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A NEW FRACTIONAL ORDER POINCARE'S INEQUALITY WITH WEIGHTS

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Abstract. We derive a new Sawyer's type sufficient condition for the fractional order Poincare inequality with weights

$$\Big(\int_{\Omega} |f(x)-\bar{f}_{v,\Omega}|^q v(x)\,dx\Big)^{\frac{1}{q}} \leqslant C\Big(\iint_{\Omega\times\Omega} |f(x)-f(y)|^p\,\omega(x,y)\,dxdy\Big)^{\frac{1}{p}}$$

to hold in a non-regular domain $\Omega\subset R^n$ of finite volume, where $\omega(x,y)=|x-y|^{-n-\alpha p}\omega_0(x,y)$, $0<\alpha<1$, $q\geqslant p>1$, $f\in C(\Omega)$, and $v(\cdot)$, $\omega(\cdot,\cdot)$ are positive measurable functions such that $\omega^{1-p'}(x,\cdot)\,v^{p'}(\cdot)\in L(\Omega)$ a.e. $x\in\Omega$ and $\bar{f}_{v,\Omega}=\frac{1}{v(\Omega)}\int_{\Omega} fv\,dx$.

1. Introduction

The aim of this paper is to further investigate the fractional order weighted Poincare's inequality

$$\Big(\int_{\Omega} |f(x)-\bar{f}_{v,\Omega}|^q v(x)dx\Big)^{\frac{1}{q}} \leqslant C\Big(\iint_{\Omega} |f(x)-f(y)|^p\omega(x,y)dxdy\Big)^{\frac{1}{p}}, \tag{1}$$

in a bounded domain Ω for $q\geqslant p>1$ and a continuous function $f\in C(\Omega)$. In this inequality v,ω are positive measurable functions, Ω is a finite volume non-regular domain in $R^n, n\geqslant 1$ and $\bar{f}_{v,\Omega}=\frac{1}{v(\Omega)}\int_{\Omega} f(x)v(x)\,dx$. It is a well-known open Problem to find necessary and sufficient conditions on the weights $v=v(x)$ and $\omega=\omega(x,y)$ so that (1) holds in very simple domains (see the book [25] and the references therein). However, such kind of inequalities and sufficiency conditions on a domain Ω (e.g. c-John domain, a domain satisfying fat condition on its complementary set, measure density condition domains, etc.) are subject of many studies. Fractional inequalities have important applications e.g. in the study of Brownian motion, in the censored stable processes, in the investigation of transience and boundary behavior of the underlying

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Levi and Markov processes, where the right hand side of the inequalities appears in a Dirichlet form (see, e.g. [3, 9, 19, 21, 36]). One important reason for that is that such inequalities find application in interpolation theory, boundedness of maximal function in Lorentz spaces, and in the study of compactness problems for non-smooth domains.

We continue by giving an elementary background of this type of inequalities. The Hardy inequality on finite interval $(0,l)$ reads

$$\int_0^l \Big(\frac{1}{x}\int_0^x f(t)\,dt\Big)^p dx \leqslant \Big(\frac{p}{p-1}\Big)^p \int_0^l f(x)^p\,dx, \tag{2}$$

where f is a positive measurable function on $(0,l)$ and $p>1$. The constant $\big(\frac{p}{p-1}\big)^p$ is sharp and nowadays it is also known that the integral on the right hand side in (2) can be replaced by $\int_0^l f(x)^p\big[1-\big(\frac{x}{l}\big)^{\frac{p-1}{p}}\big]dx$.

After the change of variable $y=l-x$ in the left hand side of (2) we get that it is equal to

$$\int_0^l \Big(\frac{1}{l-y}\int_y^l g(s)ds\Big)^p dy = \int_0^l \Big(\frac{1}{l-y}\int_0^{l-y} f(t)dt\Big)^p dy = \int_0^l \Big(\frac{1}{x}\int_0^x f(t)dt\Big)^p dx,$$

where $g(t)=f(l-t)$. Changing again the variable $t=l-s$ on the right hand side of (2), we find that it is equal to

$$\Big(\frac{p}{p-1}\Big)^p \int_0^l f(t)^p\,dt = \Big(\frac{p}{p-1}\Big)^p \int_0^l f(l-s)^p\,ds = \Big(\frac{p}{p-1}\Big)^p \int_0^l g(s)^p\,ds.$$

Therefore, by (2), it yields that

$$\int_0^l \Big(\frac{1}{l-y}\int_y^l g(s)\,ds\Big)^p dy \leqslant \Big(\frac{p}{p-1}\Big)^p \int_0^l g(s)^p\,ds. \tag{3}$$

Let u be an absolutely continuous function on R_+ satisfying that $u(0)=u(l)=0$. Then

$$\frac{u(x)}{x(l-x)} = \frac{1}{l}\Big(\frac{u(x)}{x}+\frac{u(x)}{l-x}\Big) = \frac{1}{lx}\int_0^x u'(x)dx + \frac{1}{l(l-x)}\int_x^l (-u'(x))dx.$$

By now applying the triangle inequality of norms together with the inequalities (2) and (3) we get the inequality

$$\int_0^l \Big|\frac{u(x)}{x(l-x)}\Big|^p dx \leqslant \Big(\frac{2p}{l(p-1)}\Big)^p \int_0^l |u'(x)|^p\,dx \tag{4}$$

for all absolutely continuous functions $u(x)$ on the interval $(0,l)$, with $u(0)=u(l)=0$.

