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OPERATOR NORM AND NUMERICAL RADIUS ANALOGUES OF COHEN'S INEQUALITY

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Abstract. Let D be an invertible multiplication operator on $L^2(X,\mu)$, and let A be a bounded operator on $L^2(X,\mu)$. In this note we prove that $\|A\|^2 \leqslant \|DA\|\,\|D^{-1}A\|$, where $\|\cdot\|$ denotes the operator norm. If, in addition, the operators A and D are positive, we also have $w(A)^2 \leqslant w(DA)\,w(D^{-1}A)$, where w denotes the numerical radius.

1. Introduction

Let A be a nonnegative matrix and D a diagonal matrix with positive diagonal entries. J. E. Cohen [1, inequality (3.7)] proved that

$$r(A)^2 \leqslant r(DA)\,r(D^{-1}A) \tag{1}$$

where r denotes the spectral radius. In fact, he proved slightly more general inequality [1, inequality (3.6)]. Let D_1, ..., D_m be diagonal matrices with positive diagonal entries such that $D_1 \cdots D_m = I$, where I is the identity matrix. Then

$$r(A)^m \leqslant r(D_1A)\,r(D_2A)\cdots r(D_mA). \tag{2}$$

This inequality is important in the theory of population dynamics in Markovian environments; see [2]. In this note we consider this inequality with the spectral radius replaced by the operator norm and by the numerical radius. In fact, we introduce a more general setting.

Throughout the note, let μ be a σ-finite positive measure on a set X. We consider bounded (linear) operators on the complex Banach space $L^p(X,\mu)$ $(1 \leqslant p \leqslant \infty)$. The adjoint of an operator A on $L^p(X,\mu)$ is denoted by A^*. An operator A on $L^p(X,\mu)$ is said to be *positive* if it maps nonnegative functions to nonnegative ones. Given operators A and B on $L^p(X,\mu)$, we write $A \geqslant B$ if the operator $A - B$ is positive. The norm in $L^p(X,\mu)$ and the operator norm are denoted by $\|\cdot\|_p$ and $\|\cdot\|$, respectively. The *numerical radius* of an operator A on $L^2(X,\mu)$ is defined by

$$w(A) := \sup\{|\langle Af, f\rangle| : f \in L^2(X,\mu), \|f\|_2 = 1\}.$$

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If, in addition, A is positive, then we have

$$w(A) = \sup\{\langle Af, f\rangle : f \in L^2(X,\mu), f \geqslant 0, \|f\|_2 = 1\}.$$

Indeed, this follows from the estimate

$$|\langle Af, f\rangle| \leqslant \int_X |Af|\,|f|\,d\mu \leqslant \langle A|f|, |f|\rangle$$

that holds for any $f \in L^2(X,\mu)$. It is well-known [4] that

$$r(A) \leqslant w(A) \leqslant \|A\|$$

for all bounded operators A on $L^2(X,\mu)$.

We will make use of the following generalized Hölder's inequality; see e.g. [3, p. 196, Exercise 31], or [5] for its proof.

LEMMA 1.1. *Assume that $r \in [1,\infty]$ and $p_1, \ldots, p_m \in [1,\infty]$ satisfy the equality*

$$\sum_{i=1}^{m} \frac{1}{p_i} = \frac{1}{r},$$

where (as usual) we interpret $1/\infty$ as 0. If $f_i \in L^{p_i}(X,\mu)$ for $i = 1,\ldots,m$, then $f_1 \cdots f_m \in L^r(X,\mu)$ and

$$\|f_1 \cdots f_m\|_r \leqslant \|f_1\|_{p_1} \cdots \|f_m\|_{p_m}.$$

2. Results

We begin with the operator norm analogue of Cohen's inequality (2).

