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THE POLAR ORLICZ–BRUNN–MINKOWSKI INEQUALITIES

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Abstract. In this paper, we establish some Orlicz-Brunn-Minkowski type inequalities for (dual) quermassintegrals of polar bodies and star dual bodies, respectively.

1. Introduction

The Brunn-Minkowski inequality for quermassintegrals can be stated as follows: Let K and L be convex bodies (compact convex sets with nonempty interior) in $\mathbb{R}^n$ and let $0 \leqslant i \leqslant n-1$. Then

$$\left(\frac{W_i(K)}{W_i(K+L)}\right)^{\frac{1}{n-i}} + \left(\frac{W_i(L)}{W_i(K+L)}\right)^{\frac{1}{n-i}} \leqslant 1, \tag{1.1}$$

with equality if and only if K and L are homothetic. Here $K+L = \{x+y : x \in K, y \in L\}$, and $W_i(K)$ denotes the i-th quermassintegral of K. The case $i=0$ of (1.1) is the classical Brunn-Minkowski inequality. It works as the cornerstone of the Brunn-Minkowski theory. There are a huge amount of work on its generalizations and on its connections with other areas. An excellent survey on this inequality is provided by Gardner [3].

The L_p-Minkowski addition $+_p$ was introduced by Firey [2]. Let $\mathscr{K}_o^n$ denote the set of convex bodies in $\mathbb{R}^n$ that contain the origin in their interiors. For $K, L \in \mathscr{K}_o^n$ and $p \geqslant 1$, the L_p-Minkowski addition $+_p$ is defined by

$$h_{K+_pL}(x)^p = h_K(x)^p + h_L(x)^p,$$

for $x \in \mathbb{R}^n$, where h_M denotes the support function of the set M. In the mid 1990's, it was shown in [11, 12], that when L_p-addition is combined with volume the result is an embryonic L_p-Brunn-Minkowski theory. The L_p-Brunn-Minkowski inequality

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for quermassintegrals was established by Lutwak [11]: Let $K,L \in \mathscr{K}_o^n$, $p \geqslant 1$, and let $0 \leqslant i \leqslant n-1$. Then

$$\left(\frac{W_i(K)}{W_i(K+_p L)}\right)^{\frac{p}{n-i}} + \left(\frac{W_i(L)}{W_i(K+_p L)}\right)^{\frac{p}{n-i}} \leqslant 1,$$

with equality if and only if K and L are dilates.

The Orlicz-Brunn-Minkowski theory was launched by Lutwak, Yang and Zhang in a series of papers [8, 13, 14]. This theory has been considerably developed in the recent years. In 2014, Gardner, Hug and Weil [5] introduced the concept of the Orlicz addition. Let Φ be the class of convex, strictly increasing functions, $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$. For $K,L \in \mathscr{K}_o^n$, and $\phi \in \Phi$, the Orlicz addition $+_\phi$ is defined by

$$\phi\left(\frac{h_K(x)}{h_{K+_\phi L}(x)}\right) + \phi\left(\frac{h_L(x)}{h_{K+_\phi L}(x)}\right) = \phi(1), \tag{1.2}$$

for $x \in \mathbb{R}^n$. In particular, if $\phi(t) = t^p$ $(p \geqslant 1)$, then $+_\phi = +_p$.

Xiong and Zou [21] established the following Orlicz-Brunn-Minkowski inequality for quermassintegrals. Let $K,L \in \mathscr{K}_o^n$, $\phi \in \Phi$, and $0 \leqslant i \leqslant n-1$. Then

$$\phi\left(\left(\frac{W_i(K)}{W_i(K+_\phi L)}\right)^{\frac{1}{n-i}}\right) + \phi\left(\left(\frac{W_i(L)}{W_i(K+_\phi L)}\right)^{\frac{1}{n-i}}\right) \leqslant \phi(1). \tag{1.3}$$

If ϕ is strictly convex, equality holds if and only if K and L are dilates. The case $i = 0$ was established by Gardner, Hug and Weil [5] (see also Xi, Jin and Leng [20]).