J. Necas, in [34], proposed the following extension of (4) for any n dimensional bounded Lipschitz domain Ω for $p>1$:

$$\int\limits_{\Omega} \frac{|f(x)|^p}{\operatorname{dist}(x,\partial\Omega)^p}\,dx \leqslant C_{n,p}(\Omega)\int\limits_{\Omega} |\nabla f|^p\,dx, \quad f\in C_0^\infty(\Omega), \tag{5}$$

where ∇f denotes the gradient of f. Moreover, for a fractional scale $\alpha\in(0,1)$ extension of (5), it was suggested the following modeling inequality

$$\int\limits_{\Omega} \frac{|f(x)|^p}{\operatorname{dist}(x,\partial\Omega)^{\alpha p}}\,dx \leqslant C_{n,p,\alpha}(\Omega)\iint\limits_{\Omega\times\Omega} \frac{|f(x)-f(y)|^p}{|x-y|^{n+\alpha p}}\,dxdy, \quad \alpha p>1, \tag{6}$$

where $\operatorname{dist}(x,\partial\Omega)$ denotes the distance from x to the boundary $\partial\Omega$ of Ω.

In general, inequality (6) does not hold in non-regular domains. If $\alpha p<1$, then there are examples showing that such inequality fails even for smooth domains (see e.g. [12, 39]). For this inequality to hold in Lipshitz domains see [4, 10] and for fat complementary condition domains see [8].

Another extensions of Hardy's inequality (4) was concentrated around its following fractional order analogue. If $n\geqslant 1$, $0<\alpha<1$, then the inequality

$$\int\limits_{\Omega} \frac{|f(x)|^p}{|x|^{\alpha p}}\,dx \leqslant C_{n,\alpha,p}\iint\limits_{\Omega\times\Omega} \frac{|f(x)-f(y)|^p}{|x-y|^{n+\alpha p}}\,dxdy \tag{7}$$

holds for all $f\in C_0^\infty(\Omega)$ in the case $1\leqslant p<\frac{n}{\alpha}$, and for all $f\in C_0^\infty(\Omega\setminus\{0\})$ in the case $p>\frac{n}{\alpha}$. Note that the domain in this inequality need not to be bounded or smooth (for the case $1\leqslant p<\frac{n}{\alpha}$ of this inequality see also [28] and [27, Theorems 1, 3]). For a more exact description of the history and current status of such fractional order Hardy inequalities we refer to Chapter 5 of the new book [25], see also the references therein.

The first study of fractional order Hardy's type inequalities (7) was started in [17, 20] and after that such research was continued e.g. in [6, 7, 16, 18, 22, 23, 24, 26, 40] -essentially in one-dimensional cases. An essential use on boundedness and derivative identities for Hardy's operator and its conjugate was made in those studies. For the inequality (7) see also R.L. Frank and R. Seiringer [14, Theorem 1.1] for exact constant problem, B.Dyda and A.V. Vahakangas [13, Corollary 2] for a generalization of $|x|^{-\alpha p}$ to regularly varying functions.

There are many generalizations of (7) to the general weighted cases. See Chapter 5 of the book [25] and the references therein. Moreover, it is still an open problem to find necessary and sufficient conditions on the weights $v(x)$ and $\omega(x,y)$ so that the weighted version of (7) holds even in the one-dimensional case (see p.295 in [25]). Our study in this paper concerns Poincare's inequality in non-regular domains. We refer to R. Hurri-Syrjanen and her coauthors concerning the fractional order Poincare's inequality in the c-John domains (see [37, 38])

$$\Big(\int\limits_{\Omega} |f(x)-\bar{f}_{\Omega}|^{q_*}\,dx\Big)^{\frac{1}{q_*}} \leqslant C\Big(\iint\limits_{\Omega\times\Omega} \frac{|f(x)-f(y)|^p}{|x-y|^{n+\alpha p}}\,dxdy\Big)^{\frac{1}{p}}, \tag{8}$$

where $0 < \alpha < 1$, $\frac{1}{p} - \frac{1}{q_*} = \frac{\alpha}{n}$ and $\bar{f}_\Omega = \frac{1}{|\Omega|} \int_\Omega f(x)\,dx$. See also [11]. For this inequality to hold in smooth domains see V.G. Mazya and T.O. Shaposhnikova [32, 33].

In this paper we are influenced by some ideas from [27, 28, 30, 31]. However, we derive a much more general inequality which, in particular, do not use Muckenhoupt's condition A_∞. This allows us to consider more general weights and we can avoid additional difficulties with checking the weight assumptions in order to derive effective sufficient conditions. The obtained much weaker condition (10) below on the pair of weights $v(x)$ and $\omega(x,y)$ in Theorem 1 could be considered as a Sawyer's type condition for fractional order Poincare's inequality (1). Another essential fact is that in our conditions we do not use integration over all balls, instead we only need to consider the parts contained in domain Ω (see e.g. Theorem 1). Note especially the Poincare-Sobolev type inequalities play a fundamental role in the study of qualitative properties of partial differential equations (especially [15] and also e.g. [1, 2, 29, 30]).

