,t}^{(k)})$ and $\sigma_{ap}(\widehat{T}_{s,t}^{(k)}) = \sigma_p(\widehat{(T^\circ)}_{s,t}^{(k)})$.

Proof. Since $\widehat{(T^\circ)}_{s,t}^{(k)} = (\widehat{T}_{s,t}^{(k)})^\circ$ by Theorem 3, we obtain from Proposition 1 that

$$\sigma(\widehat{T}_{s,t}^{(k)}) = \sigma((\widehat{T}_{s,t}^{(k)})^\circ) = \sigma(\widehat{(T^\circ)}_{s,t}^{(k)})$$

and

$$\sigma_{ap}(\widehat{T}_{s,t}^{(k)}) = \sigma_p((\widehat{T}_{s,t}^{(k)})^\circ) = \sigma_p(\widehat{(T^\circ)}_{s,t}^{(k)}),$$

as we desired. $\square$

For a bounded sequence $\{\alpha_n\}_{n=0}^\infty$ of positive real numbers, the *weighted shift* with weights $\{\alpha_n\}_{n=0}^\infty$ is the operator $W : \mathscr{H} \to \mathscr{H}$ defined by $We_n = \alpha_n e_{n+1}$ for all $n \geqslant 0$, where $\{e_n\}_{n=0}^\infty$ denotes an orthonormal basis for $\mathscr{H}$, which will be fixed from now on. We finally consider the convergence of iterated weighted mean transforms of weighted shifts. We first note that the iterated weighted mean transforms of a weighted shift is also a weighted shift, which is obtained from easy computations.

LEMMA 2. Let W be a weighted shift on $\mathscr{H}$ with weights $\{\alpha_n\}$ of positive real numbers, and let the numbers $s \geqslant 0$ and $t > 0$. For a positive integer k, the k-th iterated weighted mean transform $\widehat{W}_{s,t}^{(k)}$ is the weighted shift with weights $\{\sum_{j=0}^k \binom{k}{j} s^{k-j} t^j \alpha_{n+j}\}_{n=0}^\infty$.

THEOREM 4. Let W be a weighted shift on $\mathscr{H}$ with monotone decreasing weights $\{\alpha_n\}$ of positive real numbers, and let the numbers $s \geqslant 0$ and $t > 0$ satisfy $s + t = 1$. Then the sequence $\{\widehat{W}_{s,t}^{(k)}\}_{k=1}^\infty$ converges to $(\inf_n \alpha_n)U$ in the norm topology, where U denotes the shift such that $Ue_n = e_{n+1}$ for all $n \geqslant 0$.

Proof. Put $\beta = \inf_n \alpha_n$. Since $\{\alpha_n\}_{n=0}^{\infty}$ is a decreasing sequence of positive real numbers, we see that

$$\sum_{j=0}^{k} \binom{k}{j} s^{k-j} t^j \alpha_j \geqslant \sum_{j=0}^{k} \binom{k}{j} s^{k-j} t^j \beta = (s+t)^k \beta = \beta$$

and

$$\|\widehat{W}_{s,t}^{(k)} - \beta U\| = \sum_{j=0}^{k} \binom{k}{j} s^{k-j} t^j \alpha_j - \beta = \sum_{j=0}^{k} \binom{k}{j} s^{k-j} t^j (\alpha_j - \beta)$$

by Lemma 2. Let $\varepsilon > 0$ be arbitrary. Since $\lim_{n\to\infty} \alpha_n = \beta$, choose a positive integer N such that $0 < \alpha_N - \beta < \varepsilon$. Assume that k is any integer with $k > 2N$. Observe that

$$\begin{aligned}\|\widehat{W}_{s,t}^{(k)} - \beta U\| &\leqslant (\alpha_0 - \beta) \sum_{j=0}^{N-1} \binom{k}{j} s^{k-j} t^j + (\alpha_N - \beta) \sum_{j=N}^{k} \binom{k}{j} s^{k-j} t^j \\ &< (\alpha_0 - \beta) M^k \sum_{j=0}^{N-1} \binom{k}{j} + \varepsilon\end{aligned}$$

where $M := \max\{s,t\}$. Since $\sum_{j=0}^{N-1} \binom{k}{j} \leqslant N \binom{k}{N}$, it follows that

$$\|\widehat{W}_{s,t}^{(k)} - \beta U\| < (\alpha_0 - \beta) N M^k \binom{k}{N} + \varepsilon \leqslant (\alpha_0 - \beta) N \frac{M^k k!}{(k-N)!} + \varepsilon.$$

Since $0 < M < 1$, the series $\sum_{k=1}^{\infty} \frac{M^k k!}{(k-N)!}$ is convergent by the ratio test and hence $\lim_{k\to\infty} \frac{M^k k!}{(k-N)!} = 0$. Since $\varepsilon > 0$ was arbitrary, we have $\lim_{k\to\infty} \|\widehat{W}_{s,t}^{(k)} - \beta U\| = 0$. □

COROLLARY 6. Let W be a weighted shift in $\mathscr{L}(\mathscr{H})$ with monotone increasing weights $\{\alpha_n\}$ of positive real numbers, and let the numbers $s \geqslant 0$ and $t > 0$ satisfy $s + t = 1$. Then $\{\widehat{W}_{s,t}^{(k)}\}_{k=1}^{\infty}$ converges to $(\sup_n \alpha_n) U$ in the norm topology, where U denotes the shift such that $Ue_n = e_{n+1}$ for all $n \geqslant 0$.

Proof. Set $\gamma = \sup_n \alpha_n$. Since $\{\alpha_n\}$ is monotone increasing, we know that the sequence $\{\sum_{j=0}^{k} \binom{k}{j} s^{k-j} t^j \alpha_{n+j}\}_{n=0}^{\infty}$ is also monotone increasing. Hence, we obtain from Lemma 2 that

$$\|\widehat{W}_{s,t}^{(k)} - \gamma U\| = \gamma - \sum_{j=0}^{k} \binom{k}{j} s^{k-j} t^j \alpha_j = \sum_{j=0}^{k} \binom{k}{j} s^{k-j} t^j (\gamma - \alpha_j)$$

for all k. Given $\varepsilon > 0$, there exists a positive integer N such that $0 < \gamma - \alpha_N < \varepsilon$. Let k be an integer with $k > 2N$, and set $M = \max\{s,t\}$. Applying the proof of Theorem 4, one can derive that

$$\|\widehat{W}_{s,t}^{(k)} - \gamma U\| < (\gamma - \alpha_0) N \frac{M^k k!}{(k-N)!} + \varepsilon$$

for all k. Since $\lim_{k\to\infty}\frac{M^k k!}{(k-N)!}=0$ and $\varepsilon>0$ was arbitrary, we complete the proof. $\square$

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