One aim of this paper is to establish the following Orlicz-Brunn-Minkowski type inequality for dual quermassintegrals of polar bodies. From now on, K^* will denote the polar body of K.

THEOREM 1.1. *Let $K,L \in \mathscr{K}_o^n$, $\phi \in \Phi$, and $0 \leqslant i \leqslant n-1$. Then*

$$\phi\left(\left(\frac{\widetilde{W}_i(K^*)}{\widetilde{W}_i((K+_\phi L)^*)}\right)^{-\frac{1}{n-i}}\right) + \phi\left(\left(\frac{\widetilde{W}_i(L^*)}{\widetilde{W}_i((K+_\phi L)^*)}\right)^{-\frac{1}{n-i}}\right) \leqslant \phi(1).$$

If ϕ is strictly convex, equality holds if and only if K and L are dilates.

The dual Brunn-Minkowksi theory for star bodies was initiated by Lutwak [10] in 1970's. In the dual Brunn-Minkowski theory, mixed volumes and Minkowski addition are replaced by dual mixed volumes and radial addition, respectively. Gardner, Hug, Weil and Ye [6] introduced the concept of radial Orlicz additions. Let $\mathscr{S}^n$ denote the set of star bodies with respect to the origin in $\mathbb{R}^n$, i.e., the family of all star-shaped sets with positive and continuous radial function. Let $\widetilde{\Phi}$ be the set of all continuous functions $\psi : [0,\infty) \to [0,\infty)$ that are strictly increasing and such that $\psi(0) = 0$ and $\lim_{t\to\infty} \psi(t) = \infty$. Let $\widetilde{\Psi}$ be the set of all continuous functions $\psi : (0,\infty) \to$

$[0,\infty)$ that are strictly decreasing and such that $\lim_{t\to 0^+}\psi(t)=\infty$ and $\lim_{t\to\infty}\psi(t)=0$. For $K,L\in\mathscr{S}^n$ and $\psi\in\widetilde{\Phi}\cup\widetilde{\Psi}$, the radial Orlicz addition $\widetilde{+}_\psi$ is defined by

$$\psi\left(\frac{\rho_K(x)}{\rho_{K\widetilde{+}_\psi L}(x)}\right)+\psi\left(\frac{\rho_L(x)}{\rho_{K\widetilde{+}_\psi L}(x)}\right)=\psi(1), \tag{1.4}$$

for $x\in\mathbb{R}^n\backslash\{o\}$.

We also establish the following dual Orlicz-Brunn-Minkowski inequality for quermassintegrals of polar bodies, which is the dual form of Theorem 1.1.

THEOREM 1.2. *Let $K,L\in\mathscr{K}_o^n$, $\psi\in\widetilde{\Psi}$ such that $\phi(t)=\psi(t^{-1})$ is strictly convex, and $0\leqslant i\leqslant n-1$. Then*

$$\psi\left(\left(\frac{W_i(K^*)}{W_i((K\widetilde{+}_\psi L)^*)}\right)^{-\frac{1}{n-i}}\right)+\psi\left(\left(\frac{W_i(L^*)}{W_i((K\widetilde{+}_\psi L)^*)}\right)^{-\frac{1}{n-i}}\right)\leqslant\psi(1),$$

with equality if and only if K and L are dilates.

We would like to notice that both Theorem 1.1 and Theorem 1.2 are stated in a non-natural setting: Theorem 1.1 deals with dual quermassintegrals and classical Orlicz addition whereas Theorem 1.2 does it for classical quermassintegrals and radial Orlicz addition. Unfortunately, we have not been able to obtain here the suitable versions of these results in their usual framework.

Another aim of this paper is to establish the following Orlicz-Brunn-Minkowski type inequality for dual quermassintegrals of star dual bodies. From now on, K^o will denote the dual star body of K.