THEOREM 2.1. *Let $D_1, \ldots, D_m$ be multiplication operators on $L^p(X,\mu)$ ($1 \leqslant p \leqslant \infty$) such that $D_1 \cdots D_m = I$. Let A be a bounded operator A on $L^p(X,\mu)$. Then*

$$\|A\|^m \leqslant \|D_1 A\|\,\|D_2 A\| \cdots \|D_m A\|. \tag{3}$$

If A is the adjoint operator of an operator, then we also have

$$\|A\|^m \leqslant \|AD_1\|\,\|AD_2\| \cdots \|AD_m\|. \tag{4}$$

Proof. For each $i = 1,\ldots,m$, let d_i be the function in $L^\infty(X,\mu)$ such that $D_i f = d_i f$ for all $f \in L^p(X,\mu)$. Therefore, $d_1 \cdots d_m = 1$ a.e. on X. There is no loss of generality in assuming that $A \neq 0$. Choose an arbitrary number $c \in (0, \|A\|)$. Then there exists a function $f \in L^p(X,\mu)$ such that $\|f\|_p = 1$ and the function $g := Af$ has norm more than c. Since $\|D_i A\| \geqslant \|D_i A f\|_p = \|d_i g\|_p$, we have

$$\|D_1 A\|\,\|D_2 A\| \cdots \|D_m A\| \geqslant \|d_1 g\|_p\,\|d_2 g\|_p \cdots \|d_m g\|_p.$$

Now Lemma 1.1 gives the inequality

$$\|d_1 g\|_p \,\|d_2 g\|_p \cdots \|d_m g\|_p \geqslant \|(d_1 g)\cdots(d_m g)\|_{p/m} = \|g^m\|_{p/m} = \|g\|_p^m > c^m \,.$$

It follows that

$$\|D_1 A\|\,\|D_2 A\| \cdots \|D_m A\| > c^m.$$

Since the number $c \in (0, \|A\|)$ is arbitrary, we obtain the inequality (3).

To prove the inequality (4), we assume first that $p < \infty$. Then the adjoint operators A^*, D_1^*, …, D_m^* are operators on the Banach space $L^q(X,\mu)$, where $q \in (1,\infty]$ is the conjugate exponent to p. Applying the inequality (3) for them, we have

$$\|A\|^m = \|A^*\|^m \leqslant \|D_1^* A^*\| \cdots \|D_m^* A^*\| = \|(AD_1)^*\| \cdots \|(AD_m)^*\| = \|AD_1\| \cdots \|AD_m\|,$$

proving the inequality (4) in this case.

Assume now that $p = \infty$. We are assuming that there exists an operator B on $L^1(X,\mu)$ such that $B^* = A$. Let E_i $(i = 1,\dots,m)$ be the multiplication operator on $L^1(X,\mu)$ with the function d_i, so that $E_i^* = D_i$. Applying the inequality (3) for the operators B, E_1, …, E_m, we obtain that

$$\|A\|^m = \|B\|^m \leqslant \|E_1 B\| \cdots \|E_m B\| = \|(E_1 B)^*\| \cdots \|(E_m B)^*\| = \|AD_1\| \cdots \|AD_m\|.$$

This completes the proof of the theorem.

COROLLARY 2.2. *Let D be an invertible multiplication operator on $L^p(X,\mu)$ ($1 \leqslant p < \infty$), and let A be a bounded operator on $L^p(X,\mu)$. Then*

$$\|A\|^2 \leqslant \|DA\|\,\|D^{-1}A\| \tag{5}$$

and

$$\|A\|^2 \leqslant \|AD\|\,\|AD^{-1}\|.$$

We now turn to the numerical radius analogue of Cohen's inequality.

THEOREM 2.3. *Let D_1, …, D_m be positive multiplication operators on $L^2(X,\mu)$ such that $D_1 \cdots D_m \geqslant I$. Then, for any positive operator A on $L^2(X,\mu)$,*

$$w(A)^m \leqslant w(D_1 A)\, w(D_2 A) \cdots w(D_m A) \tag{6}$$

and

$$w(A)^m \leqslant w(AD_1)\, w(AD_2) \cdots w(AD_m). \tag{7}$$

Proof. For each $i = 1,\dots,m$, let d_i be the function in $L^\infty(X,\mu)$ such that $D_i f = d_i f$ for all $f \in L^2(X,\mu)$. By the assumption, we have $d_1 \cdots d_m \geqslant 1$ a.e. on X. There is no loss of generality in assuming that $A \neq 0$. Choose an arbitrary number $c \in (0, w(A))$. Then there exists a nonnegative function $f \in L^2(X,\mu)$ such that