The main result of this paper (Theorem 1) is presented in Section 2. Moreover, some applications, comparisons with related results and remarks can be found there. The detailed proof of the main result is given in Section 3.

2. The main result with applications

In this paper we use the following notation:

By $C, C_1, C_2, \ldots$ we denote different constants that may change the values, which are unessential for purposes of the paper, at each appearance.

By Ω we denote an open domain in R^n and by $Q(x,r)$ we denote the Euclidean ball with center x and radius r. For a measurable set $E \subset R^n$ the $|E|$ denotes its Leg-esgue measure. For a measurable function v and measurable set E, $v(E)$ denotes the integral of this function over the set E, i.e. $v(E) = \int_E v(x)dx$. $C(\Omega)$ denotes the class of continuous functions in Ω.

DEFINITION 1. For a domain Ω we define σ_Ω as the system of balls:

$$\sigma_\Omega = \Big\{ Q = Q(x,t) : x \in \Omega, 0 < t < d(\Omega) \Big\}. \tag{9}$$

The main result of this paper reads:

THEOREM 1. (Main) *Let $q \geqslant p > 1, n \in Z_+, 0 < \alpha < 1$ and $\Omega \subset R^n$ be a domain with finite volume. Suppose that the positive measurable functions $v(x), \omega(x,y)$ are such that*

$$v(\cdot), \ \ \omega^{1-p'}(x,\cdot)\, v^{p'}(\cdot) \ \in L(\Omega) \qquad a.e. \quad x \in \Omega,$$

where $\omega(x,y) = |x-y|^{-n-\alpha p}\omega_0(x,y)$.

Then for the inequality (1) to hold $\forall f \in C(\Omega)$ it is sufficient that

$$\frac{1}{|Q \cap \Omega|} \Big(\iint\limits_{Q\cap\Omega} \omega(x,y)^{1-p'} v(y)^{p'} dxdy \Big)^{1/p'} \leqslant A \Big(\int\limits_{Q\cap\Omega} v(x)\,dx \Big)^{1/q'} \tag{10}$$

for some $A > 0$ and for all $Q \in \sigma_\Omega$, where, in (1), the constant $C = C_0 A$ with C_0 depending only on n, p and q.

Observe that we do not require that the non-regular domain Ω a priory satisfy any regularity condition like the conditions (11) or (12) below.

Let the bounded domain $\Omega \subset R^n$ satisfy the following measure density condition: there exists a constant $\delta > 0$ such that

$$|Q \cap \Omega| \geqslant \delta |Q| \tag{11}$$

for any ball $Q \in \sigma_\Omega$.

Inserting in Theorem 1 $v(x) = 1$, $\omega = |x-y|^{-n-\alpha p}$ with $0 < \alpha < 1$ we get the following assertion for such domains. We remark that this fact is usually deduced from the extension assertion by Y. Zhou [41]. However, Corollary 1 gives a direct proof (cf. also [11]).

COROLLARY 1. *Let $n \in Z_+, 0 < \alpha < 1, 1 < p < \frac{n}{\alpha}$, $\frac{\alpha}{n} - \frac{1}{p} + \frac{1}{q_*} = 0$ and $\Omega \subset R^n$ be a domain with finite volume and satisfying property (11). Then the inequality (8) holds for any function $f \in C(\Omega)$.*

Concerning this inequality in smooth domains see e.g [27, 32]. It is extended to the c-John domains by R. Hurry-Syrjanen, B. Dyda et al. [11, 37, 38]. Moreover, J. Bourgain, H. Brezis, and P. Mironescu even found the optimal constant C in (8) when Ω is a cube, see [5, Theorem 1].

Let $s \geqslant 1$, Ω be a domain satisfying the following property: there exists a $c > 0$ such that for any $Q \in \sigma_Q$ it holds that

$$|Q \cap \Omega| \geqslant c\, d_Q^{ns} \cdot \tag{12}$$

Inserting in Theorem 1 $v(x) = 1$, $\omega(x,y) = |x-y|^{-n-\alpha p}$ i.e. $\omega_0(x,y) \equiv 1$ with $0 < \alpha < 1$ we get the following statement for a domain Ω satisfying condition (12).

COROLLARY 2. *Let $n \in Z_+, 0 < \alpha < 1, 1 < p < \frac{n(2s-1)}{\alpha}$, $\frac{\alpha}{n} - \frac{2s-1}{p} + \frac{s}{\tilde{q}} = 0$ and $\Omega \subseteq R^n$ be a finite volume domain satisfying (12). Then the inequality*

$$\Big(\int_\Omega |f(x) - \bar{f}_\Omega|^{\tilde{q}}\, dx\Big)^{\frac{1}{\tilde{q}}} \leqslant C\,\Big(\iint_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{n+\alpha p}}\, dxdy\Big)^{\frac{1}{p}}$$

holds for any function $f \in C(\Omega)$, with $\bar{f}_\Omega = \frac{1}{|\Omega|}\int_\Omega f(x)\, dx$.