THEOREM 1.3. *Let $K,L\in\mathscr{S}^n$, $\psi\in\widetilde{\Phi}\cup\widetilde{\Psi}$, and $0\leqslant i\leqslant n-1$. If $\psi_0(t)=\psi(t^{-\frac{1}{n-i}})$ is concave, then*

$$\psi\left(\left(\frac{\widetilde{W}_i(K^o)}{\widetilde{W}_i((K\widetilde{+}_\psi L)^o)}\right)^{-\frac{1}{n-i}}\right)+\psi\left(\left(\frac{\widetilde{W}_i(L^o)}{\widetilde{W}_i((K\widetilde{+}_\psi L)^o)}\right)^{-\frac{1}{n-i}}\right)\geqslant\psi(1),$$

while if ψ_0 is convex, the inequality is reversed. If ψ_0 is strictly concave (or convex, as appropriate), equality holds if and only if K and L are dilates.

2. Notation and background material

A convex body is a compact convex set of $\mathbb{R}^n$ with nonempty interior. For a convex body K, the support function $h_K:\mathbb{R}^n\to\mathbb{R}$ is defined by $h_K(x)=\sup\{x\cdot y: y\in K\}$, where $x\cdot y$ denotes the standard inner product of x and y in $\mathbb{R}^n$. For $0\leqslant i\leqslant n-1$, let $W_i(K)$ denote the ith quermassintegral of K. It has the following integral representation:

$$W_i(K)=\frac{1}{n}\int_{S^{n-1}}h_K(u)dS_{n-i-1}(K,u),$$

where $S_{n-i-1}(K,\cdot)$ is $(n-i-1)$th surface area measure of K. In particular, $W_0(K)=V(K)$, $nW_1(K)=S(K)$, and $W_n(K)=V(B)$, where B is the unit ball in $\mathbb{R}^n$ and V,S denote, respectively, the volume and the surface area of the set involved. For a general reference about quermassintegrals we refer the reader to [18].

A compact set $K\subset\mathbb{R}^n$ is a star-shaped set (with respect to the origin) if the intersection of every straight line through the origin with K is a line segment.

The radial function $\rho_K:\mathbb{R}^n\backslash\{0\}\to\mathbb{R}$ of a compact star-shaped set K (with respect to the origin) is defined by $\rho_K(x)=\max\{\lambda\geqslant 0:\lambda x\in K\}$. If ρ_K is positive and continuous, then we call K a star body (with respect to the origin).

Given star bodies $K_1,\dots,K_n$ in $\mathbb{R}^n$, the dual mixed volume $\widetilde{V}(K_1,\dots,K_n)$ is defined by (see [10])

$$\widetilde{V}(K_1,\dots,K_n)=\frac{1}{n}\int_{S^{n-1}}\rho_{K_1}(u)\cdots\rho_{K_n}(u)dS(u), \tag{2.1}$$

where $dS(u)$ is the spherical Lebesgue measure of S^{n-1}. If $K_1=\cdots=K_{n-i}=K$ and $K_{n-i+1}=\cdots=K_n=B$, then the dual mixed volume $\widetilde{V}(\underbrace{K,\dots,K}_{n-i},\underbrace{B,\dots,B}_{i})$ is written as $\widetilde{W}_i(K)$ and is called the dual quermassintegral of K. In particular, $\widetilde{W}_0(K)=V(K)$ and $\widetilde{W}_n(K)=V(B)$. The dual mixed quermassintegral $\widetilde{W}_i(K,L)$ is defined by

$$(n-i)\widetilde{W}_i(K,L)=\lim_{\varepsilon\to 0^+}\frac{\widetilde{W}_i(K\widetilde{+}\varepsilon\cdot L)-\widetilde{W}_i(K)}{\varepsilon}.$$

And it has the following integral representation:

$$\widetilde{W}_i(K,L)=\frac{1}{n}\int_{S^{n-1}}\rho_K(u)^{n-i-1}\rho_L(u)dS(u).$$

In particular, $\widetilde{W}_i(K,K)=\widetilde{W}_i(K)$.

The dual Minkowski inequality for dual mixed quermassintegrals states that (see [4]): Let K,L be star bodies in $\mathbb{R}^n$ and let $0\leqslant i<n-1$. Then

$$\widetilde{W}_i(K,L)^{n-i}\leqslant\widetilde{W}_i(K)^{n-i-1}\widetilde{W}_i(L),$$

with equality if and only if K and L are dilates.