$\|f\|_2 = 1$ and the nonnegative function $g := Af$ satifies the inequality $\langle g, f\rangle > c$. Since $w(D_iA) \geqslant \langle D_iAf, f\rangle = \langle d_ig, f\rangle = \|\sqrt{d_igf}\|_2^2$, we have

$$w(D_1A)\,w(D_2A)\cdots w(D_mA) \geqslant \left(\|\sqrt{d_1gf}\|_2\,\|\sqrt{d_2gf}\|_2\cdots\|\sqrt{d_mgf}\|_2\right)^2.$$

Using Lemma 1.1 we obtain the inequality

$$\|\sqrt{d_1gf}\|_2\,\|\sqrt{d_2gf}\|_2\cdots\|\sqrt{d_mgf}\|_2 \geqslant \left(\int_X (d_1\cdots d_m)^{1/m} gf\,d\mu\right)^{m/2} \geqslant \left(\int_X gf\,d\mu\right)^{m/2} > c^{m/2}.$$

It follows that

$$w(D_1A)\,w(D_2A)\cdots w(D_mA) > c^m.$$

Since the number $c \in (0, w(A))$ is arbitrary, we get the inequality (6).

To prove (7), we apply (6) for the adjoint operator A^*:

$$w(A)^m = w(A^*)^m \leqslant w(D_1A^*)\cdots w(D_mA^*) = w(AD_1)\cdots w(AD_m).$$

COROLLARY 2.4. *Let D be an invertible positive multiplication operator on $L^2(X,\mu)$, and let A be a positive operator on $L^2(X,\mu)$. Then*

$$w(A)^2 \leqslant w(DA)\,w(D^{-1}A) \tag{8}$$

and

$$w(A)^2 \leqslant w(AD)\,w(AD^{-1}).$$

The following example shows that in Theorem 2.3 and Corollary 2.4 we cannot omit the assumption that multiplication operators are positive. The same example also shows that Cohen's inequality does not hold without the positivity assumption.

EXAMPLE 2.5. Define the matrices A and D by

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

One can verify that $r(A) = \|A\| = 2$. Since $r(A) \leqslant w(A) \leqslant \|A\|$, we conclude that $w(A) = 2$ as well. Since

$$DA = D^{-1}A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

is unitarily equivalent to a multiple of a Jordan nilpotent J and $w(J) = 1/2$, we have $w(D^{-1}A) = w(DA) = \|DA\|/2 = 1$, and so the inequality (8) does not hold. Since $r(D^{-1}A) = r(DA) = 0$, the inequality (1) is not true either.

One may ask whether the inequality

$$w(A)^2 \leqslant w(DA)\,w(AD^{-1})$$

holds for an invertible positive multiplication operator D on $L^2(X,\mu)$ and for a positive operator A on $L^2(X,\mu)$. The following example show that this is not the case.

EXAMPLE 2.6. Define the matrices A and D by

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix},$$

where $d \in (0,1)$. Then $w(A) = w(AD^{-1}) = 1/2$ and $w(DA) = d/2$, so that $w(A)^2 > w(DA)\,w(AD^{-1})$.

We conclude this note by posing an open question.

QUESTION 2.7. Is Cohen's inequality (1) true for operators on the space $L^2(X,\mu)$? That is, does it the inequality

$$r(A)^2 \leqslant r(DA)\,r(D^{-1}A) \tag{9}$$

hold for an invertible positive multiplication operator D on $L^2(X,\mu)$ and for a positive operator A on $L^2(X,\mu)$?

If the operator A has rank one, the answer is affirmative. Namely, if $A = u \otimes v$ for some nonnegative functions u and v in $L^2(X,\mu)$ and if the positive function $\varphi \in L^\infty(X,\mu)$ corresponds to the multiplication operator D, then we have

$$r(A) = \int_X uv\,d\mu = \int_X \sqrt{\varphi uv}\sqrt{uv/\varphi}\,d\mu,$$
$$r(DA) = \int_X \varphi uv\,d\mu\ , \quad r(D^{-1}A) = \int_X uv/\varphi\,d\mu,$$

so that (9) holds by the Cauchy-Schwarz inequality.

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