It is easily seen that the exponent of maximal integration for domains with property (12) fall down if to consider the domains with $s > 1$, i.e. $\tilde{q} = \frac{spn}{(2s-1)n-\lambda p} < q_* = \frac{pn}{n-\lambda p}$ and equals to the Sobolev's power for $s = 1$. This result extends (8) to domains satisfying (12) covering those obtained by H. Surjanen for s-John domains.

REMARK 1. Using our approaches in this paper we can produce the Poincare's type inequality (1) under the condition

$$\frac{1}{|Q|^2}\Big(\int_Q v dx\Big)^{\frac{1}{q}}\Big(\iint_{Q\times Q}\omega^{1-p'}(x,y)dxdy\Big)^{\frac{1}{p'}}\leqslant A \tag{13}$$

for the pair of weights $v(x)\in A_\infty$ and $\omega(x,y):=|x-y|^{-n-\alpha p}\omega_0(x,y)$, exponents $q\geqslant p\geqslant 1$ and the domain satisfying condition (11)(cf. [27, Theorem 3]). Inserting in it $q=p>1$, $v=\operatorname{dist}(x,\partial\Omega)^{-\alpha p}$, $\omega(x,y)=|x-y|^{-n-\alpha p}$ and assuming the following estimate $\forall Q\in\sigma_\Omega$ on John domains

$$\int_Q \operatorname{dist}(x,\partial\Omega)^{-\alpha p}dx\leqslant CQ^{1-\frac{\alpha p}{n}},$$

we obtain the inequality

$$\int_\Omega \frac{|f(x)-\overline{f}_{v,\Omega}|^p}{\operatorname{dist}(x,\partial\Omega)^{\alpha p}}dx\leqslant C_{n,p,\alpha}(\Omega)\iint_{\Omega\times\Omega}\frac{|f(x)-f(y)|^p}{|x-y|^{n+\alpha p}}dxdy,\quad 1<\alpha p<n, \tag{14}$$

for those domains. For that we will invoke Whitney decomposition technique dividing each cube into sufficiently small dyadic sub-cubes (see e.g. [37, p. 391]).To check that $v\in A_2$, in addition, we use the elementary inequality

$$\int_Q \operatorname{dist}(x,\partial\Omega)^{\alpha p}dx\leqslant CQ^{1+\frac{\alpha p}{n}}.$$

3. Proof of Theorem 1

Proof. Let the fixed number $a\in R$ be such that

$$\min\Big\{a\in R:\ v\big(\{x\in\Omega: f(x)\leqslant a\}\big)\Big\}\geqslant\frac{1}{2}v(\Omega). \tag{15}$$

We denote $\Omega_\alpha=\{x\in\Omega:\ f(x)>a+\alpha\}$ for $\alpha>0$. Note that Ω_α is an open set since f is continuous. It is clear that

$$v\big(\{x\in\Omega: f(x)\geqslant a\}\big)\geqslant\frac{1}{2}v\big(\Omega\big).$$

Let γ be a sufficiently small positive number that will be specified later. Suppose that $\alpha>0$ is a fixed number such that Ω_α is nonempty. Choose a connected component $\Omega_\alpha^j\subset\Omega_\alpha\ \ (j=1,2,\ldots)$. We denote the parts of $\Omega_{3\alpha}$ and $\Omega_{2\alpha}$ contained in Ω_α^j by $\Omega_{3\alpha,j}$ and $\Omega_{2\alpha,j}$, respectively (the sets $\Omega_{3\alpha,j}$ and $\Omega_{2\alpha,j}$ may be disconnected).

Let the set $\Omega_{3\alpha,j} \subset \Omega_\alpha^j$ be nonempty.

For any fixed point $x \in \Omega_{3\alpha,j}$ there exists a ball $Q = Q(x,\rho(x))$ such that

$$v\left(Q \cap \Omega \setminus \Omega_\alpha^j\right) = \gamma v(Q \cap \Omega). \tag{16}$$

Indeed, if $0 < \gamma < 1,$ then the continuous function

$$F(t) = \frac{1}{\gamma} v\left(Q(x,t) \cap \Omega \setminus \Omega_\alpha^j\right) - v\left(Q(x,t) \cap \Omega\right),\ t > 0,$$

is negative for sufficiently small $t > 0$ since x is an interior point of $\Omega_{3\alpha,j}$. In view of our choice of a from (15), $F(t)$ is positive for $t = d(\Omega)$:

$$\begin{aligned} F(d(\Omega)) &= \frac{1}{\gamma} v\left(Q(x,d(\Omega)) \cap \Omega \setminus \Omega_\alpha^j\right) - v\left(Q(x,d(\Omega)) \cap \Omega\right) \\ &\geqslant \frac{1}{2\gamma} v(\Omega) - v\left(Q(x,d(\Omega)) \cap \Omega\right) = \left(\frac{1}{2\gamma} - 1\right) v(\Omega) > 0. \end{aligned}$$

By applying Cauchy's theorem, we find that

$$F(t^*) = 0 \text{ for some } t^* \in (0, d(\Omega)),$$

so setting $\rho(x) = t^*$ we can conclude that (16) holds.