Let $\psi\in\widetilde{\Phi}\cup\widetilde{\Psi}$ and K,L be star bodies in $\mathbb{R}^n$. For $0\leqslant i\leqslant n-1$, the dual mixed Orlicz-quermassintegral $\widetilde{W}_{\psi,i}(K,L)$ is defined by

$$\frac{n-i}{\psi_r'(1)}\widetilde{W}_{\psi,i}(K,L)=\lim_{\varepsilon\to 0^+}\frac{\widetilde{W}_i(K\widetilde{+}_\psi\varepsilon\cdot L)-\widetilde{W}_i(K)}{\varepsilon}. \tag{2.2}$$

Here ψ_r' denotes the right derivative of ψ.

The polar body K^* of a convex body K is the convex body defined by

$$K^*=\{x\in\mathbb{R}^n:x\cdot y\leqslant 1\text{ for all }y\in K\}.$$

It is easy to see that $(K^*)^* = K$. If K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior then, for every $u \in S^{n-1}$,

$$h_{K^*}(u) = \frac{1}{\rho_K(u)}. \tag{2.3}$$

A possible way to define the 'polar' body of a star body K was provided by Moszyńska [15] (see also [16]). Let $i : \mathbb{R}^n \backslash \{o\} \to \mathbb{R}^n \backslash \{o\}$ be defined by

$$i(x) := \frac{x}{|x|^2}.$$

Moszyńska [15] introduced the dual star body K^o of a star body K as

$$K^o = \mathrm{cl}(\mathbb{R}^n \backslash i(K)).$$

It is easy to verify that for every $u \in S^{n-1}$ (see [15]),

$$\rho_{K^o}(u) = \frac{1}{\rho_K(u)}.$$

Suppose that μ is a probability measure on a space X and $g : X \to I \subset \mathbb{R}$ is a μ-integrable function, where I is a possibly infinite interval. Jensen's inequality states that if $\phi : I \to \mathbb{R}$ is a convex function, then

$$\int_X \phi(g(x))d\mu(x) \geqslant \phi\left(\int_X g(x)d\mu(x)\right). \tag{2.4}$$

When ϕ is strictly convex, equality holds if and only if $g(x)$ is a constant for μ-almost all $x \in X$ (see [7]). If ϕ is a concave function, the inequality is reversed.

3. Proof of the main results

LEMMA 3.1. [22] *Let $K, L \in \mathscr{S}^n$ and $\psi \in \widetilde{\Phi} \cup \widetilde{\Psi}$. Then*

$$\lim_{\varepsilon \to 0^+} \frac{\rho_{K \widetilde{+}_\psi \varepsilon \cdot L}(u) - \rho_K(u)}{\varepsilon} = \frac{\rho_K(u)}{\psi_r'(1)} \psi\left(\frac{\rho_L(u)}{\rho_K(u)}\right),$$

uniformly for all $u \in S^{n-1}$.

LEMMA 3.2. *Let $K, L \in \mathscr{S}^n$, $\psi \in \widetilde{\Phi} \cup \widetilde{\Psi}$, and $0 \leqslant i \leqslant n-1$. Then*

$$\widetilde{W}_{\psi,i}(K,L) = \frac{1}{n} \int_{S^{n-1}} \psi\left(\frac{\rho_L(u)}{\rho_K(u)}\right) \rho_K^{n-i}(u)dS(u). \tag{3.1}$$

Proof. Suppose $\varepsilon > 0, K, L \in \mathscr{S}^n$, and $u \in S^{n-1}$. By Lemma 3.1, it follows that

$$\lim_{\varepsilon\to 0^+} \frac{\rho^{n-i}_{K\widetilde{+}_\psi \varepsilon\cdot L}(u) - \rho^{n-i}_K(u)}{\varepsilon} = (n-i)\rho^{n-i-1}_{K\widetilde{+}_\psi \varepsilon\cdot L}(u)|_{\varepsilon=0} \cdot \lim_{\varepsilon\to 0^+} \frac{\rho_{K\widetilde{+}_\psi \varepsilon\cdot L}(u) - \rho_K(u)}{\varepsilon}$$