1) If

$$v\left(Q \cap \Omega_{3\alpha,j}\right) \leqslant \gamma v(Q \cap \Omega), \tag{17}$$

then by using (16) it follows that

$$\begin{aligned} v(Q \cap \Omega) &= v\left(Q \cap \Omega \setminus \Omega_\alpha^j\right) + v\left(Q \cap \Omega_\alpha^j\right) \\ &\leqslant \gamma v(Q \cap \Omega) + v\left(Q \cap \Omega_\alpha^j\right). \end{aligned}$$

This fact and (17) yield that

$$v\left(Q \cap \Omega_{3\alpha,j}\right) \leqslant \frac{\gamma}{1-\gamma} v\left(Q \cap \Omega_\alpha^j\right). \tag{18}$$

2) Now, let

$$v\left(Q \cap \Omega_{3\alpha,j}\right) > \gamma v(Q \cap \Omega). \tag{19}$$

Then at least one of the following conditions holds:

$$a)\quad |Q \cap \Omega_{2\alpha,j}| \geqslant \frac{1}{2} |Q \cap \Omega| \tag{20}$$

or

$$b)\ \ |Q \cap \Omega \setminus \Omega_{2\alpha,j}| \geqslant \frac{1}{2} |Q \cap \Omega|. \tag{21}$$

Assume that a) is satisfied. Then, by using (16) and (20), we get that

$$\int\limits_{Q\bigcap\Omega\setminus\Omega_\alpha^j} v(y)dy \int\limits_{Q\bigcap\Omega_{2\alpha,j}} dx \geqslant \frac{\gamma}{2} v\left(Q\cap\Omega\right)|Q\cap\Omega|.$$

Therefore,

$$1 \leqslant \frac{2}{\gamma v(Q\cap\Omega)\,|Q\cap\Omega|} \int\limits_{Q\bigcap\Omega_{2\alpha,j}} \Big(\int\limits_{Q\bigcap\Omega\setminus\Omega_\alpha^j} v(y)dy\Big)\, dx.$$

Applying Hölder's inequality we obtain that

$$\begin{aligned}
1 &\leqslant \frac{2}{\gamma v(Q\cap\Omega)\,|Q\cap\Omega|} \int\limits_{Q\bigcap\Omega_{2\alpha,j}} \Big(\int\limits_{Q\bigcap\Omega\setminus\Omega_\alpha^j} \omega^{1/p}\cdot\omega^{-1/p} v(y)\, dy\Big)\, dx \\
&\leqslant \frac{2}{\gamma v(Q\cap\Omega)\,|Q\cap\Omega|} \Big(\int\limits_{Q\bigcap\Omega_{2\alpha,j}} \Big(\int\limits_{Q\bigcap\Omega\setminus\Omega_\alpha^j} \omega\, dy\Big) dx\Big)^{\frac{1}{p}} \times \\
&\times \Big(\int\limits_{Q\bigcap\Omega\setminus\Omega_\alpha^j} \Big(\int\limits_{Q\bigcap\Omega_{2\alpha,j}} \omega^{1-p'}\, dx\Big) v(y)^{p'} dy\Big)^{\frac{1}{p'}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
v(Q\cap\Omega_{3\alpha,j}) \leqslant \frac{2}{\gamma|Q\cap\Omega|} &\Big(\int_{Q\bigcap\Omega\setminus\Omega_\alpha^j} \Big(\int_{Q\bigcap\Omega_{2\alpha,j}} \omega(x,y)^{1-p'}\, dx\Big) v(y)^{p'}\, dy\Big)^{1/p'} \times \\
&\times \Big(\int_{Q\bigcap\Omega\setminus\Omega_\alpha^j} \Big(\int_{Q\bigcap\Omega_{2\alpha,j}} \omega(x,y)dx\Big) dy\Big)^{1/p}.
\end{aligned}$$

We use the condition (10) and find that

$$v(Q\cap\Omega_{3\alpha,j}) \leqslant \frac{2A}{\gamma} v(Q\cap\Omega)^{\frac{1}{q'}} \Big(\int\limits_{Q\bigcap\Omega\setminus\Omega_\alpha^j} dy \int\limits_{Q\bigcap\Omega_{2\alpha,j}} \omega(x,y)\, dx\Big)^{\frac{1}{p}}. \tag{22}$$

b) Now, assume that

$$|Q\cap\Omega\setminus\Omega_{2\alpha,j}| \geqslant \frac{1}{2}|Q\cap\Omega|. \tag{23}$$

We may repeat all arguments above, for example, in this case using (19) and (23) it follows that

$$\int_{Q\bigcap\Omega_{3\alpha,j}} \Big(\int_{Q\bigcap\Omega\setminus\Omega_{2\alpha,j}} dx\Big) v(y)\, dy \geqslant \frac{\gamma}{2} v(Q\cap\Omega)|Q\cap\Omega|.$$

We conclude that

$$v(Q\cap\Omega_{3\alpha,j}) \leqslant \frac{2A}{\gamma} v(Q\cap\Omega)^{\frac{1}{q'}} \Big(\int_{Q\bigcap\Omega_{3\alpha,j}} \Big(\int_{Q\bigcap\Omega\setminus\Omega_{2\alpha,j}} \omega(x,y)\, dx\Big)\, dy\Big)^{\frac{1}{p}}, \tag{24}$$

where it has been used that, according to (10), it holds that

$$\frac{1}{|Q\cap\Omega|}\Big(\int\limits_{Q\bigcap\Omega_{3\alpha,j}}\Big(\int\limits_{Q\bigcap\Omega\setminus\Omega_{2\alpha,j}}\omega(x,y)^{1-p'}dx\Big)v(y)^{p'}dy\Big)^{\frac{1}{p'}}\leqslant Av(Q\cap\Omega)^{\frac{1}{q'}}.$$

By combining (24) and (22), we get that

$$v(Q\cap\Omega_{3\alpha,j})\leqslant\frac{2A}{\gamma}v(Q\cap\Omega)^{\frac{1}{q'}}\Big[\Big(\int_{Q\cap\Omega_{3\alpha,j}}\big(\int_{Q\bigcap\Omega\setminus\Omega_{2\alpha,j}}\omega(x,y)\,dx\big)\,dy\Big)^{1/p} + \Big(\int_{Q\bigcap\Omega\setminus\Omega_\alpha^j}dy\Big(\int_{Q\bigcap\Omega_{2\alpha,j}}\omega(x,y)\,dx\Big)^{1/p}\Big]. \tag{25}$$

In case 2) by using (19) and (25) we have the inequality

$$v(Q\cap\Omega_{3\alpha,j})\leqslant\frac{2A}{\gamma^{1+\frac{1}{q'}}}v(Q\cap\Omega_{3\alpha,j})^{\frac{1}{q'}}\Big[\Big(\int_{Q\cap\Omega_{3\alpha,j}}\big(\int_{Q\bigcap\Omega\setminus\Omega_{2\alpha,j}}\omega(x,y)\,dx\big)\,dy\Big)^{1/p} + \Big(\int_{Q\bigcap\Omega\setminus\Omega_\alpha^j}dy\Big(\int_{Q\bigcap\Omega_{2\alpha,j}}\omega(x,y)\,dx\Big)^{1/p}\Big]. \tag{26}$$

It is not difficult to see that $\sup\limits_{x\in\Omega_{3\alpha,j}}\rho(x)<\infty.$ By now applying Besikoviche's covering Lemma (see e.g. [35]) to the system of balls $\{Q=Q(x,\rho(x))\}_{x\in\Omega_{3\alpha,j}}$ that covers $\Omega_{3\alpha,j}$, we find a countable subcover $\{Q^i\},\ i\in N,$ such that

$$\sum_i\chi_{Q^i}(x)\leqslant\kappa_n,\quad \Omega_{3\alpha,j}\subset\bigcup_i Q^i. \tag{27}$$

From (26) and (18) it follow that

$$v(Q^i\cap\Omega_{3\alpha,j})\leqslant\frac{2A}{\gamma^{1+\frac{1}{q'}}}v(Q^i\cap\Omega_{3\alpha,j})^{\frac{1}{q'}}\Big[\Big(\int_{Q^i\cap\Omega_{3\alpha,j}}\big(\int_{Q^i\bigcap\Omega\setminus\Omega_{2\alpha,j}}\omega(x,y)\,dx\big)\,dy\Big)^{1/p} + \Big(\int_{Q^i\bigcap\Omega\setminus\Omega_\alpha^j}dy\Big(\int_{Q^i\bigcap\Omega_{2\alpha,j}}\omega(x,y)\,dx\Big)^{1/p}\Big]+\frac{\gamma}{1-\gamma}v\left(Q^i\cap\Omega_\alpha^j\right). \tag{28}$$

Summing (28) over $i\in N,$ applying (27) and using Hölder's inequality, we obtain that

$$v(\Omega_{3\alpha,j})\leqslant\frac{2A}{\gamma^{1+\frac{1}{q'}}}\Big(\sum_i\big(v(Q^i\cap\Omega_{3\alpha,j})\big)^{\frac{p'}{q'}}\Big)^{\frac{1}{p'}}\Big[\sum_i\Big(\int\limits_{Q^i\bigcap\Omega_{3\alpha,j}}\Big(\int\limits_{Q^i\bigcap\Omega\setminus\Omega_{2\alpha,j}}\omega\,dx\Big)\,dy\Big) + \sum_i\Big(\int\limits_{Q^i\bigcap\Omega\setminus\Omega_\alpha^j}\Big(\int\limits_{Q^i\cap\Omega_{2\alpha,j}}\omega\,dx\Big)\,dy\Big)\Big]^{\frac{1}{p}}+\frac{\gamma}{1-\gamma}\sum_i v\left(Q^i\cap\Omega_\alpha^j\right).$$