$$= \frac{(n-i)\rho^{n-i}_K(u)}{\psi'_r(1)} \psi\left(\frac{\rho_L(u)}{\rho_K(u)}\right),$$

uniformly on S^{n-1}. Then, using (2.1),

$$\lim_{\varepsilon\to 0^+} \frac{\widetilde{W}_i(K\widetilde{+}_\psi \varepsilon\cdot L) - \widetilde{W}_i(K)}{\varepsilon} = \frac{n-i}{n\psi'_r(1)} \int_{S^{n-1}} \psi\left(\frac{\rho_L(u)}{\rho_K(u)}\right) \rho^{n-i}_K(u) dS(u).$$

Hence, by (2.2), we have

$$\widetilde{W}_{\psi,i}(K,L) = \frac{1}{n} \int_{S^{n-1}} \psi\left(\frac{\rho_L(u)}{\rho_K(u)}\right) \rho^{n-i}_K(u) dS(u). \quad \Box$$

Taking $L = K$ in (3.1), we obtain $\widetilde{W}_{\psi,i}(K,K) = \psi(1)\widetilde{W}_i(K)$. The case $i = 0$ of the dual Orlicz mixed quermassintegral $\widetilde{W}_{\psi,i}(K,L)$ is the dual Orlicz mixed volume $\widetilde{V}_\psi(K,L)$, which was defined by Zhu, Zhou and Xu [22] (see also [6]).

For $K \in \mathscr{S}^n$, since $\frac{1}{n}\int_{S^{n-1}} \rho^{n-i}_K(u)dS(u) = \widetilde{W}_i(K)$, then the measure μ on S^{n-1} given by $d\mu(u) = \rho^{n-i}_K(u)dS(u)/(n\widetilde{W}_i(K))$ is a probability measure. Next, we will establish the following dual Orlicz-Minkowski inequality for dual quermassintegrals.

THEOREM 3.1. *Let $K, L \in \mathscr{S}^n$, $\psi \in \widetilde{\Phi} \cup \widetilde{\Psi}$, and $0 \leqslant i \leqslant n-1$. If $\psi_0(t) = \psi(t^{\frac{1}{n-i}})$ is concave, then*

$$\frac{\widetilde{W}_{\psi,i}(K,L)}{\widetilde{W}_i(K)} \leqslant \psi\left(\left(\frac{\widetilde{W}_i(L)}{\widetilde{W}_i(K)}\right)^{\frac{1}{n-i}}\right),$$

while if $\psi_0(t)$ is convex, the inequality is reversed. When ψ_0 is strictly concave (or convex, as appropriate), equality holds if and only if K and L are dilates.

Proof. If $\psi_0(t) = \psi(t^{\frac{1}{n-i}})$ is concave, from (3.1) and (2.4), it follows that

$$\frac{\widetilde{W}_{\psi,i}(K,L)}{\widetilde{W}_i(K)} = \frac{1}{n\widetilde{W}_i(K)} \int_{S^{n-1}} \psi_0\left(\left(\frac{\rho_L(u)}{\rho_K(u)}\right)^{n-i}\right) \rho^{n-i}_K(u) dS(u)$$

$$\leqslant \psi_0\left(\frac{1}{n\widetilde{W}_i(K)} \int_{S^{n-1}} \rho^{n-i}_L(u) dS(u)\right) = \psi_0\left(\frac{\widetilde{W}_i(L)}{\widetilde{W}_i(K)}\right) = \psi\left(\left(\frac{\widetilde{W}_i(L)}{\widetilde{W}_i(K)}\right)^{\frac{1}{n-i}}\right).$$

This gives the desired inequality. When ψ_0 is strictly concave, from the equality condition of Jensen's inequality (2.4), we have that K and L are dilates.

The case in which ψ_0 is convex is completely analogous. $\Box$

REMARK 1. The case $i=0$ of Theorem 3.1 is the dual Orlicz-Minkowski inequality, which was established by Zhu, Zhou and Xu [22] (see also [6]).

The above result can be used to deduce the corresponding Orlicz-Brunn-Minkowski inequality, as follows.