Since $\frac{p'}{q'} \geqslant 1,$ we also have that

$$v(\Omega_{3\alpha,j}) \leqslant \frac{2A}{\gamma^{1+\frac{1}{q'}}} v\Big(\bigcup_i Q^i \cap \Omega_{3\alpha,j}\Big)^{\frac{1}{q'}} \Big[\Big(\int\limits_{\bigcup_i Q^i \bigcap \Omega_{3\alpha,j}} \Big(\int\limits_{\bigcup_i Q^i \bigcap \Omega \setminus \Omega_{2\alpha,j}} \omega\, dx\Big)\, dy\Big) + \Big(\int\limits_{\bigcup_i Q^i \bigcap \Omega \setminus \Omega_\alpha^j} \Big(\int\limits_{\bigcup_i Q^i \cap \Omega_{2\alpha,j}} \omega\, dx\Big)\, dy\Big)\Big]^{1/p} + \frac{\gamma}{1-\gamma} v\left(\bigcup_i Q^i \cap \Omega_\alpha^j\right).$$

Therefore,

$$v(\Omega_{3\alpha,j}) \leqslant \frac{2\kappa_n^{\frac{1}{q'}} A}{\gamma^{1+\frac{1}{q'}}} v(\Omega_{3\alpha,j})^{\frac{1}{q'}} \Big[\Big(\int\limits_{\Omega_{3\alpha,j}} \Big(\int\limits_{\Omega \setminus \Omega_{2\alpha,j}} \omega\, dx\Big)\, dy\Big) + \Big(\int\limits_{\Omega \setminus \Omega_\alpha^j} \Big(\int\limits_{\Omega_{2\alpha,j}} \omega\, dx\Big)\, dy\Big)\Big]^{1/p} + \frac{\kappa_n \gamma}{1-\gamma} v\left(\Omega_\alpha^j\right).$$

Again summing this inequality over j, keeping in mind the constructions in the beginning, and the assumption $\frac{q}{p} \geqslant 1,$ we get that

$$v(\Omega_{3\alpha}) \leqslant \frac{2\kappa_n^{\frac{1}{q'}} A}{\gamma^{1+\frac{1}{q'}}} v(\Omega_{3\alpha})^{\frac{1}{q'}} \Big[\Big(\int\limits_{\Omega_{3\alpha}} \Big(\int\limits_{\Omega \setminus \Omega_{2\alpha}} \omega\, dx\Big)\, dy\Big) + \Big(\int\limits_{\Omega \setminus \Omega_\alpha} \Big(\int\limits_{\Omega_{2\alpha}} \omega\, dx\Big)\, dy\Big)\Big]^{\frac{1}{p}} + \frac{\kappa_n \gamma}{1-\gamma} v\left(\Omega_\alpha\right). \tag{29}$$

In particular, it follows from (29) that,

$$v\big(x \in \Omega :\ f(x) - a > 3\alpha\big) \leqslant \frac{2^{1+\frac{1}{p}} \kappa_n^{\frac{1}{q'}} A}{\gamma^{1+\frac{1}{q'}}} v(\Omega_{3\alpha}^+)^{\frac{1}{q'}} \Big(\iint\limits_{\{x \in \Omega^+ :\ |f(x)-f(y)|>\alpha\}} \omega\, dxdy\Big)^{\frac{1}{p}} + \frac{c_n \gamma}{1-\gamma} v\left(\Omega_\alpha\right). \tag{30}$$

Considering the function $a - f(x)$ in place of $f(x) - a$ and the domain $\Omega^- = \{x \in \Omega :\ a - f(x) > 0\}$ in place of $\Omega^+ = \{x \in \Omega :\ f(x) - a > 0\}$, we get the analogous inequality

$$v\big(x \in \Omega :\ a - f(x) > 3\alpha\big) \leqslant \frac{2^{1+\frac{1}{p}} \kappa_n^{\frac{1}{q'}} A}{\gamma^{1+\frac{1}{q'}}} v(\Omega_{3\alpha}^-)^{\frac{1}{q'}} \Big(\iint\limits_{\{x \in \Omega^- :\ |f(x)-f(y)|>\alpha\}} \omega\, dxdy\Big)^{\frac{1}{p}} + \frac{c_n \gamma}{1-\gamma} v\left(\Omega_\alpha\right). \tag{31}$$

This fact can be proved absolutely similarly as we proved (30) so we delete the details.

It follows from (30) and (31) that

$$\begin{aligned} & v\big(x\in\Omega:\ |f(x)-a|>3\alpha\big) \\ & \leqslant \frac{2^{1+\frac{1}{p}}\kappa_n^{\frac{1}{q'}}A}{\gamma^{1+\frac{1}{q'}}} v(\Omega_{3\alpha})^{\frac{1}{q'}}\Big(\iint\limits_{\{x\in\Omega:\ |f(x)-f(y)|>\alpha\}} \omega\,dxdy\Big)^{\frac{1}{p}}+\frac{c_n\gamma}{1-\gamma}v(\Omega_\alpha).\end{aligned}$$

By integrating this and again applying Hölder's inequality we obtain that

$$\begin{aligned} & \int_0^\infty v(\Omega_{3\alpha})d\alpha^q \\ & \leqslant \frac{2^{1+\frac{1}{p}}\kappa_n^{\frac{1}{q'}}A}{\gamma^{1+\frac{1}{q'}}}\Big(\int_0^\infty v(\Omega_{3\alpha})d\alpha^q\Big)^{\frac{1}{q'}}\Big(\int_0^\infty\Big(\iint\limits_{|f(x)-f(y)|>\alpha}\omega\,dxdy\Big)^{\frac{q}{p}}d\alpha^q\Big)^{\frac{1}{q}} \\ & +\frac{\kappa_n\gamma}{1-\gamma}\int_0^\infty v(\Omega_\alpha)\,d\alpha^q.\end{aligned}$$

By using Minkowski's inequality, we find that

$$\begin{aligned} \Big(\int_0^\infty\Big(\iint\limits_{|f(x)-f(y)|>\alpha}\omega\,dxdy\Big)^{\frac{q}{p}}d\alpha^q\Big)^{\frac{1}{q}} & \leqslant\Big(\iint\limits_\Omega\Big(\int_0^{|f(x)-f(y)|}\omega(x,y)^{\frac{q}{p}}d\alpha^q\Big)^{\frac{p}{q}}dxdy\Big)^{\frac{1}{p}} \\ & =\Big(\iint\limits_\Omega|f(x)-f(y)|^p\omega(x,y)dxdy\Big)^{\frac{1}{p}}.\end{aligned} \tag{32}$$

Since

$$\int_0^\infty v\big(x\in\Omega:\ |f(x)-a|>3\alpha\big)d\alpha^q=\frac{1}{3^q}\int_\Omega|f(x)-a|^q v(x)dx$$

and

$$\int_0^\infty v(\Omega_\alpha)d\alpha^q=\int_\Omega|f(x)-a|^q v(x)dx,$$

by using the last inequality and (32), it follows that

$$\begin{aligned} & \Big(\frac{1}{3^q}-\frac{\kappa_n\gamma}{1-\gamma}\Big)\int_\Omega|f(x)-a|^q v(x)dx \\ & \leqslant\frac{2^{1+\frac{1}{p}}\kappa_n^{\frac{1}{q'}}A}{3^{\frac{1}{q'}}\gamma^{1+\frac{1}{q'}}}\Big(\int_\Omega|f(x)-a|^q v(x)dx\Big)^{\frac{1}{q'}}\Big(\iint\limits_\Omega|f(x)-f(y)|^p\omega(x,y)dxdy\Big)^{\frac{1}{p}}.\end{aligned}$$

Up to now we have only made the restriction $\gamma < 1$ on the crucial parameter γ but now we do the final restriction and choose γ as a sufficiently small positive number so that $\frac{1}{3^q} - \frac{\kappa_n \gamma}{1-\gamma} > 0$ and we can conclude that

$$\begin{aligned} &\Big(\int_\Omega |f(x) - a|^q v(x) dx \Big)^{\frac{1}{q}} \\ &\leqslant c_0 A \Big(\iint_\Omega |f(x) - f(y)|^p \omega \, dxdy \Big)^{1/p}, \end{aligned}$$

with $c_0 = \frac{2^{1+\frac{1}{p}} \kappa_n^{\frac{1}{q'}} A}{3^{\frac{1}{q'}} \gamma^{1+\frac{1}{q'}}} \left(\frac{1}{3^q} - \frac{c_n \gamma}{1-\gamma} \right)^{-1}$.

To finalize the proof of Theorem 1 it remains to prove that

$$\left\| v(.)^{\frac{1}{q}} \left(f - \overline{f}_{\Omega,v} \right) \right\|_{L^q(\Omega)} \leqslant 2 \left\| v(.)^{\frac{1}{q}} \left(f(.) - a \right) \right\|_{L^q(\Omega)}.$$

(see, e.g. inequality (3.26) in [30] or (22) in [27]). Indeed, using an elementary convexity inequality

$$(\xi + \eta)^q \leqslant 2^{q-1}(\xi^q + \eta^q), \quad \xi, \eta \geqslant 0, \quad q \geqslant 1,$$

we get that

$$\begin{aligned} &\left\| v(.)^{\frac{1}{q}} \left(f(.) - \overline{f}_{\Omega,v} \right) \right\|^q_{L^q(\Omega)} \\ &\leqslant 2^{q-1} \left\| v(.)^{\frac{1}{q}} \left(f(.) - a \right) \right\|^q_{L^q(\Omega)} + 2^{q-1} \left\| v(.)^{\frac{1}{q}} \left(a - \overline{f}_{\Omega,v} \right) \right\|^q_{L^q(\Omega)} \\ &\leqslant 2^q \left\| v(.)^{\frac{1}{q}} \left(f(.) - a \right) \right\|^q_{L^q(\Omega)}, \end{aligned}$$

since, by Hölder's inequality,

$$\left\| v(.)^{\frac{1}{q}} \left(a - \overline{f}_{\Omega,v} \right) \right\|_{L^q(\Omega)} = \left\| v(.)^{\frac{1}{q}} \left| \frac{1}{v(\Omega)} \int_\Omega v(x) \big(f(x) - a \big) dx \right| \right\|_{L^q(\Omega)} \leqslant \left\| v(.)^{\frac{1}{q}} \left(f(.) - a \right) \right\|_{L^q(\Omega)}.$$

The proof is complete.

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