THEOREM 3.2. *Let $K,L\in\mathscr{S}^n$, $\psi\in\widetilde{\Phi}\cup\widetilde{\Psi}$, and $0\leqslant i\leqslant n-1$. If $\psi_0(t)=\psi(t^{\frac{1}{n-i}})$ is concave, then*

$$\psi\left(\left(\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(K\widetilde{+}_\psi L)}\right)^{\frac{1}{n-i}}\right)+\psi\left(\left(\frac{\widetilde{W}_i(L)}{\widetilde{W}_i(K\widetilde{+}_\psi L)}\right)^{\frac{1}{n-i}}\right)\geqslant\psi(1),$$

while if $\psi_0(t)$ is convex, the inequality is reversed. When ψ_0 is strictly concave (or convex), equality holds if and only if K and L are dilates.

Proof. Let $K_\psi=K\widetilde{+}_\psi L$. If $\psi_0(t)=\psi(t^{\frac{1}{n-i}})$ is concave, from (1.4), (3.1) and Theorem 3.1, it follows that

$$\begin{aligned}\psi(1)&=\frac{1}{n\widetilde{W}_i(K_\psi)}\int_{S^{n-1}}\left(\psi\left(\frac{\rho_K(u)}{\rho_{K_\psi}(u)}\right)+\psi\left(\frac{\rho_L(u)}{\rho_{K_\psi}(u)}\right)\right)\rho_{K_\psi}^{n-i}(u)dS(u)\\&=\frac{1}{\widetilde{W}_i(K_\psi)}\widetilde{W}_{\psi,i}(K_\psi,K)+\frac{1}{\widetilde{W}_i(K_\psi)}\widetilde{W}_{\psi,i}(K_\psi,L)\\&\leqslant\psi\left(\left(\frac{\widetilde{W}_i(K)}{\widetilde{W}_i(K_\psi)}\right)^{\frac{1}{n-i}}\right)+\psi\left(\left(\frac{\widetilde{W}_i(L)}{\widetilde{W}_i(K_\psi)}\right)^{\frac{1}{n-i}}\right).\end{aligned}$$

When ψ_0 is strictly concave, equality holds if and only if K and L are dilates.

The case in which ψ_0 is convex is analogous. □

REMARK 2. The case $i=0$ of Theorem 3.2 is the dual Orlicz-Brunn-Minkowski inequality, which was established by Zhu, Zhou and Xu [22] (see also [6]).

LEMMA 3.3. [19] *Let $K,L\in\mathscr{K}_o^n$ and $\phi\in\Phi$. If $\psi(t)=\phi(t^{-1})$, then*

$$K^*\widetilde{+}_\psi L^*=(K+_\phi L)^*.$$

Proof of Theorem 1.1. Set $\psi(t)=\phi(t^{-1})$. We clearly have that $\psi\in\widetilde{\Psi}$ and, moreover, that $\psi_0(t)=\psi(t^{\frac{1}{n-i}})$ is convex. From Theorem 3.2 (for K^* and L^*) together with Lemma 3.3, we get

$$\phi(1)=\psi(1)\geqslant\psi\left(\left(\frac{\widetilde{W}_i(K^*)}{\widetilde{W}_i(K^*\widetilde{+}_\psi L^*)}\right)^{\frac{1}{n-i}}\right)+\psi\left(\left(\frac{\widetilde{W}_i(L^*)}{\widetilde{W}_i(K^*\widetilde{+}_\psi L^*)}\right)^{\frac{1}{n-i}}\right)$$

$$=\phi\left(\left(\frac{\widetilde{W}_i(K^*)}{\widetilde{W}_i((K+_\phi L)^*)}\right)^{-\frac{1}{n-i}}\right)+\phi\left(\left(\frac{\widetilde{W}_i(L^*)}{\widetilde{W}_i((K+_\phi L)^*)}\right)^{-\frac{1}{n-i}}\right).$$

The equality case follows from the equality case of Theorem 3.2. □

REMARK 3. When $\psi(t)=t^p, p\geqslant 1$, the above result for the volume case $(i=0)$ was previously stated by Firey [1]. Its natural extension for any i-th (classical) quermassintegral was recently obtained by Hernández Cifre and Nicolás [9]. The latter has been generalized to both the setting $p\geqslant 0$ (by Saroglou [17]) and the Orlicz case (by Wang and Huang [19]). Hence, all the above results involve the classical quermassintegrals, the usual framework when dealing with Minkowski/ L_p-/ Orlicz additions; however, here we provide with an alternative Orlicz version for dual quermassintegrals (which allows us to recover the previous results for $i=0$).

LEMMA 3.4. *Let $K,L\in\mathscr{K}_o^n$ and $\psi\in\widetilde{\Psi}$ such that $\phi(t)=\psi(t^{-1})$ is convex. Then*

$$K+_\phi L=(K^*\widetilde{+}_\psi L^*)^*.$$

Proof. It is clear that $\phi\in\Phi$. By the definition of the radial Orlicz addition (1.4), (2.3), and the fact that $\psi(1)=\phi(1)$, we have

$$\phi(1)=\psi(1)=\psi\left(\frac{\rho_{K^*}(x)}{\rho_{K^*\widetilde{+}_\psi L^*}(x)}\right)+\psi\left(\frac{\rho_{L^*}(x)}{\rho_{K^*\widetilde{+}_\psi L^*}(x)}\right)$$

$$=\phi\left(\frac{h_K(x)}{h_{(K^*\widetilde{+}_\psi L^*)^*}(x)}\right)+\phi\left(\frac{h_L(x)}{h_{(K^*\widetilde{+}_\psi L^*)^*}(x)}\right),$$

for all $x\in\mathbb{R}^n$. Thus, from (1.2), we get that $K+_\phi L=(K^*\widetilde{+}_\psi L^*)^*$. □

Proof of Theorem 1.2. Set $\phi(t)=\psi(t^{-1})$. We clearly have that $\phi\in\Phi$. By (1.3) for K^* and L^* together with Lemma 3.4, we get

$$\psi(1)=\phi(1)\geqslant\phi\left(\left(\frac{W_i(K^*)}{W_i(K^*+_\phi L^*)}\right)^{\frac{1}{n-i}}\right)+\phi\left(\left(\frac{W_i(L^*)}{W_i(K^*+_\phi L^*)}\right)^{\frac{1}{n-i}}\right)$$

$$=\psi\left(\left(\frac{W_i(K^*)}{W_i((K\widetilde{+}_\psi L)^*)}\right)^{-\frac{1}{n-i}}\right)+\psi\left(\left(\frac{W_i(L^*)}{W_i(K\widetilde{+}_\psi L)^*)}\right)^{-\frac{1}{n-i}}\right),$$

with equality if and only if K and L are dilates. □

LEMMA 3.5. [19] *Let* $K, L \in \mathscr{S}^n$ *and* $\psi \in \widetilde{\Phi} \cup \widetilde{\Psi}$. *If* $\varphi(t) = \psi(t^{-1})$, *then*

$$K^\circ \widetilde{+}_\varphi L^\circ = (K \widetilde{+}_\psi L)^\circ.$$

Proof of Theorem 1.3. Without loss of generality, we may consider that ψ_0 is concave. Assume also that $\psi \in \widetilde{\Phi}$, which implies that the function φ given by $\varphi(t) = \psi(t^{-1})$ belongs to $\widetilde{\Psi}$. So, by Theorem 3.2 (for φ, K° and L°) together with Lemma 3.5 we have

$$\psi(1) = \varphi(1) \leqslant \varphi\left(\left(\frac{\widetilde{W}_i(K^\circ)}{V(K^\circ \widetilde{+}_\varphi L^\circ)}\right)^{\frac{1}{n-i}}\right) + \varphi\left(\left(\frac{\widetilde{W}_i(L^\circ)}{\widetilde{W}_i(K^\circ \widetilde{+}_\varphi L^\circ)}\right)^{\frac{1}{n-i}}\right)$$

$$= \psi\left(\left(\frac{\widetilde{W}_i(K^\circ)}{\widetilde{W}_i((K \widetilde{+}_\psi L)^\circ)}\right)^{-\frac{1}{n-i}}\right) + \psi\left(\left(\frac{\widetilde{W}_i(L^\circ)}{\widetilde{W}_i((K \widetilde{+}_\psi L)^\circ)}\right)^{-\frac{1}{n-i}}\right).$$

The equality case follows from the equality case of Theorem 3.2. □

REMARK 4. The case $i = 0$ of Theorem 1.3 is the dual Orlicz-Brunn-Minkowski inequality for star dual bodies, which was obtained by Wang and Huang [19].

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REFERENCES

[1] J. FIREY, *Polar means of convex bodies and a dual to the Brunn-Minkowski theorem*, Canad. J. Math., **13** (1961), 444–453.

[2] J. FIREY, *p-means of convex bodies*, Math. Scand., **10** (1962), 17–24.

[3] R. J. GARDNER, *The Brunn-Minkowski inequality*, Bull. Amer. Math. Soc., **39** (2002), 355–405.

[4] R. J. GARDNER, *Geometric tomography*, second edition, Encyclopedia of Mathematics and its Applications, **58**, Cambridge University Press, Cambridge, 2006.

[5] R. J. GARDNER, D. HUG AND W. WEIL, *The Orlicz Brunn-Minkowski theory: a general framework, additions, and inequalities*, J. Differential Geom., **97** (2014), 427–476.

[6] R. J. GARDNER, D. HUG, W. WEIL AND D. YE, *The dual Orlicz-Brunn-Minkowski theory*, J. Math. Anal. Appl., **430** (2015), 810–829.

[7] G. H. HARDY, J. E. LITTLEWOOD AND G. PÓLYA, *Inequalities*, Cambridge Univ. Press, London, 1934.

[8] C. HABERL, E. LUTWAK, D. YANG AND G. ZHANG, *The even Orlicz Minkowski problem*, Adv. Math., **224** (2010), 2485–2510.

[9] M. A. HERNÁNDEZ CIFRE AND J. Y. NICOLÁS, *On Brunn-Minkowski-type inequalities for polar bodies*, J. Geom Anal., **26** (2014), 1–13.

[10] E. LUTWAK, *Dual mixed volumes*, Pacific J. Math., **58** (1975), 531–538.

[11] E. LUTWAK, *The Brunn-Minkowski-Firey theory I: Mixed volumes and the Minkowski problem*, J. Differential Geom., **38** (1993), 131–150.

[12] E. LUTWAK, *The Brunn-Minkowski-Firey theory II: Affine and geominimal surface areas*, Adv. Math., **118** (1996), 244–294.

[13] E. LUTWAK, D. YANG AND G. ZHANG, *Orlicz projection bodies*, Adv. Math., **223** (2010), 220–242.

[14] E. LUTWAK, D. YANG AND G. ZHANG, *Orlicz centroid bodies*, J. Differential Geom., **84** (2010), 365–387.
[15] M. MOSZYŃSKA, *Quotient star bodies, intersection bodies and star duality*, J. Math. Anal. Appl., **232** (1999), 45–60.
[16] M. MOSZYŃSKA, *Selected Topics in Convex Geometry*, Springer Verlag, 2005.
[17] C. SAROGLOU, *More on logarithmic sums of convex bodies*, Mathematika, **62** (2014), 818–841.
[18] R. SCHNEIDER, *Convex bodies: the Brunn-Minkowski theory*, second expanded edition, Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 2014.
[19] Y. WANG AND Q. HUANG, *Orlicz-Brunn-Minkowski inequality for polar bodies and dual star bodies*, Math. Inequal. Appl., **20** (2017), 1139–1144.
[20] D. XI, H. JIN AND G. LENG, *The Orlicz Brunn-Minkowski inequality*, Adv. Math., **260** (2014), 350–374.
[21] G. XIONG AND D. ZOU, *Orlicz mixed quermassintegrals*, Sci. China Math., **57** (2014), 2549–2562.
[22] B. ZHU, J. ZHOU AND W. XU, *Dual Orlicz-Brunn-Minkowski theory*, Adv. Math., **264** (2014), 700–725.

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